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ORIGINAL ARTICLE

# Periodic orbits accumulating onto elliptic tori for the $(N+1)$-body problem

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**Abstract** We prove the existence of infinitely many periodic solutions, with larger and larger minimal period, accumulating onto elliptic invariant tori for (an "outer solar-system" model of) the planar $(N+1)$-body problem.

**Keywords** $N$-body problem · Periodic orbits · Nearly-integrable Hamiltonian systems · Lower-dimensional elliptic tori · Planetary $N$-body problem

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## 1 Introduction and results

The importance of periodic solutions in Hamiltonian systems was remarked by Poincaré: "...ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissons essayer de pénétrer dans une place jusqu'ici réputée inabordable...".

Poincaré also conjectured that periodic orbits approximate any trajectory: "...voici un fait que je n'ai pu démontrer rigoureusement, mais qui me parait pourtant très vraisemblable. Étant données des équations de la forme définie dans[1] le n. 13 et une solution particulière quelconque de ces équations, on peut toujours trouver une solution périodique (dont la période peut, il est vrai, être très longue), telle que la différence entre les deux solutions soit aussi petite qu'on le veut, pendant un temps aussi long qu'on le veut."

[1] Formula no. 13 mentioned by Poincaré is the Hamilton's equation.

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A partial answer to this conjecture was given in Pugh and Robinson (1983), where it is proved that periodic orbits are dense in any regular and compact energy surface, *generically in the $C^2$ category*. Anyway, the conjecture is still *open* for given systems, in particular for the many-body problem.

As an intermediate step towards this conjecture, one may try to find periodic orbits approaching invariant manifolds. Periodic orbits accumulating on elliptic equilibria and on elliptic periodic orbits were constructed in Birkhoff and Lewis (1933), while Conley and Zehnder (1983) provided periodic orbits accumulating on maximal KAM tori.

Recently an analogous result has been proved for elliptic tori[2] of any intermediate dimension $2 \le k \le n - 1$:

**Theorem 1.1** (Berti et al. 2004) *Under suitable non-degeneracy and non-resonance assumptions between the linear and elliptic frequencies, there are infinitely many periodic orbits, whose minimal period goes to infinity, accumulating on* (*lower-dimensional*) *elliptic invariant tori of Hamiltonian systems.*

The proof of the above theorem is based on a Lyapunov–Schmidt reduction that resembles the method used in Moser (1976) (see also Moser 1977). The non-degeneracy and non-resonance assumptions are used to solve the range equation by the Fixed Point Theorem, then the kernel (or bifurcation) equation is solved by variational arguments.

It must be remarked that one of the main motivations for studying this kind of problems is Celestial Mechanics (indeed Poincaré formulated the above conjecture in his treatment of the three-body problem).

In Biasco et al. (2003) it is proved that the spatial planetary three-body problem has, for small values of the parameter $\varepsilon$ measuring the ratio between the masses of the two planets and the mass of the star, two-dimensional elliptic invariant tori, provided the osculating Keplerian major semi-axes belong to a two-dimensional set of density close to one (as $\varepsilon$ tends to zero).

In Berti et al. (2004) it is showed that the abstract Theorem 1.1 can be applied to the three-body problem, namely that, for $\varepsilon$ small enough, there exist infinitely many periodic solutions, with larger and larger minimal period, accumulating onto the two-dimensional elliptic invariant tori found in Biasco et al. (2003).

Actually the Poincaré's "periodic orbits of second kind" (see Poincaré 1982) and Fejoz's generalization away from nearly circular ellipses (see Féjoz 2002) also fall into this category (in the planar case).

Elliptic tori for the many-body problem have been considered in Biasco et al. (2006). In particular that paper focuses on a "caricature of the outer solar system". Since in the following we will study the same model, we are going to describe it in details.

Let us consider $N + 1$ "bodies" $P_0, \ldots, P_N$ mutually interacting through gravitational attraction. Such bodies are supposed to lie on a fixed plane (*planar* case). Moreover, we consider the *planetary* case, where one of the bodies, for instance $P_0$ (the "Sun"), has mass much greater than that of the other ones (the "planets"). Denoting by $m_i$ the mass of the $i$th body, we assume that, for a small parameter $\varepsilon$,

$$m_i = \varepsilon\mu_i, \quad i = 1, \ldots, N, \;\; 0 < \varepsilon < 1, \tag{1}$$

where $\mu_i$ are constants (of order 1 in $\varepsilon$).

The useful feature of the planetary problem is that one can, in first approximation, neglect the (small) forces between the planets and consider only the $N$ decoupled two-body systems

[2] We recall that a lower-dimensional invariant torus is called elliptic, or linearly stable, if the linearized system along the torus possesses purely imaginary eigenvalues. The dynamics on the torus is described by the *linear* frequencies, while the dynamics around the torus is described by the *elliptic* (or *normal*) frequencies.

formed by the Sun and the $i$th planet. As well known from Kepler's laws, for suitable initial data, each planet will revolve on a ellipse around the Sun. Thus, for all $i = 1, \ldots, N$, at any given instant, we can define the *osculating ellipse* associated to the planet $P_i$ as the Keplerian orbit given by the solution of the two-body problem $(P_0, P_i)$, with initial data given by the positions and the velocities of $P_0$ and $P_1$ at that instant. Of course, such ellipses describe the motions of the full $(N + 1)$-body problem only approximately; nevertheless, they provide a nice set of coordinates allowing, for example, to describe the true motions in terms of the eccentricities $e_i$ and the major semi-axes $a_i$ of the osculating ellipses.

We focus our attention on a planetary planar model with planets evolving from phase points corresponding to well separated nearly-circular ellipses (with eccentricities $e_i \ll 1$); here "well separated" means that (renumbering the planets according to their positions)

$$0 < a_i < \theta\, a_{i+1}, \quad 1 \leq i \leq N - 1. \tag{2}$$

for a suitable constant $0 < \theta < 1$.

We will assume that two planets (Jupiter and Saturn) have mass considerably bigger than the other ones; besides, the two big planets are supposed to have an orbit which is internal with respect to the orbits of the small planets (Uranus, Neptune...). Precisely, we will assume for concreteness that, for some $m_0 < \bar{\mu}_i < 4m_0$,

$$\begin{aligned} \mu_i &= \bar{\mu}_i \quad \text{for} \;\; i = 1, 2\,, \\ \mu_i &= \delta\,\bar{\mu}_i \quad \text{for} \;\; i = 3, \ldots, N, \qquad 0 < \delta < 1. \end{aligned} \tag{3}$$

The reason for considering this model is that it makes possible to compute asymptotically the eigenvalues of the linearized secular dynamics (we will recall them in Proposition 4.1 below). Then, one can directly prove that the (Melnikov) non-resonance condition, needed to apply elliptic KAM theory, is satisfied. Accordingly, Biasco et al. (2006) shows, for $\varepsilon$ small and for a large set of semi-axes, the existence of quasi-periodic orbits with small eccentricities filling up $N$-dimensional invariant elliptic tori. Such orbits can be seen as continuations of the "limiting" circular trajectories of the system obtained by neglecting the mutual interactions among the planets.

In this paper we show how Theorem 1.1 can be applied to the $(N + 1)$-body problem considered in Biasco et al. (2006), namely we obtain the following result (for a more precise formulation see Theorem 5.5):

**Theorem 1.2** *For $\varepsilon$ small enough, there exist infinitely many periodic solutions of increasing minimal period, accumulating onto the $N$-dimensional elliptic invariant tori of the $(N+1)$-body problem considered in Biasco et al. (2006).*

Theorem 1.1 works under several different non-resonance hypotheses (see Theorem 2.1 of Berti et al. (2004)). The simplest one requires that the dimension of the elliptic torus is equal to two (or smaller); so, in the case of the three-body problem the non-resonance hypothesis is automatically satisfied. Anyway, this is not our case (when $N \geq 3$). An alternative non-resonance hypothesis of Theorem 2.1 is that the vector formed by the linear and elliptic frequencies is non-resonant *at any order*. Proving this fact for the $(N+1)$-body problem is difficult[3] since the $N$-dimensional linear frequencies and the $N$-dimensional elliptic frequencies do not vary independently, depending only on $N$ parameters (the $N$ major semi-axes).

[3] By the KAM analysis used to prove the persistence of elliptic tori one only has that the "second order Melnikov condition" holds (see (49) below).

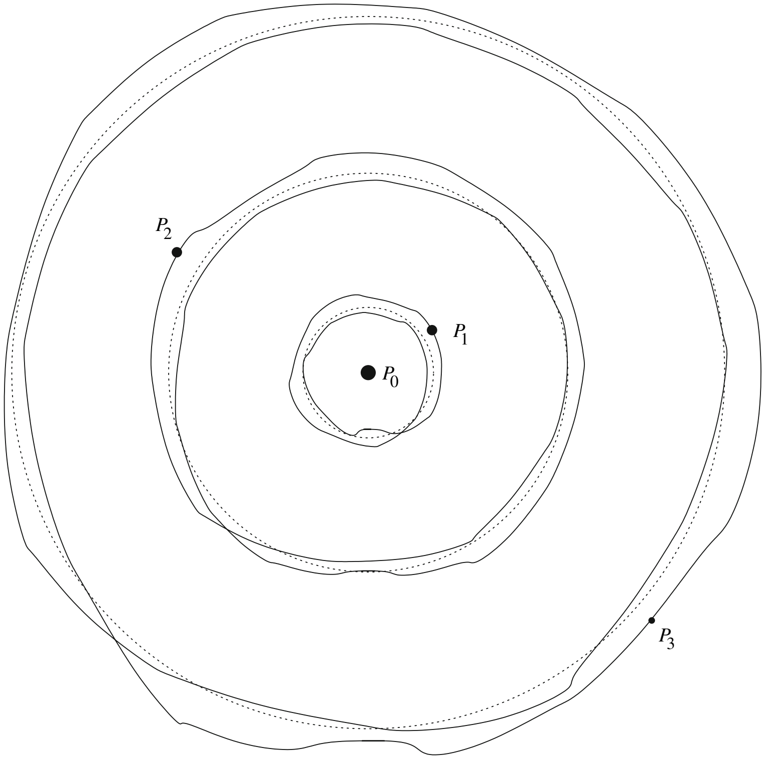


**Fig. 1** The periodic orbits of Theorem 1.2 in the case $N = 3$. The picture is drawn in an inertial frame. The star $P_0$, due to its huge mass, is almost at rest, the bigger planets $P_1$ and $P_2$, and the smaller planet $P_3$ revolve on periodic orbits close to nearly circular ellipses

To overcome this problem we first prove that Theorem 1.1 still holds assuming that the vector of the linear and elliptic frequencies is non-resonant *only at a (large but) finite order* (see Theorem 2.1 below). Then, through a careful handling of the asymptotics in Biasco et al. (2006) (see Proposition 4.1) and exploiting the analytic properties of the involved functions, we will manage to find a subset of osculating major semi-axes on which the vector of the linear and elliptic frequencies of the $(N + 1)$-body model we are considering is indeed non-resonant up to a suitable (large) finite order. Furthermore, the measure of the set of "discarded" semi-axes is proved to be polynomially small with respect to the perturbative parameters.

The paper is organized as follows: In Sect. 2 we present our modified version of Theorem 1.1. In Sect. 3 we prove a result about the measure of sub-level-sets of (non-identically-vanishing) real-analytic functions of several variables, that will be used in estimating the measure of the set of non-resonant semi-axes. In Sect. 4 we recall some results about the model of the $(N + 1)$-body problem introduced (Biasco et al. 2006); we also add some useful estimates. In Sect. 5 we finally prove that the non-degeneracy and non-resonance hypotheses of the abstract theorem of Sect. 2 are fulfilled by the $(N + 1)$-body problem we are considering.

A preliminary version of some results contained in this paper can be found in Coglitore (2007). The result stated in theorem 1.2 has been announced in Biasco and Valdinoci (2008).

## 2 Periodic orbits close to elliptic tori

We present the abstract result (Theorem 2.1) about the existence of periodic solutions, with larger and larger minimal period, accumulating onto elliptic invariant tori of Hamiltonian systems.

