Celest Mech Dyn Astr (2010) 108:147–164
DOI 10.1007/s10569-010-9298-y

ORIGINAL ARTICLE

Linear stability for some symmetric periodic simultaneous binary collision orbits in the four-body problem

Lennard F. Bakker · Tiancheng Ouyang · Duokui Yan · Skyler Simmons · Gareth E. Roberts

Received: 30 September 2009 / Revised: 28 June 2010 / Accepted: 2 July 2010 /
Published online: 28 July 2010

Abstract We apply the analytic-numerical method of Roberts to determine the linear stability of time-reversible periodic simultaneous binary collision orbits in the symmetric collinear four-body problem with masses 1, m, m, 1, and also in a symmetric planar four-body problem with equal masses. In both problems, the assumed symmetries reduce the determination of linear stability to the numerical computation of a single real number. For the collinear problem, this verifies the earlier numerical results of Sweatman for linear stability with respect to collinear and symmetric perturbations.

Keywords N-body problem · Singular periodic orbits · Stability · Schubart-like orbit · Roberts' method

Mathematics Subject Classification (2000) Primary: 70F10 · 70H12 · 70H14 · Secondary: 70F16 · 70H33

L. F. Bakker (✉) · T. Ouyang · S. Simmons
Department of Mathematics, Brigham Young University, Provo, UT 84602, USA
e-mail: bakker@math.byu.edu

T. Ouyang
e-mail: ouyang@math.byu.edu

S. Simmons
e-mail: xinkaisen@yahoo.com

D. Yan
Chern Institute of Mathematics, Nankai University, 300071 Tianjin, People's Republic of China
e-mail: duokui@math.byu.edu

G. E. Roberts
Department of Mathematics and Computer Science, College of the Holy Cross, 1 College Street, Worcester, MA 01610, USA
e-mail: groberts@radius.holycross.edu

1 Introduction

Recently, Roberts (2007) described an analytic-numerical method for determining the linear stability of a symmetric periodic orbit of a Hamiltonian system. He applied this method to the time-reversible collision-free figure-eight orbit in the equal mass three-body problem (Moore 1993; Chenciner and Montgomery 2000). (Other such choreographic solutions have been found numerically; Simó 2001). Roberts' method shows that the figure eight orbit is linearly stable. The method uses the symmetries to factor a matrix similar to the monodromy matrix for the periodic orbit into an integer power of the product of two involutions. One of the two involutions depends on the linearized dynamics along only a part of the periodic orbit. For the figure eight this part is one-twelve of the full orbit since it has a symmetry group isomorphic to the group $D_3 \times \mathbb{Z}_2$ of order 12. (Here the dihedral group D_k is the group of symmetries of the regular k-gon.) The eigenvalues of the product of the two involutions are then reduced to the numerical computation of a few real numbers.

Schubart (1956) numerically discovered a singular periodic orbit in the collinear equal mass three-body problem. The orbit alternates between binary collisions. Hénon extended Schubart's numerical investigations to the case of unequal masses (Hénon 1977). Only recently did Venturelli (2008) and Moeckel (2008) analytically prove the existence of the Schubart orbit when the outer masses are equal and the inner mass is arbitrary. The linear stability of the Schubart orbit was determined numerically by Hietarinta and Mikkola (1993) revealing that linear stability occurs for some but not all of the choices of the three masses. The role that the Schubart orbit plays in the overall structure of the phase space is considered in Saito and Tanikawa (2007), Orlov et al. (2008), and Saito and Tanikawa (2009) through extensive numerically studies.

Sweatman (2002, 2006) numerically found and determined the linear stability of a Schubart-like orbit in the symmetric collinear four-body problem with masses $1, m, m$, and 1 for $m > 0$. This Schubart-like periodic orbit alternates between simultaneous binary collisions (SBC) and inner binary collisions. Ouyang and Yan (2010) proved analytically the existence and symmetries of this orbit. In the regularized setting, this periodic orbit has a symmetry group isomorphic to D_2, of which both of the generators are time-reversing symmetries. The role that the Schubart-like orbit plays in the overall structure of phase space is considered numerically in Sekigucki and Tanikawa (2004).

Ouyang et al. (2008) numerically found and then analytically proved the existence and symmetry of a singular periodic orbit in a symmetric planar four-body problem with equal masses in which the four bodies alternate between different simultaneous binary collisions. In the regularized setting, this periodic orbit has a symmetry group isomorphic to D_4, of which one of the generators is a time-reversing symmetry. Bakker et al. (2010) have numerically continued this singular periodic orbit and its symmetry with respect to the origin to a simultaneous binary collision orbit in the planar pairwise symmetric four-body, showing that when the masses are $1, m, 1$, and m (identified in a counterclockwise manner) with $0 < m \leq 1$, linear stability seems to hold only when $0.538 \leq m \leq 1$.

In this paper we apply the method of Roberts to show the linear stability of the Schubart-like orbit in the symmetric collinear four-body $1, m, m, 1$ problem for certain values of $m > 0$, and of the singular periodic orbit in the symmetric planar equal mass four-body problem. The regularization of the collisions in these singular periodic orbits is achieved by a generalized Levi-Civita type transformation and an appropriate scaling of time, as adapted from Aarseth and Zare (1974). In the first setting, the linear stability is determined from the regularized equations for perturbations of the Schubart-like orbit that are both symmetrical and collinear. In the second setting, the linear stability is determined from the regularized equations for perturbations of the planar singular orbit that preserve the positions of the four equal masses relative to the symmetry with respect to the origin and also relative to the reflection across the 45 degree line in the plane. In both cases, the assumed symmetries mean that the regularized equations have only one degree of freedom (after rescaling for Energy/time) and so reduce the determination of linear stability to a single real number which we find using a numerical computation. Our linear stability analysis determines values of m in the interval [0, 50] in the symmetric collinear problem for which the singular periodic orbit is linearly stable, and also shows that the planar singular periodic orbit is linearly stable. Much of this linear stability analysis is contained in Ph.D. Thesis of Yan (2009). The linearly stable examples above support and extend the conjecture made by Roberts (2007) that the only linearly stable periodic orbits in the equal mass n-body problem are those that exhibit a time-reversing symmetry.

Our linear stability analysis has confirmed Sweatman's determination of the stability of the Schubart-like orbit for perturbations that are symmetrical and collinear. Sweatman used a numerical perturbation technique to assess the collinear and the transverse linear stability of the singular periodic orbit when the masses are arranged from left to right as m_1, m_2, m_2, and m_1 with the condition that $m_1 + m_2 = 2$. Our mass parameter m is related to his mass parameter m_1 by $m = (2-m_1)/m_1$. Sweatman numerically estimated the two linear stability parameters associated with collinear motion. In terms of our mass parameter m, the linear stability parameter that is associated with symmetric perturbations indicates that linear stability for collinear and symmetric perturbations occurs when the value of m is smaller than approximately 2.83 and when it is larger than approximately 35.4, and is linearly unstable otherwise. (The other linear stability parameter for collinear motion is not associated with any instability.) He also numerically estimated the two linear stability parameters for transverse perturbations (in two dimensions) which show instability for approximately $0 < m < 0.406$ and $0.569 < m < 1.02$. Our application of Roberts' method to the linear stability of the Schubart-like orbit for collinear and symmetric perturbations requires less computation than previously used techniques and only requires numerical integration of the regularized periodic orbit and its linearization.

