

# Kolmogorov and Nekhoroshev theory for the problem of three bodies

Antonio Giorgilli · Ugo Locatelli · Marco Sansottera

Received: 23 September 2008 / Revised: 20 January 2009 / Accepted: 30 January 2009 /  
Published online: 3 March 2009  
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**Abstract** We investigate the long time stability in Nekhoroshev's sense for the Sun–Jupiter–Saturn problem in the framework of the problem of three bodies. Using computer algebra in order to perform huge perturbation expansions we show that the stability for a time comparable with the age of the universe is actually reached, but with some strong truncations on the perturbation expansion of the Hamiltonian at some stage. An improvement of such results is currently under investigation.

**Keywords** Perturbation theory · KAM theory · Nekhoroshev theory · Exponential stability · Kolmogorov theorem · Sun–Jupiter–Saturn system

## 1 Introduction

The stability of the Solar System is a classical, long standing and challenging problem, already pointed out by Newton. In this article we revisit the problem in the light of the theorems of Kolmogorov and of Nekhoroshev, with the aim of proving that they apply to the problem of three bodies with the masses and orbital parameters of Jupiter and Saturn.

Let us briefly recall the historical development of our knowledge. After 1954 a possible solution was suggested by the celebrated theorem of [Kolmogorov \(1954\)](#) stating the existence of a large measure set of invariant tori for a nearly integrable Hamiltonian system, e.g., the planetary system when the mutual perturbation of the planet is taken into account.

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A. Giorgilli (✉) · M. Sansottera  
Dipartimento di Matematica, Università degli Studi di Milano, via Saldini 50, 20133 Milano, Italy  
e-mail: antonio.giorgilli@unimi.it

M. Sansottera  
e-mail: marco.sansottera@unimi.it

U. Locatelli  
Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata",  
via della Ricerca Scientifica 1, 00133, Rome, Italy  
e-mail: locatell@mat.uniroma2.it

The relevance of Kolmogorov's result for the planetary problem has been soon emphasized by Arnold (1963a) and Moser (1973). In particular Arnold worked out a proof taking into account the degeneration of the unperturbed Hamiltonian which occurs in the planetary case (see Arnold 1963b). On the other hand, Moser first gave a proof for the case of an area preserving mapping of an annulus, and a few years later pointed out that the theorem of Kolmogorov implies that the classical Lindstedt series are actually convergent (see Moser 1962, 1967, respectively).

As a matter of fact it was soon remarked by Hénon that the application of the Kolmogorov's theorem to the planetary motions is not straightforward, due to the condition that the masses of the planets should be small enough. Indeed, the available estimates could only assure the applicability, e.g., to the problem of three bodies, if the masses of the planets are less than that of a proton. On the other hand, numerical integrations (see Carpino et al. 1987; Laskar 1989, 1994) of the full Solar System over a time span of billions of years have shown that the orbits of the inner planets exhibit a chaotic evolution which is incompatible with the quasi periodic motion predicted by Kolmogorov's theorem. Furthermore the subsystem of the major planets (i.e. Jupiter, Saturn, Uranus and Neptune) shows a very small positive Lyapunov exponent (see Murray and Holman 1999), once again, this cannot fit with a motion on an invariant torus.

A second approach was suggested by Moser (1955) and Littlewood (1959a,b) and fully stated by Nekhoroshev (1977, 1979) with his celebrated theorem on exponential stability. According to this theorem the time evolution of the actions of the system (which in the planetary case are actually related to the semimajor axes, the inclinations and the eccentricities) remains bounded for a time exponentially increasing with the inverse of the perturbation parameter. Thus, although the possibility of a chaotic motion is not excluded, nevertheless a dramatic change of the orbits should not occur for such a long time, and it may be conjectured that such a time exceeds the age of the Solar System itself. But also in this case the problem of the applicability of the theorem still persists, since the analytical estimates based on Nekhoroshev's formulation or other analytical proofs give ridiculous estimates for the size of the masses of the planets.

In recent years the estimates for the applicability of both Kolmogorov's and Nekhoroshev's theorems to realistic models of some part of the Solar System have been improved by some authors. For example the applicability of Nekhoroshev's theorem to the stability of the Trojan asteroids in the vicinity of the triangular Lagrangian points has been investigated by Giorgilli et al. (1989); Celletti and Giorgilli (1991); Giorgilli and Skokos (1997); Efthymiopoulos and Sándor (2005) and by Lhotka et al. (2008), the connection between Nekhoroshev's theorem and Arnold diffusion has been considered in Efthymiopoulos (2008); the applicability of KAM theorem has been studied by Robutel (1995); Fejoz (2005); Celletti and Chierchia (2005); Gabern et al. (2005) and by Locatelli and Giorgilli (2005, 2007). In particular in the latter two articles the Sun–Jupiter–Saturn (hereafter SJS) system is investigated, and evidence is produced that an invariant torus exists in the *vicinity of the initial data* of Jupiter and Saturn, at least in the approximation of the general problem of three bodies.

In the present article we study the stability in Nekhoroshev's sense of the neighbourhood of the invariant torus for the SJS system. The aim is to give evidence, with help of a computer-assisted calculation, that the size of the neighbourhood of the invariant torus for which exponential stability holds for a time interval as long as the age of the universe is big enough to contain the *actual initial data* of Jupiter and Saturn. We should say that such an ambitious goal is still out of our actual possibilities. However, we show that our methods should allow us to achieve our program provided a sufficient computer power will be available in the next future and a further refinement of our approximation methods will be worked out. This is work for the next future.