Following the notation in Berti et al. (2004), let us consider a Hamiltonian of the form

$$\mathcal{H}_*(\mathcal{I}_*, \varphi_*, Z_*, \overline{Z}_*) \;=\; \omega \cdot \mathcal{I}_* \;+\; \Omega Z_* \cdot \overline{Z}_* \;+ \sum_{2|k|+|a+\overline{a}|\geq 3} R^*_{k,a,\overline{a}}(\varphi_*) \mathcal{I}_*^k Z_*^a \overline{Z}_*^{\overline{a}}, \tag{4}$$

where $(\mathcal{I}_*, \varphi_*) \in \mathbb{R}^n \times \mathbb{T}^n$ are action-angle variables and $(Z_*, \overline{Z}_*) \in \mathbb{C}^{2m}$ are called the normal (or elliptic) coordinates. The phase space $\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^m \times \mathbb{C}^m$ is equipped with the symplectic form $d\mathcal{I}_* \wedge d\varphi_* + \mathrm{i}\, dZ_* \wedge d\overline{Z}_*$.

$\omega \in \mathbb{R}^n$ is the vector of "linear frequencies" and $\Omega := \mathrm{diag}(\Omega_1, \ldots, \Omega_m)$ is the $m \times m$ diagonal matrix[4] of the "elliptic frequencies". This denomination comes by the fact that the Hamilton's equations given by $\mathcal{H}_*$,

$$\dot{\mathcal{I}}_* = -\partial_{\varphi_*}\mathcal{H}_*, \quad \dot{\varphi}_* = \partial_{\mathcal{I}_*}\mathcal{H}_*, \quad \dot{Z}_* = \mathrm{i}\partial_{\overline{Z}_*}\mathcal{H}_*, \quad \dot{\overline{Z}}_* = -\mathrm{i}\partial_{Z_*}\mathcal{H}_*, \tag{5}$$

admit the elliptic invariant torus

$$\mathcal{T} \;:=\; \left\{ (\mathcal{I}_*, \varphi_*, Z_*, \overline{Z}_*) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2m} \;\middle|\; \mathcal{I}_* = 0, \; Z_* = \overline{Z}_* = 0 \right\}$$

supporting the flow $t \to (0, \varphi_{*0} + \omega t, 0, 0)$.

The functions $R^*_{k,a,\overline{a}}(\varphi_*)$ can be expanded in Fourier series as

$$R^*_{k,a,\overline{a}}(\varphi_*) = \sum_{\ell \in \mathbb{Z}^n} R^*_{k,a,\overline{a},\ell}\, e^{\mathrm{i}\ell\cdot\varphi_*}. \tag{6}$$

Note that, in order to let $\mathcal{H}_*(\mathcal{I}_*, \varphi_*, Z_*, \overline{Z}_*)$ be real-analytic, it must necessarily be

$$\overline{R^*_{k\,a,\overline{a},\ell}} = R^*_{k,\overline{a},a,-\ell}. \tag{7}$$

The frequency vector $(\omega, \Omega) := (\omega_1, \ldots, \omega_n, \Omega_1, \ldots, \Omega_m)$ is assumed to satisfy the "second order Melnikov non-resonance condition"

$$\left|\omega \cdot \ell \,+\, \Omega \cdot h\right| \;\geq\; \frac{\gamma}{1+|\ell|^\tau}, \quad \forall\, \ell \in \mathbb{Z}^n, \quad \forall\, h \in \mathbb{Z}^m,\; |h| \leq 2, \quad (\ell\,, h) \neq (0,0), \tag{8}$$

for some positive constants $\gamma, \tau \in \mathbb{R}$. This implies that the linear frequency vector $\omega$ is rationally independent (actually Diophantine), while the whole frequency vector $(\omega, \Omega)$ can meet some resonance relations.

We define the symmetric "twist" matrix $\mathcal{R} \in \mathrm{Mat}(n \times n, \mathbb{R})$

$$\begin{aligned}\mathcal{R}_{ii'} &:= (1+\delta_{ii'})\; R^*_{e_i+e_{i'},0,0,0} \\ &\quad - \sum_{1\leq j\leq m} \sum_{\ell\in\mathbb{Z}^n} \frac{1}{\omega\cdot\ell+\Omega_j} \left( R^*_{e_i,e_j,0,\ell}\, R^*_{e_{i'},0,e_j,-\ell} + R^*_{e_i,0,e_j,-\ell}\, R^*_{e_{i'},e_j,0,\ell} \right),\end{aligned} \tag{9}$$

[4] In the sequel, we will often identify the diagonal matrix $\Omega$ with the vector $(\Omega_1, \ldots, \Omega_m) \in \mathbb{R}^m$ without further specifications. The expression $\Omega Z_* \cdot \overline{Z}_*$ in (4) denotes $\sum_{1\leq j\leq m} \Omega_j Z_{*j} \overline{Z}_{*j}$.

where $R^*_{k,a,\overline{a},\ell}$ are the Taylor-Fourier coefficients introduced in (6), and $\delta_{ii'}$ is the classical Krönecker symbol. We also define the matrix $\mathcal{Q} \in \mathrm{Mat}(m \times n, \mathbb{R})$ as

$$\mathcal{Q}_{ji} := R^*_{e_i,e_j,e_j,0} - \sum_{1\le i'\le n} \sum_{\ell\in\mathbb{Z}^n} \frac{\ell_{i'}}{\omega\cdot\ell+\Omega_j} \times \left( R^*_{e_i,e_j,0,\ell}\, R^*_{e_{i'},0,e_j,-\ell} + R^*_{e_i,0,e_j,-\ell}\, R^*_{e_{i'},e_j,0,\ell} \right) - \sum_{1\le j'\le m} \sum_{\ell\in\mathbb{Z}^n} \frac{1}{\omega\cdot\ell+\Omega_{j'}} \times \left( R^*_{0,e_j,e_j+e_{j'},-\ell}\, R^*_{e_i,e_{j'},0,\ell} + R^*_{0,e_j+e_{j'},e_j,\ell}\, R^*_{e_i,0,e_{j'},-\ell} \right). \tag{10}$$

We now state the aforesaid theorem concerning the existence of periodic orbits close to elliptic tori. First, let's give a last definition: for $l, M \in \mathbb{N}$ we denote the set of the vectors in $\mathbb{Z}^l$ having norm less than $M$ by

$$\mathbb{Z}^l_M := \left\{ \vec{k} \in \mathbb{Z}^l : 0 < |\vec{k}|_1 \le M \right\}. \tag{11}$$

**Theorem 2.1** *Given a Hamiltonian of the form* (4)*, let the frequency vector* $(\omega, \Omega)$ *satisfy the second order Melnikov non-resonance condition* (8) *for some positive constant* $\gamma, \tau \in \mathbb{R}$. *Let* $\mathcal{R}$ *and* $\mathcal{Q}$ *be the matrices defined in* (9) *and* (10)*, and*

$$L := \max_{1\le i\le m} \sum_{1\le j\le n} \left| (\mathcal{Q}\mathcal{R}^{-1})_{ij} \right|. \tag{12}$$

*Assume the "twist" condition* $\det \mathcal{R} \ne 0$ *and that the frequency vector* $(\omega, \Omega)$ *is non-resonant up to a sufficiently high order, i.e.*

$$(\omega, \Omega)\cdot\vec{k} \ne 0 \quad \forall \vec{k} \in \mathbb{Z}^{n+m}_M, \tag{13}$$

*where* $M = M(n, m, L) \in \mathbb{N}$ *is a suitable constant. Then* $\exists\, \eta_0 > 0$ *such that* $\forall \eta \in (0, \eta_0]$ *there exists an open set of periods* $\Theta_\eta \subset [\frac{1}{\eta^2}, +\infty)$ *such that* $\forall\, T \in \Theta_\eta$ *there exists a vector of "shifted linear frequencies"* $\widetilde{\omega} = \widetilde{\omega}(T) \in \mathbb{R}^n$, *with* $\widetilde{\omega}T \in 2\pi\mathbb{Z}^n$, $|\widetilde{\omega} - \omega| \le \mathrm{const}\, \eta^2$, *such that the Hamiltonian system* (5) *admits at least n geometrically distinct T-periodic solutions* $\varrho_\eta(t) = \left(\mathcal{I}_{*\eta}(t), \varphi_{*\eta}(t), Z_{*\eta}(t), \overline{Z}_{*\eta}(t)\right)$ *satisfying*

$(i)$ $\sup_{t\in\mathbb{R}} \left( |\mathcal{I}_{*\eta}(t)| + |Z_{*\eta}(t)| + |\overline{Z}_{*\eta}(t)| \right) \le \mathrm{const}\, \eta^2$,

$(ii)$ $\sup_{t\in\mathbb{R}} |\varphi_{*\eta}(t) - (\varphi_{*\eta}(0) + \widetilde{\omega}t)| \le \mathrm{const}\, \eta.$

*In particular, the closure of the family of periodic orbits* $\varrho_\eta$, $\eta \in (0, \eta_0]$, *contains the elliptic torus* $\mathcal{T} := \{\mathcal{I}_* = 0\,,\ \varphi_* \in \mathbb{T}^n\,,\ Z_* = \overline{Z}_* = 0\}$.
*Moreover the minimal period* $T_{\min}$ *of* $\varrho_\eta$ *satisfies* $T_{\min} \ge \mathrm{const}\, T^{1/(\tau+1)}$.

*Remark 2.2* Theorem 1.1 of Berti et al. (2004) holds under various non-resonance hypotheses, in particular if (13) is satisfied for *all* $\vec{k} \in \mathbb{Z}^{n+m}$. Actually in Berti et al. (2004) it is suggested that the same result still holds if the weaker condition (13) is satisfied (see p. 97 of Berti et al. (2004)). Here we prove it.

The proof of Theorem 2.1 is easily obtained adapting the arguments of Berti et al. (2004), that we will briefly recall here for completeness. Then we will focus on the new part of the proof, which consists in finding the set of the "admissible periods" (see Lemma 2.5 below).

*Sketch of the proof of Theorem 2.1*: First of all, as we are interested in the region of phase space near the torus $\mathcal{T}$, a small rescaling parameter $\eta > 0$ measuring the distance from $\mathcal{T}$ is introduced. Then, since $(\omega, \Omega)$ satisfies the second order Melnikov non-resonance conditions (8), in view of an averaging procedure, the Hamiltonian $\mathcal{H}_*$ is casted, in a suitable set of coordinates $(I, \phi, z, \bar{z}) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}^{2m}$, and sufficiently close to the torus $\mathcal{T}$, in a small perturbation of the integrable Hamiltonian

$$H_{\text{int}} := \omega \cdot I + \frac{\eta^2}{2} \mathcal{R} I \cdot I + \Omega z \bar{z} + \eta^2 \mathcal{Q} I \cdot z \bar{z}.$$

The Hamiltonian system generated by $H_{\text{int}}$ possesses the elliptic tori $\mathcal{T}(I_0) := \{I = I_0,\ \phi \in \mathbb{T}^n,\ z = \bar{z} = 0\}$. The torus $\mathcal{T}(I_0)$ supports the linear flow $t \to (I_0,\ \phi_0 + (\omega + \eta^2 \mathcal{R} I_0)t,\ 0,\ 0)$, whereas on the normal space the dynamic is described by $\dot{z} = \mathrm{i}(\Omega + \eta^2 \mathcal{Q} I_0) z$, $\dot{\bar{z}} = \mathrm{i}(\Omega + \eta^2 \mathcal{Q} I_0)\bar{z}$. $\widetilde{\omega} = \omega + \eta^2 \mathcal{R} I_0$ and $\widetilde{\Omega} = \Omega + \eta^2 \mathcal{Q} I_0$ are called respectively the vector of the "*shifted linear frequencies*" and of the "*shifted elliptic frequencies*".

By the "twist condition" $\det \mathcal{R} \neq 0$ the system generated by $H_{\text{int}}$ is properly nonlinear. In particular, such condition ensures that the shifted linear frequencies $\widetilde{\omega}$ vary with the actions $I_0$. Hence, it is always possible to find *completely resonant* frequencies

$$\widetilde{\omega} = \frac{1}{T} 2\pi k \in \frac{1}{T} 2\pi \mathbb{Z}^n, \tag{14}$$

for some $T$: in such case, $\mathcal{T}(I_0)$ is a completely resonant torus supporting the family of $T$-periodic motions $\mathcal{P} := \{I(t) = I_0,\ \phi(t) = \phi_0 + \widetilde{\omega} t,\ z(t) = \bar{z}(t) = 0\}$.