2 Linear Stability of Periodic Orbits

For a smooth function Γ defined on an open subset of $\mathbb{R}^{2n}$, suppose that $\gamma(s)$ is a T-periodic solution of a Hamiltonian system $z' = J\nabla\Gamma(z)$ where $' = d/ds$,

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

for I the appropriately sized identity matrix, and ∇ is the gradient operator. The fundamental matrix solution $X(s)$ of the linearized equations along $\gamma(s)$,

$$\xi' = J\nabla^2\Gamma(\gamma(s))\xi, \qquad \xi(0) = I \tag{1}$$

(where $\nabla^2\Gamma$ is the symmetric matrix of second-order partials of Γ) is symplectic and satisfies $X(s+T) = X(s)X(T)$ for all s. The matrix $X(T)$ is commonly called the monodromy matrix for γ, and it measures the non-periodicity of solutions to the linearized equations. The eigenvalues of $X(T)$ are the characteristic multipliers of γ, and determine the linear stability of the periodic solution γ. Linear stability therefore requires that all of the multipliers lie on the unit circle.

The characteristic multipliers may be obtained by solving (1) with different initial conditions. For an invertible matrix Y_0, let $Y(s)$ be the fundamental matrix solution to

$$\xi' = J\nabla^2\Gamma(\gamma(s))\xi, \qquad \xi(0) = Y_0. \tag{2}$$

By definition of $X(s)$, we know that $Y(s) = X(s)Y_0$, and so $X(T) = Y(T)Y_0^{-1}$. It follows that the matrix $Y_0^{-1}Y(T)$ is similar to the monodromy matrix i.e.,

$$X(T) = Y(T)Y_0^{-1} = Y_0(Y_0^{-1}Y(T))Y_0^{-1}.$$

Thus the eigenvalues of $Y_0^{-1}Y(T)$ are identical to the characteristic multipliers.

2.1 Stability reduction using symmetry

The monodromy matrix for a periodic solution with special types of symmetry can be factored using some linear algebra and standard techniques in differential equations. We begin by reviewing the relevant factorization and reduction theory that are applicable to a wide range of symmetric periodic orbits commonly found in Hamiltonian systems. Proofs of the following statements can be found in Roberts (2007).

Lemma 1 *Suppose that $\gamma(s)$ is a symmetric T-periodic solution of a Hamiltonian system with Hamiltonian Γ and symmetry matrix S such that:*

1. *for some positive integer N, $\gamma(s + T/N) = S\gamma(s)$ for all s;*
2. $\Gamma(Sz) = \Gamma(z)$;
3. $SJ = JS$;
4. *S is orthogonal.*

Then the fundamental matrix solution $X(s)$ to the linearization problem in (1) satisfies

$$X(s + T/N) = SX(s)S^T X(T/N).$$

Here of course, the notation S^T means the transpose of S. We mention this because we are using the letter T in two distinct ways.

Corollary 1 *Given the hypothesis of Lemma 1, the fundamental matrix solution $X(s)$ satisfies*

$$X(kT/N) = S^k \left(S^T X(T/N)\right)^k$$

for any $k \in \mathbb{N}$.

A remark here is that if $Y(s)$ is the fundamental matrix solution to Eq. (2), then for any $k \in \mathbb{N}$, the matrix $Y(kT/N)$ factors as

$$Y(kT/N) = S^k Y_0(Y_0^{-1}S^T Y(T/N))^k.$$

Lemma 2 *Suppose that $\gamma(s)$ is a T-periodic solution of a Hamiltonian system with Hamiltonian Γ and time-reversing symmetry S such that:*

1. *for some positive integer N, $\gamma(-s + T/N) = S\gamma(s)$ for all s;*
2. $\Gamma(Sz) = \Gamma(z)$;
3. $SJ = -JS$;
4. *S is orthogonal.*

Then the fundamental matrix solution $X(s)$ to the linearization problem in (1) satisfies

$$X(-s+T/N) = SX(s)S^T X(T/N).$$

Corollary 2 *Given the hypothesis of Lemma 2,*

$$X(T/N) = SB^{-1}S^T B \quad \textit{where } B = X(T/2N).$$

Several more remarks about these factorizations are needed here.

1. In the case of time-reversing symmetry matrix, S is typically block diagonal with two blocks of opposite sign, one for the position variable and one for the momenta, that is,

$$\begin{bmatrix} F & 0 \\ 0 & -F \end{bmatrix}$$

where F is orthogonal. A matrix of this form is orthogonal and anti-commutes with J.
2. A matrix satisfying properties 3 and 4 of Lemma 2 is symplectic with a multiplier of -1 since $S^T JS = -S^T SJ = -J$.
3. If $Y(s)$ is the fundamental matrix solution to (2), then a similar argument shows that $Y(-s+T/N) = SY(s)Y_0^{-1}S^T Y(T/N)$ and consequently

$$Y(T/N) = SY_0 B^{-1}S^T B, \quad \text{where } B = Y(T/2N).$$

Applying this factorization theory results in expressing the matrix $Y_0^{-1}Y(T)$, which is similar to $X(T)$, as W^k for some positive integer k, where the symplectic matrix W is the product of two involutions. If an eigenvalue of W lies on the unit circle, then so does its kth power. The symplectic matrix W is called *spectrally stable* if all of its eigenvalues lie on the unit circle.

Lemma 3 *For a symplectic matrix W, suppose there is a matrix K such that*

$$\frac{1}{2}\left(W + W^{-1}\right) = \begin{bmatrix} K^T & 0 \\ 0 & K \end{bmatrix}. \tag{3}$$

Then W is spectrally stable if and only if all of the eigenvalues of K are real and have absolute value smaller than or equal to 1.

We will show for each of the symmetric periodic orbits under consideration, there is a choice of Y_0 such that W satisfies Lemma 3. This reduces the linear stability to the computation of the eigenvalues of a 2×2 matrix K. As one of the eigenvalues of K is known to be real and have absolute value 1, the problem of *linear stability* is determined by the numerical computation of the other real eigenvalue of K and showing that, within error, it lies between -1 and 1. This is because when that other real eigenvalue of K is between -1 and 1, it is the real part of two distinct eigenvalues of modulus one for W (Roberts 2007), and so the matrix W^k is diagonalizable when restricted to the two dimensional subspace corresponding to those two distinct eigenvalues of W.