## 2 Theoretical framework

The basis of our approach is the investigation of the stability of a neighbourhood of an invariant Kolmogorov’s torus. To this end let us briefly recall the statement of Kolmogorov’s theorem.

**Theorem 1** *Consider a canonical system with Hamiltonian*

$$H(p, q) = h(p) + \varepsilon f(p, q) . \tag{1}$$

*Let us assume that the unperturbed part of the Hamiltonian is non-degenerate, i.e.,  $\det \left( \frac{\partial^2 h}{\partial p_j \partial p_k} \right) \neq 0$ , and that  $p^* \in \mathbb{R}^n$  is such that the corresponding frequencies  $\omega = \frac{\partial h}{\partial p}(p^*)$  satisfy a Diophantine condition, i.e.,*

$$|\langle k, \omega \rangle| \geq \gamma |k|^{-\tau} \quad \forall 0 \neq k \in \mathbb{Z}^n$$

*with some constants  $\gamma \geq 0$  and  $\tau \geq n - 1$ . Then for  $\varepsilon$  small enough the Hamiltonian (1) admits an invariant torus carrying quasiperiodic motions with frequencies  $\omega$ . The invariant torus lies in a  $\varepsilon$ -neighbourhood of the unperturbed torus  $\{(p, q) : p = p^*, q \in \mathbb{T}^n\}$ .*

The question is about the dynamics in the neighbourhood of the invariant torus. In order to discuss this point we need a few technical details about the Kolmogorov’s proof method. The key points, clearly outlined in the original short note [Kolmogorov \(1954\)](#), are the following. First, one picks an unperturbed invariant torus  $p^*$  for the Hamiltonian (1) characterized by diophantine frequencies  $\omega$ , and expands the Hamiltonian in power series of the actions  $p$  in the neighbourhood of  $p^*$ . Thus (with a translation moving  $p^*$  to the origin of the actions space) one gives the initial Hamiltonian the form

$$H(p, q) = \langle \omega, p \rangle + \varepsilon A(q) + \varepsilon \langle B(q), p \rangle + \frac{1}{2} \langle Cp, p \rangle + \mathcal{O}(p^2) \tag{2}$$

where  $C = \left[ \frac{\partial^2 h}{\partial p_j \partial p_k}(p^*) \right]$  is a symmetric matrix, and  $A(q)$  and  $\langle B(q), p \rangle$  are the terms independent of  $p$  and linear in  $p$  in the power expansion of the perturbation  $f(p, q)$ , respectively. The quadratic part in  $\mathcal{O}(p^2)$  is of order  $\varepsilon$ , too. The next step consists in performing a near the identity canonical transformation which gives the Hamiltonian the Kolmogorov’s normal form

$$H'(p', q') = \langle \omega, p' \rangle + \mathcal{O}(p'^2) . \tag{3}$$

As Kolmogorov points out, the invariance of the torus  $p' = 0$  is evident, due to the particular form of the normalized Hamiltonian. The whole process requires a composition of an infinite sequence of transformations, and the most difficult part is to prove the convergence of such a sequence. The point which is of interest to us is that *the transformed Hamiltonian (3) is analytic in a neighbourhood of the invariant torus  $p' = 0$ .*

Let us emphasize that the analytical form of the Hamiltonian (3) is quite similar to that of a Hamiltonian in the neighbourhood of an elliptic equilibrium, namely

$$H(x, y) = \frac{1}{2} \sum_{j=1}^n \omega_j \left( x_j^2 + y_j^2 \right) + \dots ,$$

where the dots stand for terms of degree larger than 2 in the Taylor expansion. For, introducing the action-angle variables  $p, q$  via the usual canonical transformation  $x_j = \sqrt{2p_j} \cos q_j$ ,

$y_j = \sqrt{2p_j} \sin q_j$ , the latter Hamiltonian takes essentially the form (3). Thus the exponential stability of the invariant torus  $p' = 0$  may be proved using the theoretical scheme that works fine in the case of an elliptic equilibrium, e.g., in the case the triangular Lagrangian points.

As a matter of fact, a much stronger result holds true, namely that the invariant torus is superexponentially stable, as stated in [Morbidelli and Giorgilli \(1995\)](#) and [Giorgilli and Morbidelli \(1997\)](#). However, a computer assisted method for the theory of superexponential stability seems not be currently available, so we limit our study to the exponential stability in Nekhoroshev’s sense.

### 3 Technical tools

Let us now come to the improvement of the estimates for the applicability of the theorems of Kolmogorov and Nekhoroshev. The key point is to use an explicit construction of the normal form up to a finite order with algebraic manipulation in order to reduce the size of the perturbation, and then apply a suitable formulation of the theorems.

Let us explain this point by making reference to the theorem of Kolmogorov. Starting with the Hamiltonian (2), we perform a finite number,  $r$  say, of normalization steps in order to give the Hamiltonian the normal form up to order  $r$

$$H^{(r)}(p, q) = \langle \omega, p \rangle + \frac{1}{2} \langle Cp, p \rangle + \varepsilon^r A^{(r)}(q) + \varepsilon^r \langle B^{(r)}(q), p \rangle + \mathcal{R}^{(r)}(p, q) \quad (4)$$

with  $\mathcal{R}^{(r)}(p, q) = \mathcal{O}(|p|^2)$ , so that *the perturbation is now of order  $\varepsilon^r$* .