Following (Berti et al. 2004), we define $I_0 := I_0(T)$ in dependence on the "1-dimensional parameter" $T$ in such a way that (14) is identically satisfied: for $T \geq 1/\eta^2$ we set

$$I_0 := I_0(T) := -\frac{2\pi}{\eta^2 T} \mathcal{R}^{-1} \left\langle \frac{\omega T}{2\pi} \right\rangle, \tag{15}$$

$$k := k(T) = \frac{\omega T}{2\pi} - \left\langle \frac{\omega T}{2\pi} \right\rangle, \tag{16}$$

where $\langle (x_1, \ldots, x_n) \rangle := (\langle x_1 \rangle, \ldots, \langle x_n \rangle)$ and the function $\langle \cdot \rangle : \mathbb{R} \to [-1/2, 1/2)$ is defined as $\langle x \rangle := x$ for $x \in [-1/2, 1/2)$ and it is 1-periodically extended for $x \in \mathbb{R}$.

With the choices (15), (16),

$$\widetilde{\omega} T = \omega T + \eta^2 \mathcal{R} I_0(T) T = 2\pi k \in 2\pi \mathbb{Z}^n, \tag{17}$$

and then (14) holds. In addition, $T \geq 1/\eta^2 \implies I_0(T) = O(1)$.

The idea of the proof is to find periodic solutions for the Hamiltonian system generated by $\mathcal{H}_*$ bifurcating from the ones of $H_{\text{int}}$. Nevertheless, in general, the family $\mathcal{P}$ will not persist in its entirety for the complete Hamiltonian system due to resonances among the oscillations. The key point to continue some periodic solutions of the family $\mathcal{P}$ is to choose properly the "1-dimensional parameter" $T$: the period $T$ and the "shifted elliptic frequencies" $\widetilde{\Omega}(I_0(T))$ must satisfy the following *non-resonance property*:

$$\mathcal{M} := \mathcal{M}(T) := Id_m - e^{\mathrm{i}\widetilde{\Omega} T} \quad \text{is invertible.}^5 \tag{18}$$

Before discussing this condition, we conclude the proof using a *Lyapunov-Schmidt reduction*. First, (18) and the "twist condition" allow to solve the *range equation* by means of the Contraction Mapping Theorem. Roughly, by these assumptions the manifold $\mathcal{P}$ is "non-degenerate", i.e. the only $T$-periodic solutions of $H_{int}$, close to $\mathcal{P}$, are the set $\mathcal{P}$; heuristically, this implies, by the Implicit Function Theorem, the existence, for small $\eta > 0$, of a manifold of $T$-periodic solutions of the Hamilton's equations induced by $H$. Then, the

[5] Here $\widetilde{\Omega} := \operatorname{diag}(\widetilde{\Omega}_1, \ldots, \widetilde{\Omega}_m)$.

*bifurcation equation* given by the previous Lyapunov-Schmidt reduction is written in a variational form, so that the solutions are seen as critical points of a suitable "reduced Hamiltonian action functional". Finally, by Ljusternik-Shnirelmann Category Theory (see for example Ambrosetti 1992), one manages to find at least $n$ geometrically distinct $T$-periodic solutions for the system. □

We now discuss (18) in full details. First, note that

$$\mathcal{M} \quad \text{is invertible} \quad \Longleftrightarrow \quad \widetilde{\Omega}_j T \notin 2\pi\mathbb{Z}, \quad \forall j = 1, \ldots, m \tag{19}$$

and (recalling (15))

$$\left|\mathcal{M}^{-1}\right| = \frac{1}{\min\limits_{1 \le j \le m} \left|1 - e^{\mathrm{i}\widetilde{\Omega}_j T}\right|} \le \frac{2}{\min\limits_{1 \le j \le m} \operatorname{dist}\left(\widetilde{\Omega}_j T,\, 2\pi\mathbb{Z}\right)}. \tag{20}$$

We now show how, assuming condition (13), it is possible to find an open set of "non-resonant" periods $T$: this could be done, as was already suggested in Berti et al. (2004), by means of "ergodization" arguments. We enter in details:

To begin we deal with the notion of *ergodization time*, that will play an important role in the proof. Let's set $\mathbb{T}^l := \frac{\mathbb{R}^l}{\mathbb{Z}^l}$. Given a vector $\xi \in \mathbb{R}^l$, it is well known that, if $\xi \cdot k \neq 0$ $\forall k \in \mathbb{Z}^l \setminus \{0\}$, then the trajectories of the linear flow $\{\xi t + P\}_{t \in \mathbb{R}}$ are dense on $\mathbb{T}^l$ for any initial point $P \in \mathbb{T}^l$. It is also intuitively clear that the trajectories of the linear flow $\{\xi t + P\}_{t \in \mathbb{R}}$ will make an arbitrarily fine $d$-net ($d > 0$) if $\xi$ is resonant only at a sufficiently high order, namely if $\xi \cdot \vec{k} \neq 0,\ \forall \vec{k} \in \mathbb{Z}^l_M$ for some large enough $M$ depending on $d$.

We now make more precise and quantitative these considerations.

**Definition 2.3** *Let $\xi \in \mathbb{R}^l$ and $d > 0$. The ergodization time $T_{\mathrm{erg}}(\xi, d)$ required to fill $\mathbb{T}^l$ within $d$ is*

$$T_{\mathrm{erg}}(\xi, d) := \inf \left\{ t > 0 \,\middle|\, \forall x \in \mathbb{R}^l,\ \operatorname{dist}_\infty (x, [0, t]\xi + \mathbb{Z}^l) \le d \right\}, \tag{21}$$

*where* $\operatorname{dist}_\infty$ *denotes the distance induced by the* sup-*norm in* $\mathbb{R}^l$.

*As usual, we set*

$$T_{\mathrm{erg}}(\xi, d) := +\infty \ \text{ if } \ \left\{ t > 0 \,\middle|\, \forall x \in \mathbb{R}^l,\ \operatorname{dist}_\infty (x, [0, t]\xi + \mathbb{Z}^l) \le d \right\} = \emptyset.$$

The following result is proved in Berti et al. (2003) (Theorem 4.1):[6]

**Lemma 2.4** *Let $l \in \mathbb{N}$. There exist a positive constant $C_l$ such that $\forall d > 0$, if $\xi \in \mathbb{R}^l$ satisfies the non-resonance condition*

$$\xi \cdot \vec{k} \neq 0, \quad \forall \vec{k} \in \mathbb{Z}^l_M \quad \text{with} \quad M := \left\lfloor \frac{C_l}{d} \right\rfloor, \tag{22}$$

*then the ergodization time $T_{\mathrm{erg}}(\xi, d)$ is finite.*

Now we finally explain how to achieve condition (18) on an open set of periods; the following lemma is the analogue of Lemma 4.1 of Berti et al. (2004).

**Lemma 2.5** *Let $L$ be as in (12), $d := d(L) := \min\left(\frac{1}{4}, \frac{1}{8L}\right)$, and $T_{\mathrm{e}} := T_{\mathrm{erg}}((\omega, \Omega), d)$. Suppose that the frequency vector $(\omega, \Omega)$ satisfies the non-resonance condition (13) of*

[6] For $x \in \mathbb{R}_+$ we denote by $\lfloor x \rfloor$ the so called "floor function" (or integer part) of $x$, that is the largest integer less than or equal to $x$.

*Theorem 2.1 with* $M := \left\lfloor \frac{C_{n+m}}{d} \right\rfloor = \lfloor C_{n+m} \max(4, 8L) \rfloor$, $C_{n+m}$ *being the constant defined in Lemma 2.4. Then,* $\forall t_0 > 0$, *there exists an open interval* $\mathcal{J} \subset [\, t_0\, t_0 + 2\pi T_e \,]$ *such that*

$$\left\| (\mathcal{M}(T))^{-1} \right\| \le 3\,, \quad \forall\, T \in \mathcal{J}. \tag{23}$$

*Proof* Let $t_0 > 0$. By Lemma 2.4 applied with $\xi = (\omega, \Omega)$, $l = n + m$, $d = d(L)$, $T_e$ is finite. Moreover, by definition of ergodization time in (21), there exists a time $T_0 = \frac{T}{2\pi} \in \left[\frac{t_0}{2\pi}, \frac{t_0}{2\pi} + T_e\right]$ such that

$$\begin{cases} \operatorname{dist}\left(\Omega_j T_0,\, \mathbb{Z} + \dfrac{1}{2}\right) \le d \le \dfrac{1}{4} & \forall j = 1, \ldots, m \\[2ex] \operatorname{dist}\left(\omega_j T_0,\, \mathbb{Z}\right) \le d \le \dfrac{1}{8L} & \forall j = 1, \ldots, m. \end{cases} \tag{24}$$

Noting that $\operatorname{dist}\left(\Omega_j T_0\,,\, \mathbb{Z} + \frac{1}{2}\right) \le \frac{1}{4} \iff \operatorname{dist}\left(\Omega_j T_0,\, \mathbb{Z}\right) \ge \frac{1}{4}$, we get by (15)

$$\begin{aligned} &\operatorname{dist}\left(\widetilde{\Omega}_j(T) T_0,\, \mathbb{Z}\right) = \operatorname{dist}\left(\Omega_j T_0 - (\mathcal{Q}\mathcal{R}^{-1})_j < \omega T_0 >,\, \mathbb{Z}\right) \\ &\ge \frac{1}{4} - \max_{1 \le j \le m} \left|(\mathcal{Q}\mathcal{R}^{-1})_j\right|_1 \frac{1}{8L} \ge \frac{1}{8}. \end{aligned}$$

The existence of $\mathcal{J}$ follows by continuity (of $x \mapsto |\langle x \rangle|$), (20) and $\frac{2}{3} < \frac{2\pi}{8}$. □

## 3 Sub-level-sets of real-analytic functions

In this section we provide an estimate on the measure of the sub-level-sets of (non-identically-vanishing) real-analytic functions of several variables. Such result will be used in Sect. 5.2 to control the size of the "discarded" subset of "semiaxes" where the non-resonance condition (13) is not met (see Proposition 4.3).

The statement we want to prove is the following:

**Proposition 3.1** *Let* $D \subset \mathbb{R}^N$ *be nonempty, open, connected. Let* $f : D \longrightarrow \mathbb{R}$ *be a real-analytic function that does not vanish identically. Let us define the sub-level-sets*

$$D_f(\gamma) := \left\{ x \in D \,:\, |f(x)| \le \gamma \right\}.$$

*Then, for any compact* $\mathcal{C} \subset D$ *there exists* $\alpha > 0$ *such that*

$$\operatorname{meas}\left(D_f(\gamma) \cap \mathcal{C}\right) = O\left(\gamma^{\alpha}\right). \tag{25}$$

The previous proposition directly follows by compactness and by the following result on balls.

**Lemma 3.2** *Let* $D \subset \mathbb{R}^N$ *be not empty, open, connected. Let* $f : D \longrightarrow \mathbb{R}$ *be a non-identically-vanishing real-analytic function. Then, for any* $x_0 \in D$, *there exist* $r, \alpha > 0$ *such that* $B_r(x_0) \subset D$ *and*

$$\operatorname{meas}\left(D_f(\gamma) \cap B_r(x_0)\right) = O\left(\gamma^{\alpha}\right).$$

In order to prove Lemma 3.2, we need the following result from the theory of functions of several complex variables, due to Weierstraß (see, e.g. Griffiths and Harris (1978)).

**Theorem 3.3** (Weierstraß Preparation Theorem) *Let $f(z, w) = f(z_1, \dots, z_{N-1}, w)$ be a holomorphic function defined in a neighborhood of the origin of $\mathbb{C}^N$, and assume that $f$ does not vanish identically on the $w$-direction, i.e. $\exists d \in \mathbb{N}$ such that $\partial_w^j f(0,0) = 0$ for any $1 \le j < d$, but $\partial_w^d f(0,0) \ne 0$. Then, in some neighborhood of the origin $f$ can be written uniquely as $f = g \cdot h$, where $g$ is a Weierstraß polynomial of degree $d$ in $w$, i.e.*

$$w^d + a_1(z) w^{d-1} + \cdots + a_d(z), \qquad a_j(0) = 0\,, \ \forall\, 1 \le j < d,$$

*and $h = h(z, w)$ is holomorphic with $h(0,0) \ne 0$.*

We also need the following simple result.