3 Linear stability for the collinear four-body symmetric periodic orbit

The existence and symmetry of the Schubart-like periodic orbit in the collinear four-body problem has been analytically proven (Ouyang and Yan 2010). We review it here. For $x_1 \geq x_2 \geq 0$, we assume that four masses are located at x_1, x_2, $-x_2$ and $-x_1$ with masses $1, m, m,$

and 1 respectively with $m > 0$. We also assume that the system remains collinear and symmetrically distributed about the center of mass located at the origin. The respective velocities of the four bodies are $\dot{x}_1, \dot{x}_2, -\dot{x}_2, -\dot{x}_1$ where $\dot{} = d/dt$. The Newtonian equations are

$$\ddot{x}_1 = -\frac{1}{4x_1^2} - \frac{m}{(x_1+x_2)^2} - \frac{m}{(x_1-x_2)^2}$$
$$\ddot{x}_2 = -\frac{m}{4x_2^2} - \frac{1}{(x_1+x_2)^2} + \frac{1}{(x_1-x_2)^2}$$

We recount the approach in Sweatman (2002, 2006) to regularize this system. The Hamiltonian for this system is

$$H = \frac{1}{4}w_1^2 + \frac{1}{4m}w_2^2 - \frac{1}{2x_1} - \frac{m^2}{2x_2} - \frac{2m}{x_1+x_2} - \frac{2m}{x_1-x_2},$$

where $w_1 = 2\dot{x}_1$ and $w_2 = 2m\dot{x}_2$ are the conjugate momenta to x_1 and x_2. Introduce new canonical coordinates q_1, q_2, p_1, p_2 by

$$q_1 = x_1 - x_2, \quad q_2 = 2x_2, \quad p_1 = w_1, \quad p_2 = \frac{1}{2}(w_1 + w_2).$$

The Hamiltonian in the new canonical coordinates is

$$H = \frac{1}{4}\left(1+\frac{1}{m}\right)p_1^2 - \frac{p_1p_2}{m} + \frac{p_2^2}{m} - \frac{2m}{q_1} - \frac{m^2}{q_2} - \frac{2m}{q_1+q_2} - \frac{1}{2q_1+q_2}.$$

To regularize the equations of motion, Sweatman introduced a Levi-Civita type of canonical transformation

$$Q_i^2 = q_i, \quad P_i = 2Q_ip_i \quad (i = 1, 2),$$

for the the canonical coordinates Q_1, Q_2, P_1, P_2, and then replaced time t by the new independent variable s given by

$$\frac{dt}{ds} = Q_1^2Q_2^2.$$

In the extended phase space, this produces the regularized Hamiltonian

$$\Gamma = \frac{dt}{ds}(H-E) = \frac{1}{16}\left(1+\frac{1}{m}\right)Q_2^2P_1^2 + \frac{-Q_1Q_2P_1P_2 + Q_1^2P_2^2}{4m}$$
$$- m^2Q_1^2 - 2mQ_2^2 - \frac{2mQ_1^2Q_2^2}{Q_1^2+Q_2^2} - \frac{Q_1^2Q_2^2}{2Q_1^2+Q_2^2} - EQ_1^2Q_2^2.$$

We fix the energy $E = -1$. The Hamiltonian system in the new coordinate system is

$$Q_1' = \frac{Q_2}{4}\left[\frac{1}{2}\left(1+\frac{1}{m}\right)Q_2P_1 - \frac{1}{m}Q_1P_2\right], \tag{4}$$

$$Q_2' = \frac{Q_1}{2m}\left[Q_1P_2 - \frac{1}{2}Q_2P_1\right], \tag{5}$$

$$P_1' = \frac{P_2}{4m}(Q_2P_1 - 2Q_1P_2) + 2m^2Q_1 + \frac{4mQ_1Q_2^4}{(Q_1^2+Q_2^2)^2} + \frac{2Q_1Q_2^4}{(2Q_1^2+Q_2^2)^2} - 2Q_1Q_2^2, \tag{6}$$

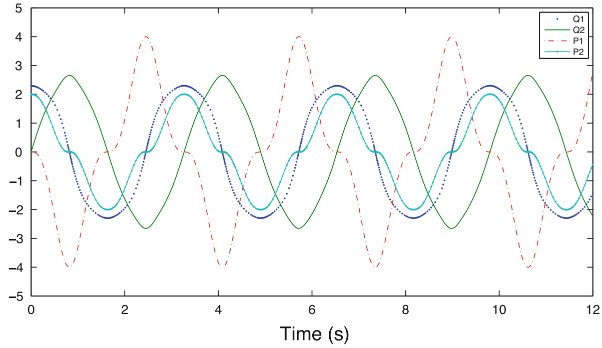

Fig. 1 The periodic solution in the coordinate system Q_1, Q_2, P_1, P_2 when $m = 1$

$$P_2' = \frac{P_1}{4}\left[\frac{Q_1 P_2}{m} - \frac{Q_2 P_1}{2}\left(1+\frac{1}{m}\right)\right] + 4mQ_2 + \frac{4mQ_1^4 Q_2}{(Q_1^2+Q_2^2)^2} + \frac{4Q_1^4 Q_2}{(2Q_1^2+Q_2^2)^2} - 2Q_1^2 Q_2, \tag{7}$$

where $'$ is the derivative with respect to s.

From the proof (Ouyang and Yan 2010) of the existence of the Schubart-like periodic orbit $Q_1(s), Q_2(s), P_1(s), P_2(s)$ of period T, there is a positive constant $R(m)$ such that

$$Q_1(0) = R(m),\ Q_2(0) = 0,\ P_1(0) = 0,\ P_2(0) = 2m^{3/2}$$

are the initial conditions of the symmetric periodic simultaneous binary collision orbit. The period T is the period of the regularized orbit and during the regularized time period from 0 to T, the actual four-body orbit completes two full periods of oscillation. The initial conditions $Q_1(0), Q_2(0), P_1(0), P_2(0)$ correspond to a binary collision of the two inner bodies. By the analytic proof of the existence of this periodic orbit, another binary collision of the two inner bodies occurs at $s = T/2$ where the conditions are

$$Q_1(T/2) = -R(m),\ Q_2(T/2) = 0,\ P_1(T/2) = 0,\ P_2(T/2) = -2m^{3/2}.$$

Simultaneous binary collisions correspond to the conditions of the periodic solution when $s = T/4$ and $s = 3T/4$, i.e., $Q_1(s) = 0$ at these values of s. Here $R(1) \approx 2.29559$. Figure 1 contains a plot of the coordinates Q_1, Q_2, P_1, P_2 of the periodic orbit when $m = 1$.