To this end we implement the normalization algorithm for the normal form of Kolmogorov step by step in powers of  $\varepsilon$ , as in the traditional expansions in Celestial Mechanics. The full justification of such a procedure, including the convergence proof, is given, e.g., in [Giorgilli and Locatelli \(1997a,b\)](#). The resulting Hamiltonian has still the form (2), with, however,  $\varepsilon$  replaced by  $\varepsilon^r$ . Thus, a straightforward application of the theorem reads, in rough terms: *if  $\varepsilon^r \leq \varepsilon_*$ , then an invariant torus exists*. The power  $r$  may considerably improve the estimate of the threshold for the applicability of the theorem. This approach has been translated in a computer assisted rigorous proof, which has been successfully applied to a few simple models (see [Celletti et al. 2000](#); [Locatelli and Giorgilli 2000](#); and [Gubern et al. 2005](#)).

Let us now come to the part concerning the estimate of the stability time which is the main contribution of the present note. To this end we remove from the Hamiltonian (4) all the contributions which are independent of or linear in the actions  $p$ , namely the terms  $\varepsilon^r A^{(r)}(q) + \varepsilon^r \langle B^{(r)}(q), p \rangle$ , which are small, thus obtaining a reduced Hamiltonian in Kolmogorov’s normal form. Moreover, we expand the perturbation  $\mathcal{R}^{(r)}(p, q)$  in power series of  $p$  and Fourier series of  $q$ , thus getting a Hamiltonian in the form

$$H(p, q) = \langle \omega, p \rangle + H_1(p, q) + H_2(p, q) + \dots \quad (5)$$

where  $H_s(p, q)$  is a homogeneous polynomial of degree  $s + 1$  in the actions  $p$  and a trigonometric series in the angles  $q$ . Here, the upper index  $r$  of  $H$  has been removed because it is now meaningless, since we use the latter Hamiltonian as an approximation of the Kolmogorov’s normal form.

On this Hamiltonian we perform a Birkhoff normalization up to a finite order, that we denote again by  $r$  although it has no relation with the order of Kolmogorov’s normalization used above. Thus we get a Birkhoff normalized Hamiltonian

$$H = \langle \omega, p \rangle + Z_1(p) + \dots + Z_r(p) + \mathcal{F}_r(p, q),$$

with  $\mathcal{F}_r(p, q)$  a power series in  $p$  starting with terms of degree  $r + 2$ . We omit the details about this part of the calculation, since there are a number of well known formal algorithms that do the job. We concentrate instead on the quantitative estimates.

Let us introduce a norm for a function  $f(p, q) = \sum_{|l|=s, k \in \mathbb{Z}^n} f_{l,k} p^l e^{i(k,q)}$  which is a homogeneous polynomial of degree  $s$  in the actions  $p$ . Precisely define

$$\|f\| = \sum_{|l|=s, k \in \mathbb{Z}^n} |f_{l,k}|. \tag{6}$$

Moreover consider the domain

$$\Delta_\varrho = \{p \in \mathbb{R}^n, |p_j| \leq \varrho, j = 1, \dots, n\}. \tag{7}$$

Then we have

$$|f(p, q)| \leq \|f\| \varrho^s \text{ for } p \in \Delta_\varrho, q \in \mathbb{T}^n.$$

Let now  $p(0) \in \Delta_{\varrho_0}$  with  $\varrho_0 < \varrho$ . Then we have  $p(t) \in \Delta_\varrho$  for  $|t| < T$ , where  $T$  is the escape time from the domain  $\Delta_\varrho$ . This is the quantity that we want to evaluate. To this end we use the elementary estimate

$$|p(t) - p(0)| \leq |t| \cdot \sup_{|p| < \varrho} |\dot{p}| < |t| \cdot \|\{p, \mathcal{F}_r\}\| \varrho^{r+2}. \tag{8}$$

The latter formula allows us to find a lower bound for the escape time from the domain  $\Delta_\varrho$ , namely

$$\tau(\varrho_0, \varrho, r) = \frac{\varrho - \varrho_0}{\|\{p, \mathcal{F}_r\}\| \varrho^{r+2}}, \tag{9}$$

which, however, depends on  $\varrho_0, \varrho$  and  $r$ . We emphasize that in a practical application, e.g., to the SJS system,  $\varrho_0$  is fixed by the initial data, while  $\varrho$  and  $r$  are left arbitrary. Thus we try to find an estimate of the escape time  $T(\varrho_0)$  depending only on the physical parameter  $\varrho_0$ . To this end we optimize  $\tau(\varrho_0, \varrho, r)$  with respect to  $\varrho$  and  $r$ , proceeding as follows. First we keep  $r$  fixed, and remark that the function  $\tau(\varrho_0, \varrho, r)$  has a maximum for

$$\varrho = \frac{r + 2}{r + 1} \varrho_0.$$

This gives an optimal value of  $\varrho$  as a function of  $\varrho_0$  and  $r$ , and so a new function

$$\tilde{\tau}(\varrho_0, r) = \sup_{\varrho \geq \varrho_0} \tau(\varrho_0, \varrho, r)$$

which is actually computed by putting the optimal value  $\varrho = \varrho_0(r + 2)/(r + 1)$  in the expression above for  $\tau(\varrho_0, \varrho, r)$ . Next we look for the optimal value  $r_{\text{opt}}$  of  $r$ , which maximizes  $\tilde{\tau}(\varrho_0, r)$  when  $r$  is allowed to change. That is, we look for the quantity