**Lemma 3.4** *Let $g(w) = w^d + a_1 w^{d-1} + \cdots + a_d$, $a_i \in \mathbb{C}$. Then*

$$\text{meas}\ \big(\{w \in \mathbb{R} \,:\, |g(w)| \le \gamma\}\big) \ \le\ 2d\,\gamma^{1/d}.$$

*Proof* Let $w_1, \dots, w_d \in \mathbb{C}$ be the (not necessarily distinct) roots of the polynomial $g$, so that $g(w) = (w - w_1)(w - w_2)\dots(w - w_d)$. For $r > 0$

$$\in \mathbb{R} \setminus \left[ \bigcup_{i=1}^{d} (\text{Re}\ w_i - r, \text{Re}\ w_i + r) \right] \implies |g(w)| \ge r^d.$$

Equivalently, $|g(w)| \le \gamma \implies w \in \bigcup_{i=1}^{d} \Big(\text{Re}\ w_i - \gamma^{1/d}, \text{Re}\ w_i + \gamma^{1/d}\Big)$ and clearly the measure of this last set is at most $2d\,\gamma^{1/d}$. □

*Proof of Lemma 3.2* Up to a translation and a rotation we can suppose that $x_0$ is the origin and that $f$ is not identically vanishing in the last coordinate. So we are in a position to apply Theorem 3.3: there exists $r > 0$ such that $B_r(0) \subset D$ and $f = g \cdot h$ on $B_r(0)$, where $g$ is a Weierstraß polynomial of degree $d \ge 1$ and $h$ is holomorphic with $\inf_{B_r(0)} |h| =: \kappa > 0$. Then,[7] by Fubini's theorem and Lemma 3.4,

$$\begin{aligned} \text{meas}_{\,N}\,\big(D_f(\gamma) \cap B_r(0)\big) &\le\ \text{meas}_{\,N}\,\big(D_g(\gamma/\kappa) \cap B_r(0)\big) \\ &\le 2d(\gamma/\kappa)^{1/d} \times\ \text{meas}_{\,N-1}\,\Big(B_r^{N-1}(0)\Big)\,. \end{aligned}$$

The statement follows taking $\alpha := 1/d$. □

We now define a non-resonance condition *up to order $M$* for real-analytic functions, that will be appropriate for our purpose:[8]

**Definition 3.5** *Let $M \in \mathbb{N}$. A real-analytic function $g = (g_1, \dots, g_l) \colon D \subset \mathbb{R}^N \longrightarrow \mathbb{R}^l$ is said to be **non-resonant up to order $M$** if*

$$\vec{k} \cdot g = k_1\, g_1 + \cdots + k_l\, g_l\ \not\equiv\ 0\,, \quad \forall \vec{k} \in \mathbb{Z}^l_M. \tag{26}$$

The next result, that is a quite direct consequence of Proposition 3.1, has a fundamental role in the proof of Proposition 5.4:

[7] We denote by $\text{meas}_{\,j}$ the $j$-dimensional Lebesgue measure on $\mathbb{R}^j$ and by $B_r^j(0)$ the $j$-dimesional ball of center the origin of $\mathbb{R}^j$ and radius $r$. For brevity $B_r(0) = B_r^N(0)$.

[8] Recall the definition of the set $\mathbb{Z}^l_M$ in (11).

**Proposition 3.6** *Let $g : D \longrightarrow \mathbb{R}^l$ be a real-analytic function, with $D \subset \mathbb{R}^N$ not empty open and connected. Fix $M \in \mathbb{N}$ and let*

$$D_g^M(\gamma) := \left\{ x \in D \ : \ |\vec{k} \cdot g(x)| \le \gamma \ \textit{for some } \vec{k} \in \mathbb{Z}^l_M \right\}. \tag{27}$$

*If $g$ is non-resonant up to order $M$, then for any compact $\mathcal{C} \subset D$*

$$\operatorname{meas}\left( D_g^M(\gamma) \cap \mathcal{C} \right) = O\left(\gamma^{\alpha}\right) \quad \textit{for some } \alpha > 0. \tag{28}$$

*Proof* For any $\vec{k}$ belonging to the *finite* set $\mathbb{Z}^l_M$, let $f_{\vec{k}} := \vec{k} \cdot g$. Since $g$ is non-resonant up to order $M$, every $f_{\vec{k}}$ is non-identically-vanishing. Hence, by Proposition 3.1 there exist positive numbers $\alpha_{\vec{k}}$ such that

$$\operatorname{meas}\left( D_{f_{\vec{k}}}(\gamma) \cap \mathcal{C} \right) = O\left(\gamma^{\alpha_{\vec{k}}}\right), \quad \forall \vec{k} \in \mathbb{Z}^l_M.$$

Since by definition (27)

$$D_g^M(\gamma) = \bigcup_{\vec{k} \in \mathbb{Z}^l_M} D_{f_{\vec{k}}}(\gamma),$$

we conclude setting $\alpha := \min_{\vec{k} \in \mathbb{Z}^l_M} \alpha_{\vec{k}}$. □

For the sake of completeness, we finally recall the Rüßmann non-degeneracy condition for analytic functions (see Rüssmann 1990).

**Definition 3.7** *A real-analytic function $g = (g_1, \ldots, g_l) : D \subset \mathbb{R}^N \longrightarrow \mathbb{R}^l$ is called* ***Rüßmann non-degenerate*** (*or, simply, non-degenerate*) *if*

$$c_1\, g_1 + \cdots + c_l\, g_l \not\equiv 0 \quad \forall\, (c_1, \ldots, c_l) \in \mathbb{R}^l \setminus \{\vec{0}\}, \tag{29}$$

*i.e. the range $g(D)$ of $g$ is not lying in any vectorial hyperplane of $\mathbb{R}^l$.*

*Remark 3.8* As it is obvious from Definition 3.5 and Definition 3.7, the Rüßmann non-degeneracy condition implies the non-resonance condition up to order $M$, $\forall M \in \mathbb{N}$.

## 4 The planetary planar $(N+1)$-body problem

In this section we recall the result from Biasco et al. (2006) concerning the existence of $N$-dimensional invariant elliptic tori (supporting quasi-periodic orbits) for the planetary planar $(N+1)$-body model described in the introduction. The scheme of the proof in Biasco et al. (2006) is as follows.

First of all the classical Hamiltonian formulation of the problem is provided. The planetary $(N+1)$-body problem is viewed as a *nearly-integrable* Hamiltonian system in the perturbative parameter $\varepsilon$; the integrable limit ($\varepsilon = 0$) consists just of the $N$ decoupled two-body systems given by the Sun and the $i$th planet.

In Poincaré (1905), carrying on the work of Delaunay, introduced a set of analytic symplectic variables for the two-body system; using a planar version of this classical result, *planar Poincaré variables* $(\Lambda_i, \lambda_i, \eta_i, \xi_i) \in (0,\infty) \times \mathbb{T} \times \mathbb{R} \times \mathbb{R}$, $i = 1, \ldots, N$ for each (Sun, $i$th planet)-system are considered. Here we only recall (for more details see, for example,

Appendix A of Biasco et al. (2006) or Chenciner (1988) where the non-planar case is consider) that the action $\Lambda_i$ is related to the major semi-axis $a_i$ of the osculating ellipse for the $i$th planet through

$$\mathrm{M}_i := 1 + \varepsilon \frac{\mu_i}{m_0}, \quad \overline{\mathrm{m}}_i := \frac{\mu_i}{m_0} \frac{1}{\sqrt{\mathrm{M}_i}}, \quad \sigma_i := \left(\frac{\mu_i}{m_0}\right)^3 \frac{1}{\mathrm{M}_i}, \quad a_i := \frac{\Lambda_i^2}{\overline{\mathrm{m}}_i^2}. \tag{30}$$

In planar Poincaré variables the planetary $(N+1)$-body Hamiltonian is

$$\mathcal{H}(\Lambda, \lambda, \eta, \xi) = \mathcal{H}_0(\Lambda) + \mathcal{H}_1(\Lambda, \lambda, \eta, \xi) \,, \quad \text{with} \quad \mathcal{H}_0(\Lambda) := -\frac{1}{2} \sum_{i=1}^{N} \frac{\sigma_i}{\Lambda_i^2},$$
$$\mathcal{H}_1 = O(\varepsilon), \tag{31}$$

and symplectic structure $\sum_{1 \le i \le N} (d\Lambda_i \wedge d\lambda_i + d\eta_i \wedge d\xi_i)$.

As customary, the variables $\lambda$ may be regarded as "fast angles"; by averaging theory one can neglect, in suitably non-resonant regions of the phase space, the fast-angle dependence up to high order in $\varepsilon$ : in first approximation the motions are governed by the *averaged Hamiltonian*.[9]

The $\lambda$-average of $\mathcal{H}_1$ is an even function of $(\eta, \xi)$. Hence, we may split the perturbation function as

$$\mathcal{H}_1 = \overline{\mathcal{H}}_1 + \widetilde{\mathcal{H}}_1, \tag{32}$$

with

$$\overline{\mathcal{H}}_1(\Lambda, \eta, \xi) := \int_{\mathbb{T}^N} \mathcal{H}_1 \frac{d\lambda}{(2\pi)^N}, \quad \int_{\mathbb{T}^N} \widetilde{\mathcal{H}}_1 \, d\lambda = 0. \tag{33}$$

Furthermore, $\overline{\mathcal{H}}_1$ may be written as

$$\overline{\mathcal{H}}_1(\Lambda, \eta, \xi) = \overline{\mathcal{H}}_{1,0}(\Lambda) + \overline{\mathcal{H}}_{1,2}(\Lambda, \eta, \xi) + \overline{\mathcal{H}}_{1,*}(\Lambda, \eta, \xi), \tag{34}$$

where $\overline{\mathcal{H}}_{1,0} := \overline{\mathcal{H}}_1(\Lambda, 0, 0)$, $\overline{\mathcal{H}}_{1,2}$ is the $(\eta, \xi)$-quadratic part of $\overline{\mathcal{H}}_1$, while $\overline{\mathcal{H}}_{1,*}$ is the "remainder of order four":

$$|\overline{\mathcal{H}}_{1,*}(\Lambda, \eta, \xi)| \le \text{const}\, |(\eta, \xi)|^4.$$

Moreover

$$\overline{\mathcal{H}}_{1,2} = \frac{1}{2} \left( Q\eta \cdot \eta + Q\xi \cdot \xi \right) \,, \tag{35}$$

$Q$ being a real, symmetric $(N \times N)$-matrix.

By the previous expression, the secular Hamiltonian system admits lower dimensional *elliptic* invariant tori run by linear flows. The crucial fact, in order to apply elliptic KAM theory and show that some of such motions (in particular those with Diophantine quasi-periodic frequency) persist, consists in proving that the eigenvalues of the matrix $Q$ are non-degenerate in the sense that they are non-vanishing and distinct (see the "Melnikov condition"in Theorem 4.1 of Biasco et al. 2006). Such non-degeneracy is checked (for $\delta$ and $\varepsilon$ suitably small) in Biasco et al. (2006), where, assuming the hypotheses (1) and (3) over the

9 The averaged Hamiltonian wil be also called *secular*, since physically it describes (looking at the first order in $\varepsilon$) the slow variation along the centuries of Keplerian ellipses which changes their shapes under perturbations due to gravitational interaction among the planets.

masses of the planets, the asymptotics of the eigenvalues are explicitly computed through direct algebraic calculations, yielding the following result.[10]

**Proposition 4.1** *There exists a (small) universal constant $\theta$ (depending only on the given masses of the planet) such that for any compact set*

$$A \subset \left\{(a_1, \dots, a_N) \,\middle|\, 0 < a_i < \theta\, a_{i+1}\,,\ i = 1, \dots, N-1\right\}$$

*there exist $0 < \delta_0 < 1$ and $0 < \varepsilon_0 < 1$ such that for all $0 < \delta < \delta_0$ and $0 \le \varepsilon < \varepsilon_0$, the eigenvalues $\{\bar\Omega_1, \dots, \bar\Omega_N\}$ of the matrix $Q$ are non vanishing and distinct and satisfy*

$$\begin{aligned} \bar\Omega_j(a) &= \varepsilon \left(\bar\Omega_j^0(a) + O\left(\sqrt{\delta},\ \varepsilon\right)\right), \quad j = 1, 2, \\ \bar\Omega_j(a) &= \varepsilon \left(\sqrt{\delta}\, \bar\Omega_j^0(a) + O(\delta,\ \varepsilon)\right), \quad 3 \le j \le N, \end{aligned} \tag{36}$$

*where $0 < b < 1/2$ is a suitable constant. Moreover the following asymptotics hold for*

$$\bar\Omega^0 := (\bar\Omega_1^0, \dots, \bar\Omega_N^0) : \left\{(a_1, \dots, a_N) \,\middle|\, 0 < a_i < a_{i+1},\ i = 1, \dots, N-1\right\} \longrightarrow \mathbb{R}^N :$$

$$\begin{aligned} \bar\Omega_1^0(a) &= \frac{3}{4}\frac{\sqrt{\bar\mu_1\bar\mu_2}}{a_2^{3/2}} \left(\frac{a_1}{a_2}\right)^{7/4} \left[1 - \frac{5}{16}\frac{a_1}{a_2} + O\left(\left(\frac{a_1}{a_2}\right)^2\right)\right], \\ \bar\Omega_2^0(a) &= \frac{3}{4}\frac{\sqrt{\bar\mu_1\bar\mu_2}}{a_2^{3/2}} \left(\frac{a_1}{a_2}\right)^{7/4} \left[1 + \frac{5}{16}\frac{a_1}{a_2} + O\left(\left(\frac{a_1}{a_2}\right)^2\right)\right], \\ \bar\Omega_j^0(a) &= \frac{3}{4}\frac{\sqrt{\bar\mu_j\bar\mu_2}}{a_j^{3/2}} \left(\frac{a_2}{a_j}\right)^{7/4} \left[1 + O\left(\left(\frac{a_1}{a_2}\right)^{7/4}\right) + O\left(\left(\frac{a_2}{a_j}\right)^2\right)\right], \quad 3 \le j \le N. \end{aligned} \tag{37}$$

*Remark 4.2* Proposition 4.1 can be obtained simply joining Eq. (3.50) of Biasco et al. (2006) and the expressions in Remark 3.2 of Biasco et al. (2006).