3.1 Stability reductions using symmetry

The Schubart-like periodic solution $\gamma(s) = (Q_1(s), Q_2(s), P_1(s), P_2(s))$ with period T in the regularized collinear symmetric problem has two time-reversing symmetries. These can be readily seen in Fig. 1, and are analytically established symmetries for this periodic orbit (Ouyang and Yan 2010) which we review here. For

$$F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

the matrix

$$S = \begin{bmatrix} F & 0 \\ 0 & -F \end{bmatrix}$$

is orthogonal and symmetric: $S^{-1} = S^T = S$. It is also an involution, i.e., $S^2 = I$. Since $S\gamma(-s+T)$ is a solution of (4) through (7), and since this solution shares the same initial conditions as $\gamma(s)$ at $s = 0$ by T-periodicity of γ, uniqueness of solutions implies that the matrix S satisfies

$$\gamma(-s+T) = S\gamma(s) \text{ for all } s.$$

Thus S is a time-reversing symmetry of $\gamma(s)$. With $N = 1$, conditions (2), (3), and (4) in Lemma 2 are satisfied, and so by Corollary 2, the monodromy matrix for γ satisfies

$$X(T) = SX(T/2)^{-1}S^T X(T/2) = SX(T/2)^{-1}SX(T/2). \tag{8}$$

Consequently, from the above equation and $S^2 = I$,

$$[SX(T)]^2 = \left[X(T/2)^{-1}SX(T/2)\right]\left[X(T/2)^{-1}SX(T/2)\right] = I.$$

Since $-S\gamma(-s+T/2)$ is a solution of (4) through (7), and as $-S\gamma(T/2)$ is the same as $\gamma(0)$, uniqueness of solutions implies that the matrix $-S$ satisfies

$$\gamma(-s+T/2) = -S\gamma(s) \text{ for all } s.$$

Thus $-S$ is another time-reversing symmetry of $\gamma(s)$. For $N = 2$, conditions (2), (3), and (4) of Lemma 2 are satisfied, and so Corollary 2 implies that

$$X(T/2) = SX(T/4)^{-1}SX(T/4). \tag{9}$$

For

$$B = X(T/4),$$

combining equations (8) and (9) gives

$$X(T) = (SB^{-1}SB)^2$$

With $A = SB^{-1}SB$ and $D = B^{-1}SB$, then

$$X(T) = A^2 = (SD)^2,$$

where $S^2 = I$ and $D^2 = I$. The two time-reversing symmetries S and $-S$ of γ are both involutions, and together they generate a D_2 symmetry group for γ.

3.2 A good basis

We have reduced the stability analysis to $T/4$, which is a quarter of the period of the regularized solution during which the actual four-body orbit completes half a period. Let $Y(s)$ be the fundamental matrix solution to the linearized equations about Schubart-like periodic orbit $\gamma(s)$ with arbitrary initial conditions Y_0. Let

$$B = Y(T/4).$$

By the third remark following Corollary 2, the matrix $Y_0^{-1}Y(T)$, which is similar to the monodromy matrix $X(T) = Y(T)Y_0^{-1}$, satisfies

$$Y_0^{-1}Y(T) = \left(\left(Y_0^{-1}SY_0\right)B^{-1}SB\right)^2.$$

The question of stability reduces to showing that the eigenvalues of

$$W = \left(Y_0^{-1}SY_0\right)B^{-1}SB$$

are on the unit circle. An appropriate choice of Y_0 will simplify the factor $Y_0^{-1}SY_0$ in W. Set

$$\Lambda = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Lemma 4 *There exists Y_0 such that*

1. *Y_0 is orthogonal and symplectic, and*
2. *$Y_0^{-1}SY_0 = \Lambda$.*

Proof Choose the third column of Y_0 to be $\gamma'(0)/\|\gamma'(0)\| = [0\ 1\ 0\ 0]^T = e_2$. (See Fig. 1 where it can be seen that $\gamma'(0)/\|\gamma'(0)\| = e_2$.) For $e_3 = [0\ 0\ 1\ 0]^T$, the matrix

$$Y_0 = [Je_2, Je_3, e_2, e_3] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

is orthogonal and symplectic. Since $S = \operatorname{diag}\{1, -1, -1, 1\}$, it follows that $Y_0^{-1}SY_0$ has the desired form. □

Setting $D = B^{-1}SB$ and choosing Y_0 as constructed in Lemma 4 gives

$$W = \left(Y_0^{-1}SY_0\right) B^{-1}SB = \Lambda D.$$

The matrices Λ and D are both involutions, i.e., $\Lambda^2 = I$, $D^2 = I$. From these it follows that

$$W^{-1} = D\Lambda.$$

The form of the inverse of the symplectic matrix B is determined by its block partition into two by two submatrices A_1, A_2, A_3, A_4:

$$B = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \text{ implies that } B^{-1} = \begin{bmatrix} A_4^T & -A_2^T \\ -A_3^T & A_1^T \end{bmatrix}.$$

Thus we have that

$$D = B^{-1}SB = \begin{bmatrix} K^T & L_1 \\ -L_2 & -K \end{bmatrix}$$

for the 2×2 matrices $K = A_3^TFA_2 + A_4^TFA_1$, $L_1 = A_4^TFA_2 + A_2^TFA_4$, and $L_2 = A_3^TFA_1 + A_1^TFA_3$. It follows that

$$W = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} K^T & L_1 \\ -L_2 & -K \end{bmatrix} = \begin{bmatrix} K^T & L_1 \\ L_2 & K \end{bmatrix},$$

and

$$W^{-1} = \begin{bmatrix} K^T & L_1 \\ -L_2 & -K \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} K^T & -L_1 \\ -L_2 & K \end{bmatrix}.$$

Hence,

$$\frac{1}{2}\left(W + W^{-1}\right) = \begin{bmatrix} K^T & 0 \\ 0 & K \end{bmatrix}.$$

We show that the first column of K is $[-1\ 0]^T$. Set $v = Y_0^{-1}\gamma'(0)$. By the choice of Y_0,

$$v = Y_0^T\gamma'(0) = \|\gamma'(0)\|e_3.$$

Since S is symmetric and Y_0 is orthogonal, then by the third remark after Corollary 2,

$$W = Y_0^{-1}SY_0B^{-1}SB = Y_0^{-1}SY_0B^{-1}S^TB = Y_0^TY(T/2).$$

Differentiation of $\gamma'(s) = J\nabla\Gamma(\gamma(s))$ with respect to s yields $\gamma'' = J\nabla^2\Gamma(\gamma(s))\gamma'(s)$. This implies that $\gamma'(s)$ is the solution of $\xi' = J\nabla^2\Gamma(\gamma(s))\xi$ with $\xi(0) = \gamma'(0)$. Since $Y(s)$ satisfies the linearized equations along $\gamma(s)$ too, we have $\gamma'(s) = Y(s)Y_0^{-1}\gamma'(0) = Y(s)v$. This implies that

$$Y_0^{-1}\gamma'(T/2) = Y_0^TY(T/2)v = Wv.$$

Since $\gamma(s)$ satisfies $\gamma(-s+T/2) = -S\gamma(s)$ for all s, then $\gamma'(-s+T/2) = S\gamma'(s)$ for all s. Setting $s = 0$ in this gives $\gamma'(T/2) = S\gamma'(0)$. Since $\gamma'(0)$ is a nonzero scalar multiple of e_2 and since $Se_2 = -e_2$, then

$$Y_0^{-1}\gamma'(T/2) = Y_0^{-1}S\gamma'(0) = -Y_0^{-1}\gamma'(0) = -v.$$

Thus $Wv = -v$, implying that -1 is an eigenvalue of W and e_3 is an eigenvector of W corresponding to this eigenvalue. Thus the first column of K is as claimed.