$$T(\varrho_0) = \max_{r \geq 1} \tilde{\tau}(\varrho_0, r),$$

which is our best estimate of the escape time, depending only on the initial data. We define the latter quantity as the estimated stability time. Here we need some further considerations in order to convince the reader that the maximum in the r.h.s. of the latter formula actually exists. This follows from the asymptotic properties of the Birkhoff’s normal form. For, according to the available analytical estimates based on Diophantine inequalities for the frequencies, the norm  $\|\{p, \mathcal{F}_r\}\|$  in the denominator of (9) is expected to grow as  $(r!)^n$ ,  $n$  being the number of degrees of freedom. Thus, for  $\varrho_0$  small enough the denominator  $\|\{p, \mathcal{F}_r\}\| \varrho_r^r$  reaches a

minimum for some  $r^n \sim 1/\varrho_0$ , which means that the wanted maximum actually exists, thus providing the optimal value  $r_{\text{opt}}$ . We also remark that although no proof exists that the analytical estimates are optimal, accurate numerical investigations based on explicit expansions show that the  $r!$  growth of the norms actually shows up (see, e.g., [Contopoulos et al. 2003](#) and [Efthymiopoulos et al. 2004](#)). Working out an analytical evaluation of the stability time on the basis of these considerations leads to an exponential estimate of Nekhoroshev's type for  $T(\varrho_0)$  (see, e.g., [Giorgilli et al. 1989](#)). Here we replace the analytical estimates with an explicit numerical optimization of  $\tilde{\tau}(\varrho_0, r)$  by just calculating it for increasing values of  $r$  until the maximum is reached.

Our aim is to perform the procedure above by using computer algebra. Thus some truncation of the functions must be introduced in order to implement the actual calculation. The most straightforward approach is the following. First we truncate the Hamiltonian (5) at a finite polynomial order in the actions. This is legitimate if the radius  $\varrho$  of the domain is small, due to the well known properties of Taylor series. However, the Fourier expansion of every term  $H_s$  still contains infinitely many contributions. Here we take advantage of the exponential decay of the Fourier coefficients of analytic functions and of some algebraic properties of the Poisson brackets. Precisely, let  $f(q) = \sum_k f_k e^{i(k,q)}$ ; the dependence of the coefficients  $f_k$  on the actions is irrelevant here. Then the exponential decay of the coefficients means that  $|f_k| \leq C e^{-|k|\sigma}$  with some positive constants  $C$  and  $\sigma$ . Thus, having fixed a positive integer  $K$  we truncate the Fourier expansion as  $f(q) = \sum_{|k| \leq K} f_k e^{i(k,q)}$ , i.e., we remove all Fourier modes  $|k| \geq K$ . This is allowed because the exponential decay assures that the neglected part is small. The interested reader may find a more detailed discussion about this method of splitting the Hamiltonian in [Giorgilli \(2003\)](#).

Coming back to our problem, we include in  $H_s(p, q)$  all Fourier coefficients with  $|k| \leq sK$ , so that  $H_s(p, q)$  is a homogeneous polynomial of degree  $s + 1$  in the actions  $p$  and a trigonometric polynomial of degree  $sK$  in the angles  $q$ . The algebraic property mentioned above is that such a splitting of the Hamiltonian is preserved by the Lie series algorithm that we apply through all our calculations. This in view of the elementary fact that the Poisson bracket between two functions,  $f_r$  and  $f_s$  say, which are homogeneous polynomial of degree  $r + 1$  and  $s + 1$ , respectively, in  $p$  and trigonometric polynomials of degree  $rK$  and  $sK$ , respectively, in  $q$  produces a new function of degree  $r + s + 1$  in  $p$  and  $(r + s)K$  in  $q$ .

A final remark concerns the estimate of the remainder  $\mathcal{F}_r$  in (8), which is an infinite series, too. Here we just calculate the first term of the remainder, namely the term of degree  $r + 1$ , and multiply its norm by a factor 2. This factor is justified in view of the fact that the analytical estimates of the same quantities involve a sum of a geometric series which, for  $\varrho$  small enough, decreases with a ratio less than  $1/2$ . Here a natural objection could be that for some strange reason the norm of the remainder at some finite order could be smaller than predicted by the analytical estimates. However, it is a common experience that after a few perturbation steps the norms of the functions take a rather regular behaviour consistent with the geometric decrease predicted by the theory. Thus, our choice appears to be justified by experience.

As a final remark we note that our way of dealing with the truncation is the most straightforward one, but it is not the sole possible. Other more refined criteria may be invented, of course, which may take into account the most important contribution while substantially reducing the number of coefficients to be calculated. In this sense our direct approach should be considered as a first attempt to check if the concept of Nekhoroshev's stability may be expected to apply to our Solar System. Although being unable to produce rigorous results in a strict mathematical sense, we believe that our method gives interesting results in the spirit of classical perturbation methods.

### 4 Application to the planetary problem of three bodies

Applying the theories of Kolmogorov and Nekhoroshev to the planetary problem is not straightforward, due to the degeneration of the Keplerian motion. In order to remove such a degeneration, a lengthy procedure is needed; this essentially requires a suitable adaptation of the canonical coordinates, paying a very particular care to the secular ones (to appreciate some deep point of view about this problem, see, e.g., Arnold 1963b and Nekhoroshev 1977).

In our approach the difficulty shows up in the part concerning the application of Kolmogorov’s theory. Once a Kolmogorov torus has been constructed, then there is no extra difficulty in applying the method of Sect. 3, due to the fact that the method is local. In the present section we give a brief sketch of the procedure for the construction of a Kolmogorov torus. The complete procedure is described in Locatelli and Giorgilli (2005, 2007), to which we refer for details.