Using the asymptotics in (36) and (37), and choosing small $\delta$ and $\varepsilon$, we can now achieve a "preliminary" non-resonance result for the vector $\bar\Omega = \bar\Omega(a)$. This will be exploited in Sect. 5 as the starting point to check the non-resonance condition (13) and apply Theorem 2.1 to the $(N+1)$-body problem. We just give the statement here. The proof will be provided later (Sect. 5.2).

**Proposition 4.3** *Fix $M \in \mathbb{N}$. There exist positive constants $\alpha^\star$ and $\delta^\star < \delta_0$ such that, for every $0 < \delta \le \delta^\star$, and $0 < \varepsilon \le \varepsilon^\star(\delta) \le \varepsilon_0$ there exists a subset of semi-axes $A^\star = A^\star(\delta) \subset A$, with*

$$\text{meas}\left(A \setminus A^\star\right) = O(\delta^{\alpha^\star}), \tag{38}$$

*such that*

$$\left|\bar\Omega(a) \cdot K\right| \ \ge\ \text{const}\, \varepsilon\, \delta^{3/4}, \quad \forall K \in \mathbb{Z}_M^N, \quad \forall a \in A^\star. \tag{39}$$

[10] By (30) we will consider $Q$ as a function of $a = (a_1, \dots, a_N)$ instead of $\Lambda$.

Going on with Biasco et al. (2006) contents, let now $\theta$, $A^\star$, $\delta^\star$ and $\varepsilon^\star$ be as above. *Fix now* $0<\delta<\delta^\star$, *which henceforth will be kept fixed. In the rest of the paper only* $\varepsilon$ *is regarded a free parameter:* at the moment, $\varepsilon$ is assumed not to exceed $\varepsilon^\star$ but later will be required to satisfy stronger smallness conditions.

In Biasco et al. (2006), several symplectic close-to-the-identity[11] changes of variables are performed,[12] casting the Hamiltonian in the form

$$\tilde{\mathcal{H}}(\tilde\Lambda,\tilde\lambda,\tilde\eta,\tilde\xi)=h_0(\tilde\Lambda)+\frac12\sum_{1\le i\le N}\tilde\Omega_i(\tilde\Lambda)(\tilde\eta_i^2+\tilde\xi_i^2)+\tilde g_0(\tilde\Lambda,\tilde\eta,\tilde\xi)+\tilde f_0(\tilde\Lambda,\tilde\lambda,\tilde\eta,\tilde\xi), \tag{40}$$

with

$$h_0=\mathcal{H}_0+O(\varepsilon),\quad \tilde\Omega_i-\bar\Omega_i=O(\varepsilon^{1+\mathrm{const}}),\quad \tilde g_0=O\left(\varepsilon|(\tilde\eta,\tilde\xi)|^3\right),$$
$$\tilde f_0=O(\varepsilon^3). \tag{41}$$

Then the Hamiltonian $\tilde{\mathcal{H}}$ is written in a form suitable to apply (elliptic) KAM theory: introducing translated variables $y:=\tilde\Lambda-\mathrm{p}(a)$ and complex variables $z,\bar z$, we define

$$H(y,\psi,z,\bar z;a):=\tilde{\mathcal{H}}\left(\mathrm{p}+y,\psi,\frac{z+\bar z}{\sqrt2},\frac{z-\bar z}{\mathrm{i}\sqrt2}\right), \tag{42}$$

where $\mathrm{p}=\mathrm{p}(a)$, with $\mathrm{p}_i:=\overline{\mathrm{m}}_i\sqrt{a_i}$ (recall (30)), and the vector of semi-axes $a$ is regarded as a parameter; the symplectic form is now $\sum_{j=1}^N dy_j\wedge d\psi_j+\mathrm{i}\sum_{j=1}^N dz_j\wedge d\bar z_j$. The Hamiltonian $H$ is seen to have the form

$$H=\mathcal{N}+P$$

where

$$\mathcal{N}=e+\omega\cdot y+\sum_{j=1}^N\Omega_j z_j\bar z_j,\ \ e=e(a):=h_0(\mathfrak{p}(a)),\ \ \omega=\omega(a):=h_0'(\mathfrak{p}(a)),$$
$$\Omega=\Omega(a):=\tilde\Omega(\mathfrak{p}(a)) \tag{43}$$

and $P$ is a perturbation. The Hamiltonian $H$ is real-analytic on

$$(y,\psi,z,\bar z;a)\in\mathcal{D}_{\bar r,\bar s}\times\mathcal{A}_{\bar d}, \tag{44}$$

where

$$(y,\psi,z,\bar z)\in\mathcal{D}_{\bar r,\bar s}:=B^N_{\bar r^2}\times\mathbb{T}_{\bar s}\times B^{2N}_{\bar r},$$

[11] With respect to some positive power of $\varepsilon$.

[12] In particular one performs first an averaging over the fast angles $\lambda$. To do that, one needs the frequency vector, whose components are $\partial_{\Lambda_i}\mathcal{H}_0(\Lambda(a))=\sqrt{\mathrm{M}_i}a_i^{-3/2}$, to be $\bar\gamma$-$\tau$ diophantine, with $\bar\gamma:=\varepsilon^{b_1}$, up to order $K=\varepsilon^{b_2}$ (recall that a vector $v\in\mathbb{R}^N$ is $\bar\gamma$-$\tau$ diophantine up to order $K$ if $|v\cdot k|\ge\bar\gamma|k|^{-\tau}$, $\forall\,|k|\le K$), with $0<b_1<1/2$ and $0<b_2<(\frac12-b_1)/(\tau+1)$. Since the above condition is equivalent to require that the vector $(\dots,a_i^{-3/2},\dots)$ is $\bar\gamma$-$\tau$ diophantine up to order $K$ (recall that $\mathrm{M}_i=1+O(\varepsilon)$), one can perform the averaging for $a$ belonging to a suitable set of semi-axes $\mathcal{A}=\mathcal{A}(\varepsilon)\subset A$ (corresponding to diophantine frequencies) verifying meas $(A\setminus\mathcal{A})=O(\varepsilon^{b_1})$.

with, for $n \in \mathbb{Z}$, $\rho > 0$, $B^n_\rho := \{x \in \mathbb{C}^n \text{ s.t. } |x| < \rho\}$ and $\mathbb{T}^n_\rho := \big\{z \in \mathbb{C}^n \;\big|\; \mathrm{Re}\, z \in \mathbb{T}^n,\ |\mathrm{Im}\, z_j| < \rho,\ \forall\, 1 \le j \le n\big\}$, $\mathcal{A}_{\bar d} := \{a \in \mathbb{C}^N \text{ s.t. } \mathrm{dist}\,(a, \mathcal{A}) < \bar d\}$, for suitable

$$\bar r := \text{ const } \varepsilon^{3/4}, \quad \bar s := \text{ const }, \quad \bar d := \text{ const} \sqrt{\varepsilon}, \tag{45}$$

finally $\mathcal{A} = \mathcal{A}(\varepsilon) \subset A$ is a suitable set of semi-axes such that the symplectic transformations carried out in Biasco et al. (2006) hold (see footnote 12). We have

$$\text{meas}\,(A \setminus \mathcal{A}) \le \varepsilon^{\text{const}}. \tag{46}$$

We can now state the KAM theorem on the conservation of elliptic tori.

**Theorem 4.4** *For $\varepsilon$ small enough, let's say $\varepsilon < \varepsilon^\sharp$ for some $0 < \varepsilon^\sharp < \varepsilon^\star$, there exists a Cantor set $\mathcal{A}^\sharp = \mathcal{A}^\sharp(\varepsilon) \subset \mathcal{A} \subset A$, with*

$$\text{meas}\,\big(A \setminus \mathcal{A}^\sharp\big) = O(\varepsilon^{\text{const}}), \tag{47}$$

*such that for any vector of semi-axes $a \in \mathcal{A}^\sharp$ it is possible to find a real-analytic symplectic transformation*

$$\Phi\colon\ \mathcal{D}_{\bar r/2, \bar s/2} \longrightarrow \mathcal{D}_{\bar r, \bar s}$$

*casting the $(N+1)$-body Hamiltonian $H$ into the normal form*

$$\begin{aligned} H_*(y_*, \psi_*, z_*, \bar z_*; a) &:= H \circ \Phi = \mathcal{N}_*(y_*, z_*, \bar z_*; a) + R_*(y_*, \psi_*, z_*, \bar z_*), \\ \mathcal{N}_* &:= e_* + \omega(a) \cdot y_* + \Omega_*(a) z_* \bar z_*, \quad R_* := \sum_{2|k|+|a+\bar a| \ge 3} R^*_{ka\bar a}(\psi_*) y_*^k z_*^a \bar z_*^{\bar a}. \end{aligned} \tag{48}$$

*Moreover*:

(*i*) *the frequency vector $(\omega, \Omega_*) = (\omega, \Omega_*)(a)$ meets the "second order Melnikov non-resonance condition"*

$$\begin{aligned} \Big|\omega(a) \cdot \ell + \Omega_*(a) \cdot h\Big| &\ge \frac{\text{const }\varepsilon}{(1 + |\ell|^\tau)}, \\ \forall\, \ell \in \mathbb{Z}^n, \quad \forall\, h \in \mathbb{Z}^n,\ |h| \le 2, &\quad (\ell, h) \ne (0,0), \qquad \forall\, a \in \mathcal{A}^\sharp. \end{aligned} \tag{49}$$

(*ii*) *$\Phi$ has the form*

$$\begin{aligned} y &= y_* + Y(y_*, \psi_*, z_*, \bar z_*) \\ \psi &= \psi_* + X(\psi_*) \\ z &= z_* + Z(\psi_*, z_*, \bar z_*) \\ \bar z &= \bar z_* + \bar Z(\psi_*, z_*, \bar z_*) \end{aligned} \tag{50}$$

*where*

$$Y := \sum_{2|k|+|a+\bar a| \le 2} Y_{ka\bar a}(\psi_*) y_*^k z_*^a \bar z_*^{\bar a}, \qquad Z := \sum_{|a+\bar a| \le 1} Z_{a\bar a}(\psi_*) z_*^a \bar z_*^{\bar a}, \tag{51}$$

*and, denoting by $\|\cdot\|_* := \sup_{\mathcal{D}_{\bar r/2, \bar s/2}} |\cdot|$,*

$$\|Y\|_*, \quad \frac{\bar r^2}{\bar s}\|X\|_*, \quad \bar r\|Z\|_*, \quad \bar r\|\bar Z\|_* \le \text{ const } \varepsilon^2. \tag{52}$$

(*iii*) *the elliptic frequencies in the normal form satisfy*

$$\left|\Omega_*(a) - \Omega(a)\right| \le \text{const } \varepsilon^{3/2}. \tag{53}$$

*In particular,* $\forall\, a \in \mathcal{A}^\sharp$,

$$\mathcal{T} := \{y_* = 0\} \times \mathbb{T}^N \times \{z_* = \bar{z}_* = 0\}$$

*is an* $N$*-dimensional elliptic invariant torus foliated by the quasi-periodic (Diophantine) flows* $t \to \psi_* + \omega t$ *with frequency* $\omega = \omega(a)$.