We show that the form of K is

$$K = \begin{bmatrix} -1 & * \\ 0 & c_2^T(SJc_4) \end{bmatrix},$$

where c_i is the ith column of $B = Y(T/4)$. From the form of W in terms of K, L_1, and L_2, note that the $(2,2)$ entry of K is the $(4,4)$ entry of W. Since B is symplectic, its inverse is $B^{-1} = -JB^TJ$. The matrix S satisfies $SJ = -JS$. Thus

$$W = \Lambda D = \Lambda B^{-1}SB = -\Lambda JB^TJSB = \Lambda JB^TSJB.$$

Using the submatrices A_1, A_2, A_3, A_4 for the block partition of B, and the definitions of Λ and J gives

$$W = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}SJB = \begin{bmatrix} A_2^T & A_4^T \\ A_1^T & A_3^T \end{bmatrix}SJB.$$

The $(4,4)$ entry of W is thus $c_2^T(SJc_4)$.

The eigenvalues of K are -1 and the quantity $c_2^T(SJc_4)$ that depends on m. Lemma 3 and the comments that follow it now imply the following linear stability result.

Theorem 1 *The Schubart-like orbit with masses* $1, m, m, 1$ *is linearly stable with respect to collinear and symmetric perturbations if and only if* $-1 < c_2^T(SJc_4) < 1$.

3.3 Numerical calculations

With an absolute error tolerance of 1×10^{-12} in a one-dimensional grid search for $Q(0)$, our numerical results for $m = 1$ showed that the initial condition

$$Q_1(0) = R(1) = 2.295592258717, \quad Q_2(0) = 0, \quad P_1(0) = 0, \quad P_2(0) = 2$$

leads to a periodic simultaneous binary collision periodic orbit (as in Fig. 1) whose period T satisfies $T/4 = 0.817348080989685$. Here T is the period of the regularized orbit. Using

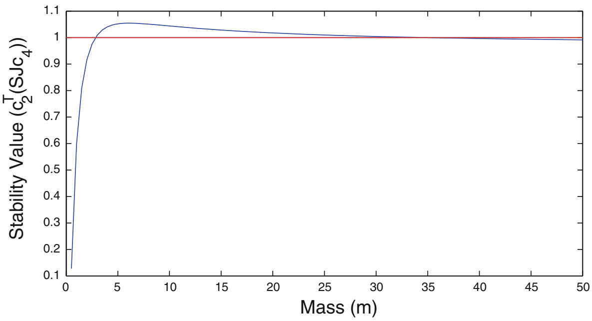

Fig. 2 The value of $c_2^T(SJc_4)$ for values of m between 0 and 50

MATLAB and a Runge–Kutta–Fehlberg algorithm, we computed the columns of the matrix $Y(T/4)$ with an absolute error tolerance of 4×10^{-6}. From this, we got

$$c_2^T(SJc_4) = 0.598490.$$

For values of m between 0 and 50 at 0.01 increments, we numerically computed the value of $R(m)$ in the initial conditions and the value of the period T (with an absolute error tolerance of 4×10^{-6}), and the values of $c_2^T(SJc_4)$ (with an absolute error tolerance of 1×10^{-6}). The results of these computations are contained in Fig. 2.

A closer look at the numerical data in Fig. 2 for where the value of $c_2^T(SJc_4)$ is close to 1 gives estimates of the two values of m where the stability of the periodic orbit changes. The first critical value of m is approximately $m = 2.83$, and the second critical value of m is approximately $m = 35.4$. For the other values of m in $[0, 50]$, the eigenvalues of K are distinct. From the estimates of $c_2^T(SJc_4)$ for m in $[0, 50]$, Theorem 1 implies that there exists small positive constants $\epsilon_i, i = 1, 2, 3, 4$ such that the periodic simultaneous binary collision orbit in the collinear symmetric four body problem with masses $1, m, m, 1$ is linearly stable when $m < 2.83 - \epsilon_1$ and $35.4 + \epsilon_2 < m \leq 50$, and is linearly unstable when $2.83 + \epsilon_3 < m < 35.4 - \epsilon_4$. This result confirms the linear stability analysis of Sweatman (2006) for m between 0 and 50, asserting that the periodic orbit is linearly unstable when m is between 2.83 and 35.4. Simulations of the periodic orbit when m is between 2.83 and 35.4 indicate that the linear instability is manifested slowly over time.

4 Linear stability for the planar symmetric periodic orbit

The existence and symmetries of a singular periodic orbit in the planar four-body problem with equal masses have been analytically proven (Ouyang et al. 2008). We review it here. If (x_1, x_2) is the position of the first body with $|x_2| \leq x_1$, then the positions of the remaining three bodies are (x_2, x_1), $(-x_1, -x_2)$, and $(-x_2, -x_1)$, with the center of mass at the origin. The positions of the four equal masses remain symmetric with respect to the origin, and also with respect to the reflection across the line $x_2 = x_1$. When each body has mass $m = 1$, the Newtonian equations for this planar four-body problem are

$$(\ddot{x}_1, \ddot{x}_2) = -\left[\frac{(x_1 - x_2, x_2 - x_1)}{2^{3/2}(x_1 - x_2)^3} + \frac{(x_1, x_2)}{4(x_1^2 + x_2^2)^{3/2}} + \frac{(x_1 + x_2, x_1 + x_2)}{2^{3/2}(x_1 + x_2)^3}\right].$$

The initial conditions for the periodic orbit, and the periodic orbit are illustrated in Fig. 3 where the horizontal axis is x_1 and the vertical axis is x_2. On the left, the initial positions

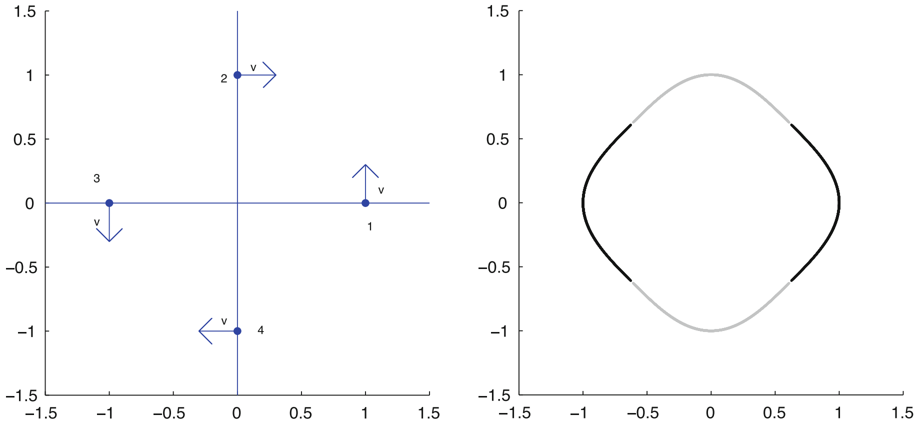

Fig. 3 The initial conditions (*left*) and the trajectories (*right*) for the four equal masses

and velocities with norm v of the four bodies are shown. On the right, the shape of the orbit is shown. Bodies 1 and 3 trace out the darker colored curves, while bodies 2 and 4 trace out the lighter curves. Simultaneous binary collisions occur at the points where the darker and lighter curves meet.