Following a traditional approach, we first reduce the integrals of motion (i.e. the linear and angular momenta); therefore, we separate the fast variables (essentially the semimajor axes and the mean anomalies) from the slow ones (the eccentricities and the inclinations with the conjugated longitudes of the perihelia and of the nodes). This is usually done in Poincaré variables by writing a reduced Hamiltonian of the form

$$H^R(\Lambda, \lambda, \xi, \eta) = F^{(0)}(\Lambda) + \mu F^{(1)}(\Lambda, \lambda, \xi, \eta) \tag{10}$$

with

$$\mu = \max\{m_1 / m_0, m_2 / m_0\}$$

where  $m_0$  is the mass of the star,  $m_1$  and  $m_2$  are the masses of the planets,  $\Lambda_j, \lambda_j$  are the fast variables and  $\xi_j, \eta_j$  are the slow (Cartesian-like) variables. Here, obviously, the values of the index  $j = 1, 2$  correspond to the internal planet and to the external one, respectively.

On this Hamiltonian we perform a procedure which is the natural extension of the one devised by Lagrange and Laplace in order to calculate the secular motion of the eccentricities and the inclinations and the conjugated angles.

The first step is the identification of a good unperturbed invariant torus for the fast angles  $\lambda$ , setting for a moment the slow variables  $\xi, \eta$  to zero. Here is a short description.

- (i) Having fixed a frequency vector  $n^* \in \mathbb{R}^2$ , we determine the corresponding action values  $\Lambda^*$  corresponding to a torus which is invariant for an integrable approximation of the system, where the dependency on both the fast angles  $\lambda$  and on the secular coordinates  $\xi, \eta$  is dropped. This can be done by solving the equation

$$\left. \frac{\partial \langle H^R \rangle_\lambda}{\partial \Lambda_j} \right|_{\substack{\Lambda = \Lambda^* \\ \xi, \eta = 0}} = n_j^*, \quad j = 1, 2.$$

Here  $\langle H^R \rangle_\lambda = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} H^R d\lambda_1 d\lambda_2$  is the average of the Hamiltonian  $H^R$  with respect to the fast angles. The explicit value of  $n^*$  is chosen so that it reflects the true mean motion frequencies of the planets (see next section for our values). Having solved the previous equation with respect to the unknown vector  $\Lambda^*$ , we expand  $H^R$  in power series of  $\Lambda - \Lambda^*$ . With a little abuse of notation we denote again by  $\Lambda$  the new variables.

- (ii) We perform two further canonical transformations which make the torus  $\Lambda = \xi = \eta = 0$  to be invariant up to order 2 in the masses. Indeed, these changes of coordinates are borrowed from the Kolmogorov’s normalization algorithm, but we look for a Kolmogorov’s normal form with respect to the fast variables only, considering the



slow ones essentially as parameters, although they are changed too. More precisely, we determine generating functions of the form  $\chi_j(\Lambda, \lambda, \xi, \eta) = \Lambda^{j-1}g_j(\lambda, \xi, \eta)$  for  $j = 1, 2$ , where  $g_j(\lambda, \xi, \eta)$  includes a finite order expansion both in Fourier modes with respect to the fast angles  $\lambda$  and in polynomial terms of the slow variables  $\xi, \eta$ . The aim of this step is to reduce the size of terms independent of or linear in the fast actions so that it is of the same order as the rest of the perturbation. We denote by  $H^T$  the resulting Hamiltonian, which is still trigonometric in the fast angles  $\lambda$  and polynomial in  $\Lambda, \xi, \eta$ .

The next goal is to determine a good invariant torus for the slow variables  $\xi, \eta$ . To this end we combine the classical Lagrange’s calculation of the secular frequencies with a Birkhoff’s procedure that takes into account the nonlinearity.

- (iii) We consider the secular system, namely the average  $\langle H^T \rangle$  of the Hamiltonian  $H^T$  resulting from the step (ii) above. Acting only on the quadratic part of the Taylor expansion of  $\langle H^T \rangle$  in  $\xi, \eta$  we determine a first approximation of the secular frequencies, and transform the Hamiltonian so that its quadratic part has a diagonal form. This part of the calculation follows the lines of Lagrange’s theory, but the calculation is worked out at the second order approximation in the masses. The diagonalization of the quadratic part requires a linear canonical transformation, which is a standard matter. Thus the quadratic part in  $\xi, \eta$  of the resulting Hamiltonian has the form  $\frac{1}{2} \sum_j \nu_j (\xi_j^2 + \eta_j^2)$ , where  $\nu$  are the secular frequencies and we denote again by  $\xi, \eta$  the slow variables.
- (iv) We perform a Birkhoff’s normalization up to order 6 in  $\xi, \eta$ . This gives a normalized secular Hamiltonian  $H^B$  which in action–angle variables  $\xi_j = \sqrt{2I_j} \cos \phi_j, \eta_j = \sqrt{2I_j} \sin \phi_j$  takes the form

$$H^B = \nu \cdot I + h^{(4)}(I) + h^{(6)}(I) + F(\Lambda, I, \phi) ,$$

where  $h^{(4)}$  and  $h^{(6)}$  are polynomials of degree 2 and 3 in  $I$ , respectively. This step removes the degeneration of the secular motion, thus allowing us to take into account the nonlinearity of the secular part of the problem.