*Remark 4.5* The proof of Theorem 4.4 goes exactly as in Biasco et al. (2006): the main result is achieved using the KAM Theorem of Pöschel (1989).

Actually, for our purpose it is more convenient to refer to the more detailed version of the KAM Theorem of Pöschel (1989) stated in Theorem 5.1 of Berti et al. (2004); here the form of the KAM transformation (Eqs. 50, 51) is provided. The (indispensable) estimates (49) and (52) are obtained respectively from equations (119) and (123) of Berti et al. (2004), choosing the involved parameters as in (45)[13] and $\alpha =$ const $\varepsilon$. See again (Coglitore 2007) for a detailed display of the proof.

Anyway, in Berti et al. (2004) no estimate on the final elliptic frequencies $\Omega_*$ is provided: the low number (m = 2) of elliptic directions in the three-body problem allows to apply a result analogous to Theorem 2.1 without need to check the non-resonance condition (13) (involving $\Omega_*$). Instead, since we are interested in the $N$-body problem (with $N \ge 3$), for which the aforesaid original theorem of Berti et al. (2004) is no longer applicable, according to Theorem 2.1 we do have to exclude resonances (up to a certain order) among the linear and the elliptic frequencies.

To obtain (53) we have used the results contained in the paragraph called **Estimates to Theorem A** on p. 37 of Pöschel (1989) (see also Eqs. 2.90, 2.101) of Coglitore (2007)).

## 5 Abundance of periodic solutions in the planetary planar $(N+1)$-body problem

In this section we prove that, for $\varepsilon$ and $\delta$ small enough and $a$ varying in a suitable subset of $A$ (that will of course depend on $\varepsilon$ and $\delta$), the non-degeneracy "twist" condition and the non-resonance condition (13) of Theorem 2.1 are fulfilled for the normal form Hamiltonian of the planetary planar $(N+1)$-body problem we have dealt with in Sect. 4. We could then apply Theorem 2.1 in such setting proving our final result, see Theorem 5.5.

First of all we study the matrices $\mathcal{R}$ and $\mathcal{Q}$.

### 5.1 Checking non-degeneracy condition

We check here the invertibility of the "twist" matrix $\mathcal{R}$; at the same time, we control the size[14] of $|\mathcal{Q}|$ and $|\mathcal{R}^{-1}|$.

[13] Note that the particular choice of the "radius" $\bar{r}$ we have made in (45), namely $\bar{r} =$ const $\varepsilon^{3/4}$, is different from the analogous one in Berti et al. (2004), that is $\bar{r} =$ const $\varepsilon^{1/2}$; this is tantamount to further narrowing the domain of the Hamiltonian $H$. The reason why we did so is that by our setting we have managed to get the estimate in (53). Instead, $\bar{r} \sim \varepsilon^{1/2}$, would give rise in (53) to a term $O(\varepsilon)$, that would be completely unuseless since, from (36), $\Omega \sim \varepsilon$.

[14] The reason for doing this is that the matrices $\mathcal{R}$ and $\mathcal{Q}$ are involved in the definition of the constant $L$ in (12). As it will be clear in the sequel (see also Sect. 5.3), we need $L$ to be uniformly bounded in $\varepsilon$.

**Proposition 5.1** *For $\varepsilon < \varepsilon^\sharp$ small enough, if $\mathcal{R}$ and $\mathcal{Q}$ are the matrices defined in* (9) *and* (10) *associated to the planetary planar $(N+1)$-body problem Hamiltonian $H_*$ in* (48)*, then*

$$\det \mathcal{R} \neq 0, \qquad |\mathcal{R}^{-1}| = O(1), \qquad |\mathcal{Q}| = O(\varepsilon^{1/2}). \tag{54}$$

*Proof* (*Sketch:* The definitions of matrices $\mathcal{R}$ and $\mathcal{Q}$ involve some Taylor-Fourier coefficients $R^*_{k,a,\bar{a},\ell}$ of the function $R_*$ appearing into the normal form (48). To estimate them, we first obtain each $R^*_{k,a,\bar{a}}(\varphi_*)$ as a suitable partial derivative of the function $H_*$ evaluated in $(0, \psi_*, 0, 0)$; we could then write $H_* = H \circ \Phi$, as in Theorem 4.4, in order to estimate the derivatives thanks to the special form of the canonical transformation $\Phi$ (while we know $H$ explicitly). Finally, making use of classical estimate on the Fourier coefficients of analytic functions, we prove (54)).

From the normal form (48), by Taylor's formula, it follows that:

$$\begin{aligned}
R^*_{e_i+e_{i'},0,0}(\psi_*) &= \frac{1}{1+\delta_{(i,i')}}\,\frac{\partial^2 H_*}{\partial y_{*i}\partial y_{*i'}}\,(0,\psi_*,0,0),\\
R^*_{e_i,e_j,0}(\psi_*) &= \frac{\partial^2 H_*}{\partial y_{*i}\partial z_{*j}}\,(0,\psi_*,0,0),\\
R^*_{e_i,0,e_j}(\psi_*) &= \frac{\partial^2 H_*}{\partial y_{*i}\partial \bar{z}_{*j}}\,(0,\psi_*,0,0),\\
R^*_{e_i,e_j,e_j}(\psi_*) &= \frac{\partial^3 H_*}{\partial y_{*i}\partial z_{*j}\partial \bar{z}_{*j}}\,(0,\psi_*,0,0),\\
R^*_{0,e_j,e_j+e_{j'}}(\psi_*) &= \frac{1}{1+\delta_{(j,j')}}\,\frac{\partial^3 H_*}{\partial z_{*i}\partial \bar{z}_{*j}\partial \bar{z}_{*j'}}\,(0,\psi_*,0,0),\\
R^*_{0,e_j+e_{j'},e_j}(\psi_*) &= \frac{1}{1+\delta_{(j,j')}}\,\frac{\partial^3 H_*}{\partial z_{*j}\partial z_{*j'}\partial \bar{z}_{*j}}\,(0,\psi_*,0,0).
\end{aligned} \tag{55}$$

So, defining

$$x := (x_1, \ldots, x_{3N}) := (y_1, \ldots, y_N, z_1, \ldots, z_N, \bar{z}_1, \ldots, \bar{z}_N)\,,$$
$$x_* := (x_{*1}, \ldots, x_{*3N}) := (y_{*1}, \ldots, y_{*n}, z_{*1}, \ldots, z_{*N}, \bar{z}_{*1}, \ldots, \bar{z}_{*n})\,,$$

we are interested in partial derivatives of the form[15]

$$\left.\frac{\partial^\alpha H_*(x_*, \psi_*)}{\partial x_*}\right|_{(x_*,\psi_*)=(\vec{0},\psi_*)}$$

where $\alpha \in \mathbb{Z}^{3N}$ is a multi-index with norm $|\alpha|_1 = 2, 3$.

We proceed by writing $H_*(x_*, \psi_*) = H \circ \Phi(x_*, \psi_*)$, where $\Phi(x_*, \psi_*)$ is the symplectic transformation given by Theorem 4.4 and $H$ has been defined in (42).

[15] Here and in the next few formulas there is a slight "misuse of notation": we will usually write $(x, \psi)$ and $(x_*, \psi_*)$ in place of $(y, \psi, z, \bar{z})$ and $(y_*, \psi_*, z_*, \bar{z}_*)$.

Using the "chain rule" for the derivatives of composite functions, we have

$$\frac{\partial H_*}{\partial x_{*i}}(\vec{0},\psi_*) = \sum_{1\le a\le 3N} \frac{\partial H}{\partial x_a}\frac{\partial \Phi_a}{\partial x_{*i}},$$

$$\frac{\partial^2 H_*}{\partial x_{*i}\partial x_{*j}}(\vec{0},\psi_*) = \sum_{1\le a,b\le 3N} \frac{\partial^2 H}{\partial x_a \partial x_b}\frac{\partial \Phi_a}{\partial x_{*i}}\frac{\partial \Phi_b}{\partial x_{*j}} + \sum_{1\le a\le 3N}\frac{\partial H}{\partial x_a}\frac{\partial^2 \Phi_a}{\partial x_{*i}\partial x_{*j}},$$

$$\begin{aligned}\frac{\partial^3 H_*}{\partial x_{*i}\partial x_{*j}\partial x_{*l}}(\vec{0},\psi_*) = &\sum_{1\le a,b,c\le 3N} \frac{\partial^3 H}{\partial x_a \partial x_b \partial x_c}\frac{\partial \Phi_a}{\partial x_{*i}}\frac{\partial \Phi_b}{\partial x_{*j}}\frac{\partial \Phi_c}{\partial x_{*l}}\\ &+ \sum_{1\le a,b\le 3N} \frac{\partial^2 H}{\partial x_a \partial x_b}\frac{\partial^2 \Phi_a}{\partial x_{*i}\partial x_{*j}}\frac{\partial \Phi_b}{\partial x_{*l}}\\ &+ \sum_{1\le a,b\le 3N} \frac{\partial^2 H}{\partial x_a \partial x_b}\frac{\partial^2 \Phi_a}{\partial x_{*i}\partial x_{*l}}\frac{\partial \Phi_b}{\partial x_{*j}}\\ &+ \sum_{1\le a,b\le 3N} \frac{\partial^2 H}{\partial x_a \partial x_b}\frac{\partial^2 \Phi_a}{\partial x_{*j}\partial x_{*l}}\frac{\partial \Phi_b}{\partial x_{*i}}\\ &+ \sum_{1\le a\le 3N} \frac{\partial H}{\partial x_a}\frac{\partial^3 \Phi_a}{\partial x_{*i}\partial x_{*j}\partial x_{*l}},\end{aligned} \tag{56}$$

where the derivatives of the functions $H$ and $\Phi$ are understood to be evaluated respectively at $(x,\psi) = \Phi(\vec{0},\psi_*)$ and at $(x_*,\psi_*) = (\vec{0},\psi_*)$ (and so will be in the sequel). We remark that in the previous formulas, $\psi_*$ has been considered as a fixed parameter due to the particular form of the symplectic transformation $\Phi$; in fact (by (50)) $\psi = \psi_* + X(\psi_*)$ is independent on $x_*$.

From (51) and (52) there follows:

$$\begin{aligned}|Y| &\le \text{const}\,\frac{\mathcal{P}}{\alpha} = \text{const}\,\varepsilon^2,\\ \left|\frac{\partial Y}{\partial(z_*,\bar{z}_*)}\right| &\le \text{const}\,\frac{\mathcal{P}}{\alpha\bar{r}} = \text{const}\,\varepsilon^{5/4},\\ \left|\frac{\partial Y}{\partial y_*}\right|,\ \left|\frac{\partial(Z,\overline{Z})}{\partial(z_*,\bar{z}_*)}\right|,\ \left|\frac{\partial^2 Y}{\partial^2(z_*,\bar{z}_*)}\right| &\le \text{const}\,\frac{\mathcal{P}}{\alpha\bar{r}^2} = \text{const}\,\varepsilon^{1/2},\end{aligned} \tag{57}$$

having made use of standard[16] "Cauchy estimates" on the domain $D^2_{\frac{r}{2}} \times \mathbb{T}^2_{\frac{s}{2}} \times D^4_{\bar{r}}$, with $\bar{r} = \text{const}\,\varepsilon^{3/4}$, as in (45). The first and second order partial derivatives of $Y, Z, \overline{Z}$ not appearing in (57), as well as the third order derivatives, are null.

From (42), that allows us to know $H$ apart from $O(\varepsilon)$ terms, we get:

$$\left.\frac{\partial H}{\partial x_a}\right|_{\Phi(\vec{0},\psi_*)} = \begin{cases} \dfrac{\sigma_i}{(J_{0i})^3} + O(\varepsilon) & \text{if } a = i \le N\\ O(\varepsilon) & \text{otherwise}\end{cases}$$

[16] Cauchy estimates allow to bound $n$-derivatives of analytic functions on a set $A$ in terms of their sup-norm on larger domains $A \subset A'$ divided by $\text{dist}(\partial A, \partial A')^n$.

$$\frac{\partial^2 H}{\partial x_a \partial x_b}\bigg|_{\Phi(\vec{0},\psi_*)} = \begin{cases} \dfrac{-3\sigma_i}{(J_{0i})^4} + O(\varepsilon) & \text{if } a = b = i \le N \\ O(\varepsilon) & \text{otherwise} \end{cases} \tag{58}$$

$$\frac{\partial^3 H}{\partial x_a \partial x_b \partial x_c}\bigg|_{\Phi(\vec{0},\psi_*)} = \begin{cases} \dfrac{12\sigma_i}{(J_{0i})^5} + O(\varepsilon) & \text{if } a = b = c = i \le N \\ O(\varepsilon) & \text{otherwise} \end{cases}$$

(Note that we have used the first of (57) to neglect the dependence on $y_i(\vec{0}, \psi_*) = Y_i(\vec{0}, \psi_*) = O(\varepsilon^2)$ where necessary).