We adapt the approach in Sweatman (2002, 2006) to regularize this system. The Hamiltonian for this system is

$$H = \frac{1}{8}\left(w_1^2 + w_2^2\right) - \frac{\sqrt{2}}{x_1 - x_2} - \frac{\sqrt{2}}{x_1 + x_2} - \frac{1}{\sqrt{x_1^2 + x_2^2}},$$

where $w_1 = 4\dot{x}_1$ and $w_2 = 4\dot{x}_2$ are the conjugate momentum. In terms of the canonical coordinates (q_1, q_2, p_1, p_2) defined by

$$q_1 = x_1 - x_2, \quad q_2 = x_1 + x_2, \quad w_1 = p_1 + p_2, \quad w_2 = p_2 - p_1,$$

the Hamiltonian becomes

$$H = \frac{1}{4}\left(p_1^2 + p_2^2\right) - \frac{\sqrt{2}}{q_1} - \frac{\sqrt{2}}{q_2} - \frac{\sqrt{2}}{\sqrt{q_1^2 + q_2^2}}.$$

The Levi-Civita type of canonical transformation used to regularize the collinear problem now applies to the planar four body equal mass problem. In terms of the canonical coordinates (Q_1, Q_2, P_1, P_2) defined by

$$q_i = Q_i^2, \quad P_i = 2Q_i p_i \quad (i = 1, 2),$$

and the new time variable s defined by

$$\frac{dt}{ds} = Q_1^2 Q_2^2,$$

the Hamiltonian in extended phase space becomes

$$\Gamma = \frac{dt}{ds}(H - E) = \frac{1}{16}(P_1^2 Q_2^2 + P_2^2 Q_1^2)$$
$$-\sqrt{2}(Q_1^2 + Q_2^2) - \frac{\sqrt{2}Q_1^2 Q_2^2}{\sqrt{Q_1^4 + Q_2^4}} - E Q_1^2 Q_2^2 \quad (10)$$

where E is the total energy of the Hamiltonian H. The differential equations in terms of the new coordinates $\{Q_1, Q_2, P_1, P_2\}$ are

$$Q_1' = \frac{1}{8} P_1 Q_2^2 \quad (11)$$
$$Q_2' = \frac{1}{8} P_2 Q_1^2 \quad (12)$$
$$P_1' = -\frac{1}{8} P_2^2 Q_1 + 2\sqrt{2} Q_1 + \frac{2\sqrt{2} Q_1 Q_2^2}{\sqrt{Q_1^4 + Q_2^4}} - \frac{2\sqrt{2} Q_1^5 Q_2^2}{(Q_1^4 + Q_2^4)^{\frac{3}{2}}} + 2E Q_1 Q_2^2 \quad (13)$$
$$P_2' = -\frac{1}{8} P_1^2 Q_2 + 2\sqrt{2} Q_2 + \frac{2\sqrt{2} Q_2 Q_1^2}{\sqrt{Q_1^4 + Q_2^4}} - \frac{2\sqrt{2} Q_2^5 Q_1^2}{(Q_1^4 + Q_2^4)^{\frac{3}{2}}} + 2E Q_2 Q_1^2. \quad (14)$$

Unlike the collinear problem, we do not fix the value of E here. For each $\zeta > 0$ there exists $v_0 > 0$ such that the initial conditions

$$Q_1(0) = \zeta, \quad Q_2(0) = \zeta, \quad P_1(0) = -4v_0, \quad P_2(0) = 4v_0, \quad (15)$$

lead to a periodic solution with a minimal period T (Ouyang et al. 2008). Here T is the period of the regularized orbit, and during the regularized time period from 0 to T, the actual four-body orbit completes two full periods of oscillation. From $\Gamma = 0$, the value of E is determined by this choice of ζ and v_0. In the analytical proof, this periodic orbit satisfies

$$Q_1(T/4) = -\zeta, \quad Q_2(T/4) = \zeta, \quad P_1(T/4) = -4v_0, \quad P_2(T/4) = -4v_0.$$

Simultaneous binary collisions correspond to $s = T/8, 5T/8$ i.e., when $Q_1(s) = 0$, and to $s = 3T/8, 7T/8$, i.e., when $Q_2(s) = 0$. For $\zeta = 1$, $4v_0 = 2.57486992651942$, and $T/8 = 1.62047369909693$. Figure 4 illustrates the coordinates (Q_1, Q_2, P_1, P_2) of this periodic solution.

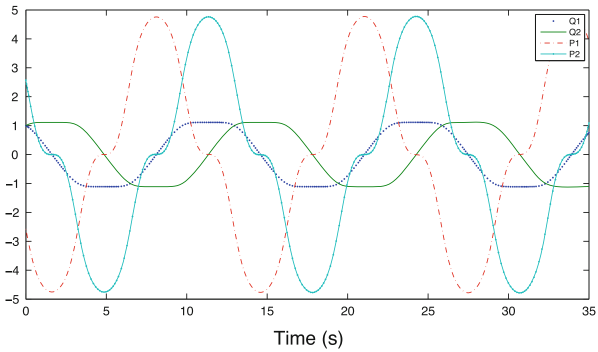

Fig. 4 The periodic solution in the coordinate system Q_1, Q_2, P_1, P_2

4.1 Stability reductions using symmetry

The symmetric periodic planar orbit $\gamma(t) = Q_1(t), Q_2(t), P_1(t), P_2(t)$ with period T has a time-reversing symmetry and a time-preserving symmetry. These symmetries can be observed in Fig. 4, and are analytically established (Ouyang et al. 2008). We review them here. For

$$F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

the matrices

$$S_F = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}, \quad S_G = \begin{bmatrix} G & 0 \\ 0 & -G \end{bmatrix}$$

satisfy $S_F^{-1} = S_F^T$, $S_F^2 \neq I$, $S_F^3 \neq I$, $S_F^4 = I$, $S_G^2 = I$, $S_G^T = S_G$, and $(S_F S_G)^2 = I$. Since $\gamma(s + T/4)$ and $S_F\gamma(s) = (-Q_2(s), Q_1(s), -P_2(s), P_1(s))$ are solutions of (11) through (14) and share the same initial conditions when $s = 0$, uniqueness of solutions implies that

$$\gamma(s + T/4) = S_F\gamma(s) \text{ for all } s.$$

Thus S_F is a time-preserving symmetry of $\gamma(s)$. With $N = 4$, conditions (2), (3), and (4) of Lemma 1 are satisfied, so that Corollary 1 (with $k = 4$) and $S_F^4 = I$ imply that

$$X(T) = S_F^4 \left(S_F^T X(T/4)\right)^4 = \left(S_F^T X(T/4)\right)^4.$$

Since $\gamma(-s + T/4)$ and $S_G\gamma(s)$ are solutions of (11) through (14) and share the same initial conditions when $s = 0$, uniqueness of solutions implies that

$$\gamma(-s + T/4) = S_G\gamma(s) \text{ for all } s.$$

Thus S_G is a time-reversing symmetry for $\gamma(s)$. With $N = 4$, conditions (2), (3), and (4) of Lemma 2 are satisfied, and so Corollary 2 implies that

$$X(T/4) = S_G\,[X(T/8)]^{-1}\,S_G^T X(T/8) = S_G\,[X(T/8)]^{-1}\,S_G X(T/8).$$

Let

$$B = X(T/8).$$

Combining the factorization of $X(T)$ that involves S_F and the factorization of $X(T/4)$ that involves S_G gives the factorization

$$X(T) = \left(S_F^T S_G B^{-1} S_G B\right)^4.$$

Setting

$$Q = S_F^T S_G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

and $D = B^{-1} S_G B$ results in the factorization

$$X(T) = (QD)^4$$

where Q and D are both involutions. The symmetries S_F and S_G generate a D_4 symmetry group for the periodic orbit $\gamma(s)$.