- (v) Having fixed the slow frequencies  $g^*$  so that they reflect the true frequencies of the system, we determine a secular torus  $I^*$  corresponding to these frequencies, by using the integrable approximation of  $H^B$ . This is done by solving for  $I$  the equation

$$\frac{\partial h^{(4)}}{\partial I_j}(I) + \frac{\partial h^{(6)}}{\partial I_j}(I) = g_j^* - \nu_j^* , \quad j = 1, 2 .$$

The values  $\Lambda^*$  and  $I^*$  so determined provide the first approximation of the Kolmogorov’s invariant torus. Reintroducing the fast angles and performing on the original Hamiltonian  $H^R$  all the transformations that we have done throughout our procedure (i–v) we get a Hamiltonian of the form (2) which is the starting point for Kolmogorov’s normalization algorithm. After a number of Kolmogorov’s steps the Hamiltonian takes the form (4), thus giving a good approximation of an invariant torus with frequencies  $n^*$  and  $g^*$ . The latter form is precisely the output of the calculation illustrated in Giorgilli and Locatelli (1997b), and by removing all terms which are independent of or linear in the actions  $p$  it provides a Hamiltonian as that in (5). This is the starting point for our algorithm evaluating the stability time in the neighbourhood of the invariant torus.



**Table 1** Physical parameters for the Sun–Jupiter–Saturn system taken from JPL at the Julian Date 2451220.5

	Jupiter ( $j = 1$ )	Saturn ( $j = 2$ )
Mass $m_j$	$(2\pi)^2/1047.355$	$(2\pi)^2/3498.5$
Semi-major axis $a_j$	5.20092253448245	9.55716977296997
Mean anomaly $M_j$	6.14053316064644	5.37386251998842
Eccentricity $e_j$	0.04814707261917873	0.05381979488308911
Perihelion argument $\omega_j$	1.18977636117073	5.65165124779163
Inclination $i_j$	0.006301433258242599	0.01552738031933247
Longitude of the node $\Omega_j$	3.51164756250381	0.370054908914043

**Table 2** The frequencies of the unperturbed torus in the SJS system corresponding to the initial data and physical parameters in Table 1

	Jupiter	Saturn
Fast frequencies	$n_1^* = 0.52989041594442$	$n_2^* = 0.21345444291052$
Secular frequencies	$g_1^* = -0.00014577520419$	$g_2^* = -0.00026201915143$

The values are calculated via frequency analysis on the orbits obtained by direct integration of the equations for the problem of three bodies

## 5 Application to the Sun–Jupiter–Saturn system

We come now to the application of our procedure to the SJS system. Let us first define the model. We consider the general problem of three bodies with the Newtonian potential. Thus, the contribution due to the other planets of the Solar System is not taken into account in our approximation. The expansion of the Hamiltonian is a classical matter, so we skip the details, just recalling that all the expansions have been done via algebraic manipulation, using a package developed on purpose by the authors.

The choice of the model plays a crucial role in determining the frequencies of the torus, that we calculate by integrating the Newton equations for the problem of three bodies and applying the frequency analysis (see, e.g., Laskar 1999) to the computed orbit. As initial data we take the orbital elements of Jupiter and Saturn as given by JPL<sup>1</sup> for the Julian Date 2451220.5. This is the point where the connection with the physical parameters of our Solar System is made. The physical parameters and the orbital elements are reported in Table 1. The calculated frequencies are given in Table 2.

The choice of the Julian Date 2451220.5 in order to set the initial data is completely arbitrary, of course, its sole justification being that such data are directly available from JPL. Choosing different dates or different determinations of the planet's elements could lead to a slightly different determination of the frequencies, and so also of the invariant torus. However, we emphasize that the aim of the present work is precisely to give a long time stability result which applies to a neighbourhood of the invariant torus. The size of such a neighbourhood should be large enough to cover the unavoidable uncertainty in determining the initial data for the SJS system. This is a delicate matter, of course, because the JPL data reflect the dynamics of the full Solar System, while our study is concerned only with the model of three

<sup>1</sup> The data about the planetary motions provided by the Jet Propulsion Laboratory are publicly available starting from the webpage <http://www.jpl.nasa.gov/>

**Table 3** Estimates of the uncertainties on the initial values of the canonical coordinates  $(\Lambda, \lambda, \xi, \eta)$ 

	$\Delta\Lambda_j$	$\Delta\lambda_j$	$\Delta\xi_j$	$\Delta\eta_j$
Jupiter ( $j = 1$ )	$1.8 \times 10^{-6}$	$6.6 \times 10^{-5}$	$1.1 \times 10^{-5}$	$2.8 \times 10^{-6}$
Saturn ( $j = 2$ )	$1.7 \times 10^{-6}$	$3.0 \times 10^{-5}$	$3.3 \times 10^{-6}$	$3.2 \times 10^{-6}$

These evaluations are derived from the comparison of different sets of JPL's DE

bodies. However, we may get some hint on the size of the uncertainty precisely by looking at the JPL data.

As everybody knows, the initial positions and velocities of the planets are usually taken from the Development Ephemeris of the Jet Propulsion Laboratory (for short, JPL's DE). There are several sets of these ephemerides, each version of them being based on more and more observational data, which take benefit from the improvement of the techniques. Thus, each new version of the JPL's DE is expected to improve the precision of the data with respect to the older ones and, then, one can approximately evaluate the error of the older versions by comparison with the most recent one (see the initial discussion in [Standish 1990b](#)). The positions and velocities of the planets given by five different sets of JPL's DE are listed in table 15 of Standish's paper [Standish \(1990a\)](#). For each kind of these data we can determine a narrow interval containing all of them and we can calculate the Keplerian orbital elements corresponding to the extrema of such intervals. By applying all the necessary transformations we translate these data into uncertainties for the Poincaré's canonical coordinates  $(\Lambda, \lambda, \xi, \eta)$  that have been used in order to write the Hamiltonian (10). These uncertainties are reported in Table 3. This provides us with a first approximation of the neighbourhood of our initial data that contains all JPL's DE reported in Standish's paper. We should now apply all the canonical transformations needed in order to construct an invariant torus close to the SJS orbit. However we remark that all such transformations are very smooth, being analytic, volume preserving, and most of them are close to identity, so that they add just a small correction with respect to the data in Table 3. Thus we may confidently expect that at some time the phase space point representing the position of the SJS system lies in a neighbourhood of our approximated invariant torus the size of which is evaluated to be  $\mathcal{O}(10^{-6})$  for the fast actions and  $\mathcal{O}(10^{-5})$  for the secular coordinates.