On the other side, combining (50) with the estimates in (57) we obtain:

$$\frac{\partial \Phi_a}{\partial x_{*i}}\bigg|_{(\vec{0},\psi_*)} = \begin{cases} 1 + O(\varepsilon^{1/2}) & \text{if } a = i \\ O(\varepsilon^{5/4}) & \text{if } a \le N,\ i > N \\ 0 & \text{if } a > N,\ i \le N \\ O(\varepsilon^{1/2}) & \text{otherwise} \end{cases}$$

$$\frac{\partial^2 \Phi_a}{\partial x_{*i} \partial x_{*j}}\bigg|_{(\vec{0},\psi_*)} = \begin{cases} O(\varepsilon^{1/2}) & \text{if } a \le N,\ i > N \\ 0 & \text{otherwise} \end{cases} \tag{59}$$

$$\frac{\partial^3 \Phi_a}{\partial x_{*i} \partial x_{*j} \partial x_{*l}}\bigg|_{(\vec{0},\psi_*)} = 0.$$

Now, recalling the expression in (55) and (56), from the estimates in (58) and (59), it is straightforward (even if lengthy) to prove that:

- $R^*_{e_i+e_{i'},0,0}(\psi_*) = \delta_{(i,i')} \dfrac{-3\sigma_i}{2(J_{0i})^4} + O(\varepsilon^{1/2})$,
- $R^*_{e_i,e_j,0}(\psi_*),\ R^*_{e_i,0,e_j}(\psi_*) = O(\varepsilon)$ (the leading term in the second of (56) is in the first summation for $a = i,\ b = N + j$ and $a = i,\ b = 2N + j$ respectively),
- $R^*_{0,e_j,e_j+e_{j'}}(\psi_*),\ R^*_{0,e_j+e_{j'},e_j}(\psi_*) = O(\varepsilon)$ (the leading term in the third of (56) is in the first summation for $a = N + j,\ b = 2N + j,\ c = 2N + j'$ and $a = N + j,\ b = N + j',\ c = 2N + j$ respectively),
- $R^*_{e_i,e_j,e_j}(\psi_*) = O(\varepsilon^{1/2})$ (the leading term in the third of (56) is in the fourth summation for $a = b = i$).

Evaluating the Fourier coefficients[17] of the above functions, which, by Theorem 4.4, are analytic on $\mathbb{T}^n_{\bar{s}/2}$ (with $\bar{s} =$ const defined in (45)), we have:

$$R^*_{e_i+e_{i'},0,0,0} = \delta_{(i,i')} \frac{-3\sigma_i}{2(J_{0i})^4} + O(\varepsilon^{1/2}), \qquad R^*_{e_i,e_j,e_j,0} = O(\varepsilon^{1/2}),$$
$$R^*_{e_i,e_j,0,\ell},\ R^*_{e_i,0,e_j,\ell},\ R^*_{0,e_j,e_j+e_{j'},\ell},\ R^*_{0,e_j+e_{j'},e_j,\ell} = O(\varepsilon)\, e^{-|\ell|\bar{s}/2}. \tag{60}$$

[17] The classical estimate $|f_\ell| \le$ const $e^{-|\ell|s}$ for the $\ell$-Fourier coefficient of a function analytic on $\mathbb{T}^n_s$ holds.

Moreover, the second order Melnikov condition (49), together with (60), implies

$$\sum_{1\le j\le m}\sum_{\ell\in\mathbb{Z}^n}\frac{1}{\omega\cdot\ell+\Omega_{*j}}\left(R^*_{e_i,e_j,0,\ell}\,R^*_{e_{i'},0,e_j,-\ell}+R^*_{e_i,0,e_j,-\ell}\,R^*_{e_{i'},e_j,0,\ell}\right),$$

$$\sum_{1\le i'\le n}\sum_{\ell\in\mathbb{Z}^n}\frac{\ell_{i'}}{\omega\cdot\ell+\Omega_{*j}}\left(R^*_{e_i,e_j,0,\ell}\,R^*_{e_{i'},0,e_j,-\ell}+R^*_{e_i,0,e_j,-\ell}\,R^*_{e_{i'},e_j,0,\ell}\right),$$

$$\sum_{1\le j'\le m}\sum_{\ell\in\mathbb{Z}^n}\frac{1}{\omega\cdot\ell+\Omega_{*j'}}\left(R^*_{0,e_j,e_j+e_{j'},-\ell}\,R^*_{e_i,e_{j'},0,\ell}+R^*_{0,e_j+e_{j'},e_j,\ell}\,R^*_{e_i,0,e_{j'},-\ell}\right)$$

$$\le\ \sum_{\ell\in\mathbb{Z}^n}\frac{1+|\ell|^\tau}{\varepsilon}\,e^{-|\ell|\bar{s}}\,O(\varepsilon^2)\ =\ O(\varepsilon),$$

and hence the summations in (9) and (10) are negligible with respect to the first addenda.

Hence, we have obtained that the entries of the matrix $\mathcal{R}$ and $\mathcal{Q}$ corresponding to the $(N+1)$-body problem Hamiltonian $H_*$ are

$$\mathcal{R}_{ii}=\frac{-3\sigma_i}{2(J_{0i})^4}+O(\varepsilon^{1/2}),\quad \mathcal{R}_{ii'}=O(\varepsilon^{1/2})\ \text{ for } i\ne i',\quad \mathcal{Q}_{ji}=O(\varepsilon^{1/2}),$$

and the proposition follows taking $\varepsilon<\varepsilon^\sharp$ small enough. □

## 5.2 Checking non-resonance condition

In this section, that is the very core of the present paper, we show that, if the perturbative parameters $\delta$ and $\varepsilon$ are small enough, the non-resonance condition (13) of Theorem 2.1 holds for any vector of semi-axes $a$ belonging to a suitable subset of $\widetilde{A}(\delta,\varepsilon)\subset\mathcal{A}^\sharp(\varepsilon)$ (defined in Theorem 4.4) whose measure tends to be full as $\delta,\varepsilon\to 0^+$. This will be done in Proposition 5.4.

First of all we prove Proposition 4.3, that has been stated before (see Sect. 4):

*Proof of Proposition 4.3* Writing $K=(K_1,\dots,K_N)$, we distinguish two cases: $(K_1,K_2)\ne(0,0)$ or $(K_1,K_2)=(0,0)$. Suppose first $(K_1,K_2)\ne(0,0)$, we need the following result:[18]

**Lemma 5.2** *The function*

$$g^{(1)}:\ a\in\{0<a_i<a_{i+1}\}\ \longrightarrow\ \left(\bar{\Omega}_1^0(a),\bar{\Omega}_2^0(a)\right)\in\mathbb{R}^2$$

*is non-degenerate.*[19]

Applying[20] Proposition 3.6 to the function $g^{(1)}$ with $\gamma:=2\delta^{1/4}$, we find:

$$\left|g^{(1)}(a)\cdot(K_1,K_2)\right|\ =\ \left|\left(\bar{\Omega}_1^0(a),\bar{\Omega}_2^0(a)\right)\cdot(K_1,K_2)\right|\ \ge\ 2\delta^{1/4},$$
$$\forall\,(K_1,K_2)\in\mathbb{Z}^2_M,\qquad \forall\,a\in A_1:=A_1(\delta):=A\setminus D^M_{g^{(1)}}\left(2\delta^{1/4}\right),\tag{61}$$

with $D^M_{g^{(1)}}\left(2\delta^{1/4}\right)$ defined as in (27). Furthermore, from (28),

$$\text{meas}\,(A\setminus A_1)=O(\delta^{\alpha_1})\quad\text{for some }\alpha_1>0.\tag{62}$$

[18] In order non to interrupt the proof of the theorem, we will prove Lemma 5.2 later.

[19] In the sense of Definition 3.7. The same holds for the next Lemma.

[20] See Remark 3.8.

By (61), recalling the asymptotics in (36), we get

$$|\bar{\Omega}(a)\cdot K| \ \geq\ \varepsilon|g^{(1)}(a)\cdot(K_1,K_2)| - \varepsilon M O(\sqrt{\delta},\varepsilon)\ \geq\ 2\varepsilon\delta^{1/4} - O(\varepsilon\sqrt{\delta},\varepsilon^2)$$
$$\geq\ \varepsilon\delta^{1/4}\quad \forall\, K\in\mathbb{Z}^N_M\cap\big\{(K_1,K_2)\neq(0,0)\big\},\quad \forall\, a\in A_1, \tag{63}$$

when $\delta\leq\delta_1$ and $\varepsilon\leq\varepsilon_1(\delta)$, for some suitable (small) $\delta_1\leq\delta_0$, and $\varepsilon_1(\delta)\leq\varepsilon_0$.

Let us suppose now $(K_1,K_2)=(0,0)$, we need the following result:

**Lemma 5.3** *The function*

$$g^{(2)}\,:\ a\in\{0<a_i<a_{i+1}\}\ \longrightarrow\ \big(\bar{\Omega}^0_3(a),\ldots,\bar{\Omega}^0_N(a)\big)\in\mathbb{R}^{N-2}$$

*is non-degenerate.*

Applying Proposition 3.6 to the function $g^{(2)}$ with $\gamma:=2\delta^{1/4}$, we get

$$\Big|g^{(2)}(a)\cdot(K_3,\ldots,K_N)\Big| = \big|\big(\bar{\Omega}^0_3(a),\ldots,\bar{\Omega}^0_N(a)\big)\cdot(K_3,\ldots,K_N)\big|\geq 2\delta^{1/4},$$
$$\forall\,(K_3,\ldots,K_N)\in\mathbb{Z}^{N-2}_M,\quad \forall\, a\in A_2:=A_2(\delta):=A\setminus D^M_{g^{(2)}}\big(2\delta^{1/4}\big), \tag{64}$$

where, $D^M_{g^{(2)}}\big(\delta^{1/4}\big)$ is defined in (27). As above, (28) yields

$$\text{meas}\,(A\setminus A_2)=O(\delta^{\alpha_2})\quad\text{for some }\alpha_2>0. \tag{65}$$

By (64) and (36), we obtain

$$|\bar{\Omega}(a)\cdot K|\ \geq\ \varepsilon\sqrt{\delta}|g^{(2)}(a)\cdot(K_3,\ldots,K_N)| - \varepsilon M O(\delta,\varepsilon)\geq\ 2\varepsilon\delta^{3/4} - O(\varepsilon\delta,\varepsilon^2)$$
$$\geq\varepsilon\delta^{3/4}\quad\forall\, K\in\mathbb{Z}^N_M\cap\big\{(K_1,K_2)=(0,0)\big\},\quad\forall\, a\in A_2, \tag{66}$$

when $\delta\leq\delta_2$ and $\varepsilon\leq\varepsilon_2(\delta)$, for some suitable (small) $\delta_2\leq\delta_0$, and $\varepsilon_2(\delta)\leq\varepsilon_0$.