4.2 A good basis

We have reduced the stability analysis to $T/8$, which is an eighth of the period of the regularized orbit during which the actual four-body orbit completes a quarter period. Let $Y(s)$ be the fundamental matrix solution to the linearized equations about the planar periodic orbit $\gamma(s)$ with arbitrary initial conditions Y_0. Let

$$B = Y(T/8).$$

By remarks following Corollaries 1 and 2, the matrix $Y_0^{-1}Y(T)$, which is similar to the monodromy matrix $X(T) = Y(T)Y_0^{-1}$, satisfies

$$Y_0^{-1}Y(T) = (Y_0^{-1}S_F^T S_G Y_0 B^{-1} S_G B)^4 = (Y_0^{-1} Q Y_0 B^{-1} S_G B)^4.$$

The question of linear stability reduces to showing that the eigenvalues of

$$W = Y_0^{-1} Q Y_0 B^{-1} S_G B$$

are on the unit circle. Recall that

$$\Lambda = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Lemma 5 *There exists Y_0 such that*

1. *Y_0 is orthogonal and symplectic, and*
2. *$Y_0^{-1} Q Y_0 = \Lambda$.*

Proof Choose the third column of Y_0 to be

$$\frac{\gamma'(0)}{\|\gamma'(0)\|} = \frac{1}{c}\begin{bmatrix} -a & a & b & b \end{bmatrix}^T$$

where $a = v_0\zeta^2/2$, $b = E\zeta^3 = (2v_0^2 - 2\sqrt{2} - 1)\zeta$ and $c = \sqrt{2a^2 + 2b^2}$. Let $\mathrm{col}_i(Y_0)$ denote the ith column of Y_0. Define

$$\mathrm{col}_1(Y_0) = J \cdot \mathrm{col}_3(Y_0) = \frac{1}{c}\begin{bmatrix} b & b & a & -a \end{bmatrix}^T.$$

We now choose $\mathrm{col}_4(Y_0)$ such that $\mathrm{col}_4(Y_0)$ is orthogonal to $\mathrm{col}_3(Y_0)$, and $\mathrm{col}_4(Y_0)$ is one of the eigenvectors of Q with respect to its eigenvalue of -1. Since the eigenspace of Q corresponding to its eigenvalue of -1 is

$$\mathrm{span}\left\{\begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T\right\},$$

define

$$\mathrm{col}_4(Y_0) = \frac{1}{c}\begin{bmatrix} b & -b & a & a \end{bmatrix}^T$$

and

$$\mathrm{col}_2(Y_0) = J \cdot \mathrm{col}_4(Y_0) = \frac{1}{c}\begin{bmatrix} a & a & -b & b \end{bmatrix}^T.$$

The matrix

$$Y_0 = \frac{1}{c}\begin{bmatrix} b & a & -a & b \\ b & a & a & -b \\ a & -b & b & a \\ -a & b & b & a \end{bmatrix},$$

is both symplectic and orthogonal and it satisfies $Y_0^{-1} Q Y_0 = \Lambda$. □

Setting $D = B^{-1} S_G B$ and choosing Y_0 to be the matrix constructed in Lemma 5 gives $W = \Lambda D$. The matrices Λ and D are involutions (the latter because $S_G^2 = I$). As in Sect. 3.2, $W^{-1} = D\Lambda$, and there is a 2×2 matrix K such that

$$\frac{1}{2}\left(W + W^{-1}\right) = \begin{bmatrix} K^T & 0 \\ 0 & K \end{bmatrix}.$$

We show that the first column of K is $[1\ 0]^T$. Since $S_G^T = S_G$, $Y_0^{-1} = Y_0^T$, it follows by the third remark following Corollary 2 that

$$W = Y_0^{-1} S_F^T S_G Y_0 B^{-1} S_G B = Y_0^{-1} S_F^T Y(T/4) = Y_0^T S_F^T Y(T/4).$$

Set $v = Y_0^{-1}\gamma'(0)$. By the choice of the matrix Y_0,

$$v = Y_0^{-1}\gamma'(0) = Y_0^T\gamma'(0) = \begin{bmatrix} 0 \\ 0 \\ ||\gamma'(0)|| \\ 0 \end{bmatrix} = ||\gamma'(0)||e_3.$$

Because $\gamma'(s)$ is a solution to the linearized equation $\xi' = J\nabla^2\Gamma(\gamma(s))\xi$ and because $Y(s)$ is a fundamental matrix solution with $Y(0) = Y_0$, then $\gamma'(s) = Y(s)Y_0^{-1}\gamma'(0)$ for all s. Hence,

$$Wv = Y_0^T S_F^T Y(T/4)v = Y_0^T S_F^T \gamma'(T/4). \tag{16}$$

Since γ satisfies $\gamma(s + T/4) = S_F\gamma(s)$ for all s and $S_F^{-1} = S_F^T$, it then follows that

$$\gamma'(s) = S_F^{-1}\gamma'(s + T/4) = S_F^T\gamma'(s + T/4).$$

Setting $s = 0$ in this gives $\gamma'(0) = S_F^T\gamma'(T/4)$, and consequently that

$$Y_0^T S_F^T\gamma'(T/4) = Y_0^T\gamma'(0) = Y_0^{-1}\gamma'(0) = v. \tag{17}$$

Equations (16) and (17) now combine to show that $Wv = v$, i.e, that 1 is an eigenvalue of W and e_3 is an eigenvector for W corresponding to this eigenvalue. The first column of K is as claimed. Since $D = B^{-1}S_G B$ and $W = \Lambda D$, the form of K follows from Sect. 3.2:

$$K = \begin{bmatrix} 1 & * \\ 0 & c_2^T(S_G J c_4) \end{bmatrix},$$

where c_i is the ith column of $B = Y(T/8)$.

The eigenvalues of K are 1 and the quantity $c_2^T(JS_G c_4)$. Lemma 3 and the comments that follow it now imply the following linear stability result.