Let us now come back to the actual calculation. The Kolmogorov's normal form has been computed up to order 17, with the generating function exhibiting a good geometric decay. Furthermore, we have compared the orbit on the approximate invariant torus with that produced by a direct numerical integration of the equations of motion, thus finding a quite good agreement between them, as shown in Table 4. Here, we omit the details about these lengthy calculations, since a complete report has been already given in [Locatelli and Giorgilli \(2007\)](#).

The calculation of Kolmogorov's normal form produces a Hamiltonian which is analytic in the neighbourhood of the approximated invariant torus. Our program performs the calculation of this Hamiltonian with the polynomial series in the actions truncated at order 3 and the trigonometric series truncated at order 34 (see [Locatelli and Giorgilli 2007](#) for more details). On this Hamiltonian we would like to apply the procedure of Sect. 3. However, a major obstacle raises up: the number of coefficients in the series that we have calculated is more than 7 100 000. Such a huge number of coefficients can not be handled in a Birkhoff normalization procedure. For, referring to the discussion at the end of Sect 3 we should set the parameter  $K$  for the truncation of trigonometric series to 34, thus getting a truncation at trigonometric degree 68, 102,  $\dots$ ,  $34r$ ,  $\dots$  at successive order. A rough estimate of the

**Table 4** Maximal discrepancies about the orbital elements of the SJS system between a numerical integration and the semi-analytic one, that is based on the construction of the invariant torus corresponding to the frequencies values given in Table 2

	$\text{Max}_t \left\{ \left  \frac{\Delta a_j(t)}{a_j(t)} \right  \right\}$	$\text{Max}_t \{  \Delta \lambda_j(t)  \}$	$\text{Max}_t \left\{ \left  \frac{\Delta e_j(t)}{e_j(t)} \right  \right\}$	$\text{Max}_t \{  \Delta \omega_j(t)  \}$
Jupiter	$1.5 \times 10^{-6}$	$5.0 \times 10^{-4}$	$1.3 \times 10^{-3}$	$1.3 \times 10^{-3}$
Saturn	$6.8 \times 10^{-6}$	$1.1 \times 10^{-3}$	$4.3 \times 10^{-3}$	$7.3 \times 10^{-3}$

The maximal relative errors on the semi-major axis  $a_j$  and on the eccentricities  $e_j$  are reported here for both Jupiter (corresponding to  $j = 1$ ) and Saturn (i.e.,  $j = 2$ ); the same is made also for the maximal absolute errors on the “fast angle”  $\lambda_j = M_j + \omega_j$  and on the perihelion argument  $\omega_j$ . In the present case, the comparisons are made starting from the initial conditions given in Table 1 and for a time span of 100 Myr

number of generated coefficients shows that we shall soon run out of memory and of time on any available computer. Thus, we must introduce some further approximation.

In view of the considerations above we decided, as a first approach, to strictly follow the truncation scheme illustrated at the end of Sect. 3 by just lowering the value of  $K$ . We report the results of this first attempt, which in our opinion appear already to be interesting. Thus we expand the Hamiltonian in the form

$$H(p, q) = \langle \omega, p \rangle + H_1(p, q) + H_2(p, q)$$

by keeping in  $H_1$  all terms of degree 2 in the actions  $p$  and  $K$  in the angles  $q$ , and in  $H_2$  all terms of degree 3 in the actions  $p$  and  $2K$  in the angles  $q$ . The Birkhoff normalization produces a Hamiltonian of the form

$$H = \langle \omega, p \rangle + Z_1(p) + \dots + Z_r(p) + \mathcal{F}_{r+1}(p, q),$$

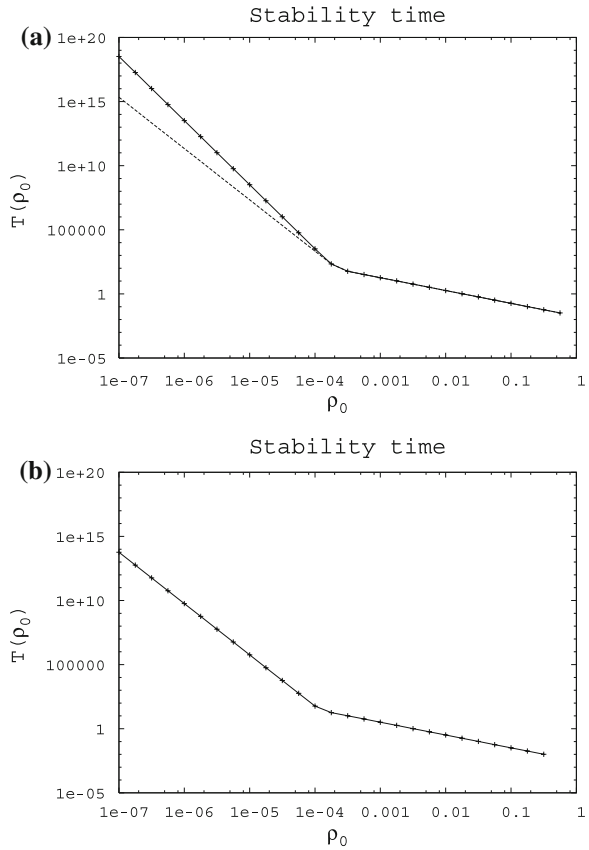
where  $\mathcal{F}_{r+1}$  denotes the term of degree  $r + 2$  in the actions  $p$  and  $(r + 1)K$  in the angles, i.e., the first term of the remainder. With a suitable choice of  $K$ , this considerably reduces the number of coefficients in the expansions thus enabling us to perform the calculation on a workstation. We emphasize however that the algorithm is a general one so that in principle it can be applied to the full Hamiltonian or, better to a Hamiltonian obtained by removing all coefficients which are very small and will likely not produce big coefficients (due to the action of small denominators) during the calculation of the Birkhoff’s normal form. The rest of the calculation closely follows the discussion in Sect. 3, so we come to illustrating the results. We performed the calculation with two different values of  $K$ , as given in the following table.