Taking

$$A^\star:=A_1\cap A_2,\qquad \delta^\star:=\min\,(\delta_1,\delta_2),\quad \varepsilon^\star:=\min\,(\varepsilon_1,\varepsilon_2),\quad \alpha^\star:=\min\,(\alpha_1,\alpha_2),$$

we conclude by (63) and (66). □

*Proof of Lemma 5.2* According to Definition 3.7, we have to check that, for any $c=(c_1,c_2)\in\mathbb{R}^2$, $c\neq\vec{0}$,

$$c\cdot g^{(1)}\ =\ c_1\,\bar{\Omega}^0_1\ +\ c_2\,\bar{\Omega}^0_2\ \not\equiv\ 0.$$

Suppose, by contradiction, that $\bar{c}_1\,\bar{\Omega}^0_1+\bar{c}_2\,\bar{\Omega}^0_2\equiv 0$ for some $(\bar{c}_1,\bar{c}_2)\neq\vec{0}$. Then, if e.g. $\bar{c}_1\neq 0$ (otherwise it would be $\bar{c}_2\neq 0$, and we could go on in a completely analogous way),

$$\frac{\bar{\Omega}^0_1}{\bar{\Omega}^0_2}\ \equiv\ -\frac{\bar{c}_2}{\bar{c}_1}\ =:\ C^{(1)}. \tag{67}$$

Now, recalling the expressions for $\bar{\Omega}^0_1(a)$ and $\bar{\Omega}^0_2(a)$ in (37), we see that

$$\frac{\bar{\Omega}^0_1(a)}{\bar{\Omega}^0_2(a)}\ =\ \frac{1-\frac{5}{16}\frac{a_1}{a_2}+O\left(\left(\frac{a_1}{a_2}\right)^2\right)}{1+\frac{5}{16}\frac{a_1}{a_2}+O\left(\left(\frac{a_1}{a_2}\right)^2\right)}. \tag{68}$$

Since the right hand member of (68) is evidently[21] a non-constant function over the domain $\{0 < a_i < a_{i+1}\}$, it is impossible for (67) to be identically verified. So, the function $g^{(1)} : a \in \{0 < a_i < a_{i+1}\} \longrightarrow \big(\bar\Omega_1^0(a), \bar\Omega_2^0(a)\big)$ must be non-degenerate. □

*Proof of Lemma 5.3* By (3.8) we have to prove that, for any $c = (c_1, \ldots, c_{N-2}) \in \mathbb{R}^{N-2}$, $c \neq \vec{0}$, the analytic function

$$f_c := c \cdot g^{(2)} = c_1 \bar\Omega_3^0 + \cdots + c_{N-2} \bar\Omega_N^0$$

is not constantly vanishing.

To this end we define the parametrized curve $\gamma : (0, 1] \longrightarrow \mathbb{R}^N$,

$$\gamma(t) := (t^2, t, t^{\alpha_1}, \ldots, t^{\alpha_{N-2}}), \tag{69}$$

with

$$\frac{7}{13} < \alpha_i := \frac{7}{13} + i\,\frac{1}{13N} < 1, \qquad \forall\, i = 1, \ldots, N-2. \tag{70}$$

Moreover, let's consider the map $G(t) : (0, 1] \longrightarrow \mathbb{R}^{N-2}$ obtained by restricting the function $g^{(2)}$ to the curve $\gamma$, i.e.

$$G(t) := \left(g^{(2)} \circ \gamma\right)(t) = \left(\bar\Omega_3^0(\gamma(t)), \ldots, \bar\Omega_N^0(\gamma(t))\right).$$

If, by contradiction, there exists $\bar c \in \mathbb{R}^{N-2} \setminus \{\vec 0\}$ such that $f_{\bar c} \equiv 0$, then it must rightly be

$$F_{\bar c}(t) := \bar c \cdot G(t) = \sum_{1 \le i \le N-2} \bar c_i \cdot \bar\Omega_{i+2}^0(\gamma(t)) = 0, \quad \forall t \in (0, 1]. \tag{71}$$

By the asymptotics for $\bar\Omega_j^0(a)$, $3 \le j \le N$, found in (37), and the definition of $\gamma$ in (69), we have

$$\bar\Omega_{i+2}^0(\gamma(t)) = \text{const}\; t^{-\beta_i}\left[1 + O\left(t^{\nu_i}\right)\right], \quad \forall\, i = 1, \ldots, N-2, \tag{72}$$

where, for $i = 1, \ldots, N-2$, we have set

$$\begin{aligned} \beta_i &:= \frac{13}{4}\alpha_i - \frac{7}{4} = \frac{13}{4}\left(\frac{7}{13} + i\,\frac{1}{13\,N}\right) - \frac{7}{4} = i\,\frac{1}{4\,N} > 0, \\ 0 < \nu_i &:= 2(1 - \alpha_i) = 2\left(1 - \left(\frac{7}{13} + i\,\frac{1}{13\,N}\right)\right) < \frac{12}{13} < \frac{7}{4}. \end{aligned} \tag{73}$$

By means of (72), (71) becomes

$$F_{\bar c}(t) = \text{const} \sum_{1 \le i \le N-2} \bar c_i \cdot t^{-\beta_i}\left[1 + O\left(t^{\nu_i}\right)\right] = 0, \quad \forall t \in (0, 1]. \tag{74}$$

Now, since, from (73), $0 < \beta_i < \beta_{i'}$ if $i < i'$, it must necessarily be $\bar c_i = 0$, for any $i = 1, \ldots, N-2$. Indeed, let otherwise

$$i_0 := \min\left\{1 \le i \le N-2 \,:\, \bar c_i \neq 0\right\}.$$

[21] For instance, $\bar\Omega_1^0(a)/\,\bar\Omega_2^0(a) \longrightarrow 1^-$ as $a_1/a_2 \to 0^+$.

Then, from (74) we would have

$$\sum_{1\le i\le N-2} \bar{c}_i \cdot t^{-\beta_i}\left[1+O\left(t^{\nu_i}\right)\right] = \sum_{i_0\le i\le N-2} \bar{c}_i \cdot t^{-\beta_i}\left[1+O\left(t^{\nu_i}\right)\right]$$

$$= t^{-\beta_{i_0}} \sum_{i_0\le i\le N-2} \bar{c}_i \cdot t^{-(\beta_i-\beta_{i_0})}\left[1+O\left(t^{\nu_i}\right)\right] = 0, \quad \forall t\in(0,1],$$

and we would come to a contradiction, since the last expression tends towards $+\infty$ as $t\to 0^+$. Hence, (71) it is not possible, i.e. $g^{(2)}$ is of course a Rüßmann non-degenerate analytic function. □

Now, in light of the previous results, it is finally straightforward to prove the non-resonance for the frequency vector for the $(N+1)$-body problem given by Theorem 4.4.

**Proposition 5.4** *Fix $M\in\mathbb{N}$, and let $(\omega,\Omega_*)=(\omega(a),\Omega_*(a))$ be the frequency vector of the normal form (48) for the planetary planar $(N+1)$-body problem Hamiltonian $H_*$ found in Theorem 4.4. For every $0<\delta\le\delta^\star$, and $0<\varepsilon\le\tilde{\varepsilon}(\delta)$, for a suitable function $\tilde{\varepsilon}(\delta)\le\varepsilon^\sharp$ (with $\delta^\star$ defined in Proposition 4.3 and $\varepsilon^\sharp$ as in Theorem 4.4), there exists a subset of semi-axes $\widetilde{\mathcal{A}}=\widetilde{\mathcal{A}}(\delta,\varepsilon)\subset\mathcal{A}^\sharp\subset A$ with*

$$\text{meas}\left(A\setminus\widetilde{\mathcal{A}}\right)=O(\delta^{\text{const}}), \tag{75}$$

$$(\omega(a),\Omega_*(a))\cdot\vec{k}\ne 0\,,\quad \forall\vec{k}\in\mathbb{Z}^{2N}_M\,,\quad \forall a\in\widetilde{\mathcal{A}}\,. \tag{76}$$

*Proof* Let $\vec{k}=(k,K)\in\mathbb{Z}^N\times\mathbb{Z}^N$. We have two cases: $k\ne 0$ or $k=0$.

Let us suppose first that $k\ne 0$, we note that the function

$$g^{(3)}\ :\ \{0<a_i<a_{i+1}\}\longrightarrow\mathbb{R}^N,\quad \text{with}\quad g^{(3)}_i(a):=a_i^{-3/2},\quad i=1,\dots,N$$

is (trivially) non-degenerate. Applying[22] Proposition 3.6 to $g^{(3)}$ with $\gamma:=\delta$, we find:

$$\left|g^{(3)}(a)\cdot k\right|\ge\delta,\quad \forall\,k\in\mathbb{Z}^N_M\,,\quad \forall\,a\in A_3=A_3(\delta):=A\setminus D^M_{g^{(3)}}(\delta)\,, \tag{77}$$

with $D^M_{g^{(3)}}(\delta)$ defined as in (27). Furthermore, from (28),

$$\text{meas}\,(A\setminus A_3)=O(\delta^{\alpha_3})\quad \text{for some constant } \alpha_3>0\,. \tag{78}$$

By (30), (31), (41) and (43), since $\sigma_i\overline{\mathrm{m}}_i^{-3}=\sqrt{\mathrm{M}_i}=1+O(\varepsilon)$, we have that $\omega=g^{(3)}+O(\varepsilon)$ and, therefore, $\omega_i(a)=\sigma_i\overline{\mathrm{m}}_i^{-3}a_i^{-3/2}+O(\varepsilon)=a_i^{-3/2}+O(\varepsilon)$. Being $\Omega_*=O(\varepsilon)$, for $\varepsilon$ small enough with respect to $\delta$, let's say $0<\varepsilon\le\varepsilon_3(\delta)\le\varepsilon^\sharp(\delta)$, we get

$$|(\omega(a),\Omega_*(a))\cdot(k,K)|\ge|g^{(3)}(a)\cdot k|-M\,O(\varepsilon)\ge\delta-O(\varepsilon)\ge\frac{1}{2}\delta>0,$$
$$\forall\,(k,K)\in\mathbb{Z}^{2N}_M\cap\{k\ne 0\},\ \forall\,a\in A_3\cap\mathcal{A}^\sharp. \tag{79}$$

Now, suppose $k=0$ (hence $K\ne 0$), by (41), (43) and (53), $\Omega_*=\bar{\Omega}+O(\varepsilon^{1+\text{const}})$. Then, by Proposition 4.3, if $0<\varepsilon\le\bar{\varepsilon}(\delta)$ for a suitable positive $\bar{\varepsilon}(\delta)\le\varepsilon^\sharp(\delta)$, we get

$$\begin{aligned}|(\omega(a),\Omega_*(a))\cdot\vec{k}|&=|\Omega_*(a)\cdot K|\ \ge\ |\bar{\Omega}\cdot K|-O(\varepsilon^{1+\text{const}})\\ &\ge\ \varepsilon\delta^{3/4}-O(\varepsilon^{1+\text{const}})\ge\frac{1}{2}\varepsilon\delta^{3/4}\ >\ 0,\\ &\quad\forall\,(k,K)\in\mathbb{Z}^{2N}_M\cap\{k=0\},\quad \forall\,a\in A^\star.\end{aligned}$$

[22] See Remark 3.8.

We conclude setting $\tilde{\varepsilon}(\delta) := \min\{\varepsilon_3, \bar{\varepsilon}\}$ and $\widetilde{\mathcal{A}} := A_3 \cap \mathcal{A}^\sharp \cap A^\star$.

The estimate (75) follows by (38), (47) and (78). □

### 5.3 Conclusions

Putting together the previous results, we can finally prove the existence of infinitely many periodic solutions of the planetary planar $(N + 1)$-body problem, as stated in the following recapitulatory theorem (for more quantitative information about the periodic orbits found, we refer to the thesis of Theorem 2.1).

**Theorem 5.5** *Consider a planetary planar* $(N+1)$*-body system* $(N \geq 3)$ *and let the masses of the planets satisfy* (1) *and* (3)*. For every compact set* $A$ *of osculating Keplerian major semi-axes, where* (2) *holds for a suitable universal constant* $\theta$ (*depending only on the given masses of the planets*)*, there exist a positive constant* $\delta^\star$ *and a positive function* $\tilde{\varepsilon}$ *such that, if* $0 < \delta \leq \delta^\star$ *and* $0 < \varepsilon \leq \tilde{\varepsilon}(\delta)$*, the system affords infinitely many periodic solutions, with minimal period increasing at infinity, clustering to the elliptic KAM tori found in Theorem* 4.4*, provided the osculating major semi-axes belong to a suitable subset* $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}(\varepsilon, \delta) \subset A$ *satisfying* $\mathrm{meas}\,\big(A \setminus \widetilde{\mathcal{A}}\big) = O(\delta^{\,\mathrm{const}})$.

*Proof* We have only to check that the Hamiltonian $H_*$ in (48) meets the hypotheses of Theorem 2.1.

The Melnikov condition (8) is satisfied by (49). Taking $\tilde{\varepsilon}(\delta)$ small enough, the twist condition $\det \mathcal{R} \neq 0$ holds by (54); moreover, we have that the constant $L$ defined in (12) is uniformly bounded in $\varepsilon$ (e.g. it is less than 1). Therefore, $M$ (defined in Theorem 2.1) can be regarded as a fixed quantity independent of $\delta$ and $\varepsilon$ (as we have done throughout the proof of Proposition 4.3) and condition (13) is satisfied by (76). Thus, we are in position to apply Theorem 2.1. The estimate on the measure of the set of "discarded semi-axes" $A \setminus \widetilde{\mathcal{A}}$ is the same as in (75). □

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