Theorem 2 *The periodic simultaneous binary collision orbit in the planar symmetric equal mass four-body problem is linearly stable for perturbations that preserve the position of the four masses relative to the symmetry with respect to the origin and relative to the reflection across the line $x_2 = x_1$, if and only if $-1 < c_2^T(S_G J c_4) < 1$.*

4.3 Numerical calculations

Having not fixed E, we used an invariant scaling of the coordinates and time in equations (11) through (14) to preselect a period T for the regularized orbit before numerically computing the initial conditions for a periodic simultaneous binary collision orbit. For $\epsilon > 0$, if $Q_1(s), Q_2(s), P_1(s), P_2(s)$ is a periodic simultaneous collision orbit of equations (11) through (14), then replacing E with $\epsilon^{-2}E$ shows that $\epsilon Q_1(\epsilon s), \epsilon Q_2(\epsilon s), P_1(\epsilon s), P_2(\epsilon s)$ is also a periodic simultaneous binary collision orbit with energy $\epsilon^{-2}E$ and period $\epsilon^{-1}T$. Furthermore, it is straight-forward to show that monodromy matrices for the periodic simultaneous binary collision orbits corresponding to values of $\epsilon \neq 1$ are all similar to that for $\epsilon = 1$. Thus the linear stability of a periodic simultaneous binary collision orbit for one $\epsilon > 0$ implies the linear stability of the periodic simultaneous binary collision orbits for all $\epsilon > 0$.

We computed the value of $c_2^T(S_G J c_4)$ for the periodic simultaneous binary collision orbit whose period is $T = 8$. This means that the first time of a simultaneous binary collision for this orbit is at $s = 1$. We set $Q_1(0) = Q_2(0) = \xi$ and $-P_1(0) = P_2(0) = \eta$, and defined a function $F(\xi, \eta)$ to be equal to the vector quantity $(Q_1(1), P_2(1))$. We used Newton's method and a good initial guess to find a root (ξ, η) of F. This involved computing the Jacobian of F which was done using the linearized equations. With an absolute error tolerance of 6×10^{-11}, this numerical method shows that the initial conditions

$$Q_1(0) = Q_2(0) = 1.62047369909693, \quad -P_1(0) = P_2(0) = 2.57486992651942,$$

lead to a periodic solution with a period of $T = 8$, and a value of $E \approx -1.142329388$. Using MATLAB and a Runge–Kutta–Fehlberg algorithm, we computed the columns of the matrix $Y(T/8)$ with an absolute error tolerance of 2.5×10^{-12}. From this we got

$$c_2^T\,(S_G J c_4) = -0.68024151010592.$$

By this estimate and Theorem 2, the periodic simultaneous binary collision orbit is linearly stable. Using the scaling of coordinates and time described above, the initial conditions for the linearly stable periodic simultaneous binary collision orbit shown in Figs. 3 and 4 are

$$Q_1(0) = Q_2(0) = 1, \quad -P_1(0) = P_2(0) = 2.57486992651942$$

with a period T satisfying $T/8 = 1.62047369909693$, and energy $E \approx -2.999682732$.

When $c_2^T(S_G J c_4)$ is real and between -1 and 1, it is the real part of an eigenvalue with unit modulus for W. The real part of $\exp(3\pi i/4)$ is $-(1/2)\sqrt{2}$. This is fairly close to the estimated value of $c_2^T\,(S_G J c_4)$. Raising $\exp(3\pi i/4)$ to the fourth power gives $\exp(3\pi i) = -1$, and so two of the eigenvalues of the monodromy matrix of the planar periodic simultaneous binary collision orbit are close to -1. The symmetry reductions used to compute the eigenvalues over just one-eighth of the regularized period and the estimate of $c_2^T\,(S_G J c_4)$ showing that it is clearly between -1 and 1, assures the linear stability of the planar periodic simultaneous binary collision orbit for perturbations that preserve the symmetry with respect to the origin and the reflection across the 45 degree line.

Acknowledgments Gareth Roberts thanks the Department of Mathematics at Brigham Young University for hosting him. Lennard Bakker, Tiancheng Ouyang, Duokui Yan, and Skyler Simmons thank Gareth Roberts for his visit and collaboration. The research of Gareth E. Roberts is supported in part by National Science Foundation grant DMS-0708741. We also thank the referees for their comments that improved the clarity and concision of the paper.

References

Aarseth, S.J., Zare, K.: A regularization of the three-body problem. Cel. Mech. **10**, 185–205 (1974)

Bakker, L.F., Ouyang, T., Yan, D., Simmons, S.: Existence and stability of symmetric periodic simultaneous binary collision orbits in the planar pairwise symmetric four-body problem. Preprint posted on arxiv (2010)

Chenciner, A., Montgomery, R.: A remarkable periodic solution of the three-body problem in the case of equal masses. Ann. Math. **152**, 881–901 (2000)

Hénon, M.: Stability of interplay orbits. Cel. Mech. **15**, 243–261 (1977)

Hietarinta, J., Mikkola, S.: Chaos in the one-dimensional gravitational three-body problem. Chaos **3**, 183–203 (1993)

Moeckel, R.: A topological existence proof for the Schubart orbits in the collinear three-body problem. Dis. Con. Dyn. Syst. Ser. B **10**, 609–620 (2008)

Moore, C.: Braids in classical dynamics. Phys. Rev. Lett. **70**, 3675–3679 (1993)

Orlov, V.V., Petrova, A.V., Tanikawa, K., Saito, M.M., Martynova, A.L.: The rectilinear three-body problem. Celest. Mech. Dynam. Astron. **100**, 93–120 (2008)

Ouyang, T., Yan, D.: Periodic solutions with alternating singularities in the collinear four-body problem. Preprint posted on arxiv. Celest. Mech. Dynam. Astron. (2010) (submitted)

Ouyang, T., Simmons, S.C., Yan, D.: Periodic solutions with singularities in two dimensions in the n-body problem. Preprint posted on arxiv. Rocky Mountain J. Math. (to appear)

Roberts, G.: Linear stability analysis of the figure-eight orbit in the three-body problem. Ergod. Th. Dynam. Syst. **27**, 1947–1963 (2007)

Saito, M.M., Tanikawa, K.: The rectilinear three-body problem using symbol sequence I Role of triple collision. Celest. Mech. Dynam. Astron. **98**, 95–120 (2007)

Saito, M.M., Tanikawa, K.: The rectilinear three-body problem using symbol sequence II: role of periodic orbits. Celest. Mech. Dynam. Astron. **103**, 191–207 (2009)

Schubart, J.: Numerische Aufsuchung periodischer Lösungen im Dreikörperproblem. Astron. Nachriften **283**, 17–22 (1956)

Sekigucki, M., Tanikawa, K.: On the symmetric collinear four-body problem. Publ. Astron. Soc. Jpn. **56**, 235–251 (2004)

Simó, C.: New families of solutions in the N-body problem. Progress in Mathematics, vol. 201, Birkhäuser 101-115 (2001)

Sweatman, W.L.: The symmetrical one-dimensional Newtonian four-body problem: a numerical investigation. Celest. Mech. Dynam. Astron. **82**, 179–201 (2002)

Sweatman, W.L.: A family of symmetrical Schubart-like interplay orbits and their stability in the one-dimensional four-body problem. Celest. Mech. Dynam. Astron. **94**, 37–65 (2006)

Venturelli, A.: A variational proof of the existence of Von Schubart's orbit. Dis. Con. Dyn. Syst. Ser. B **10**, 699–717 (2008)

Yan, D.: Four-body problem with collision singularity, Ph.D. Thesis, Brigham Young University (2009)