$K$	$r$	# of coefficients
4	5	2 494 000
6	4	3 380 000

This shows in particular the dramatic increase of the number of coefficient in the remainder  $\mathcal{F}_{r+1}$  (third column), which imposes strong constraints on the choice of the normalization order  $r$ .

A quite natural objection could be raised here. Since the most celebrated resonance of the SJS system (i.e., the mean motion resonance 5:2) has trigonometrical degree 7, it seems that some of the main resonant terms are neglected because of our choice of  $K$ . This is actually not the case, due to a technical element that we have omitted in the previous section in order to make the discussion simpler. Our sequence of transformations includes a unimodular linear

**Fig. 1** The estimated stability time. **a** Results for  $K = 4$  and Birkhoff normalization order 5 (crosses) and 4 (dashed curve). **b** Results for  $K = 6$  and Birkhoff normalization order 4



transformation on the angles, and so also on the frequencies. The action on the frequencies changes the resonance 5:2 into a 3:1 one, which is of order 4. Thus, setting  $K \geq 4$  as we did throughout all our calculations is enough in order to include the main resonant terms. The interested reader will find a detailed discussion of this point in Sect 3 of [Locatelli and Giorgilli \(2007\)](#).

Let us now come to the results. In panel (a) of Fig. 1 we report the results for  $K = 4$ . The crosses give the estimated stability time for the Birkhoff's normal form at order 5; the dashed line gives the estimated time when the Birkhoff's normal form is truncated at order 4, thus showing how relevant is the improvement when a single normalization order is added.

We can now come back to the estimate about the escape time  $T = T(\varrho_0)$ . Looking at panel (a) of Fig. 1, one can remark that we have an estimated stability time of  $10^{10}$  years, that is approximately equal to the age of the universe, for a neighbourhood of initial conditions of radius  $10^{-5}$  in actions.

It may be noted that the stability curves exhibit a sharp change of slope around  $\varrho_0 \sim 10^{-4}$ . This is because the optimal normalization order increases when the radius is decreased. Actually, further changes of slope should be expected for smaller values of  $\varrho_0$ , but due to computational limits such changes can not appear in our figure, because the optimal order exceeds the actual order of our calculation. Thus, our estimate of the stability time should be considered as a very pessimistic lower bound.

Moreover, the behaviour of the plots in Fig. 1, clearly shows that the estimate of the escape time can be substantially improved if smaller values of the radius  $\varrho_0$  can be considered. Recalling that our estimate of the size of the neighbourhood in action variables is calculated from the discrepancy among different sets of JPL's DE data, we may affirm that our neighbourhood roughly covers such a width, which is tabulated in Standish's paper quoted above.

If we try a better approximation of the Hamiltonian, setting  $K = 6$ , then we are forced to stop the Birkhoff's normalization at order 4, thus making the results definitely worse. The data for the estimated stability time are plotted in panel (b) of Fig. 1. One sees that the estimate becomes comparable with the age of the universe only in a neighbourhood of initial conditions slightly larger than  $10^{-6}$ . However, if we compare the curve in panel (b) with the dashed curve in panel (a) we see that we shall likely get substantially better results if we could compute the normal form at order 5.

Thus, our rough approximation gives results which apply to a set of initial data for the SJS system which is of the same order of magnitude as the uncertainty in JPL's data. We also emphasize that our evaluation of  $\varrho_0$  is based on observational data which are presently older than 25 years; we expect that this is quite pessimistic with respect to the features of more recent JPL's DE.

## 6 Conclusions

We have developed an effective method to compute the Kolmogorov's normal form for the problem of three bodies, and have successfully applied it to the SJS problem, using the data of our Solar System. Next, we have shown that a calculation of the stability in Nekhoroshev's sense of orbits with initial point in the neighbourhood of the torus is possible, at least if one accepts to make some strong truncation in the expansion of the Hamiltonian. A rather strong truncation allows us to get an estimated stability time comparable with the age of the universe in a neighbourhood of the torus that will likely contain the actual initial data of the SJS system.

The natural question is whether such results will remain valid if we add more and more terms in the Hamiltonian, thus making our approximation better. Answering such a question is presently beyond our limits, but in our opinion deserves to be investigated. Some improvement may be obtained by using more computer power, e.g., by performing our calculation on a cluster of computers. However, substantial improvements require also a refinement of our analytical techniques in order to be able to evaluate the error induced by our truncations, thus allowing us to introduce better computation schemes and to evaluate the reliability of our approximations.

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