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ORIGINAL ARTICLE

Secular frequencies of 3-D exoplanetary systems

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Abstract Using a 12th order expansion of the perturbative potential in powers of the eccentricities and the inclinations, we study the secular effects of two non-coplanar planets which are not in mean–motion resonance. By means of Lie transformations (which introduce an action–angle formulation of the Hamiltonian), we find the four fundamental frequencies of the 3-D secular three-body problem and compute the long-term time evolutions of the Keplerian elements. To find the relations between these elements, the main combinations of the fundamental frequencies common to these evolutions are identified by frequency analysis. This study is performed for two different reference frames: a general one and the Laplace plane. We underline the known limitations of the linear Laplace–Lagrange theory and point out the great sensitivity of the 3-D secular three-body problem to its initial values. This analytical approach is applied to the exoplanetary system υ Andromedae in order to search whether the eccentricities evolutions and the apsidal configuration (libration of $\Delta\varpi$) observed in the coplanar case are maintained for increasing initial values of the mutual inclination of the two orbital planes.

Keywords Extrasolar planets · 3-D three-body problem · Secular motion · Analytical expansion · Laplace plane

1 Introduction

The discovery of exoplanetary systems has opened a new field of research in Celestial Mechanics. As the spatial resolution of the orbits are currently impossible, studies mainly analyze the dynamics of the coplanar case. Due to the large eccentricities of the discovered exoplanets, the Laplace–Lagrange theory is unable to depict correctly the motion of these

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planets. In a previous work (Libert and Henrard 2006), we introduced, for the planar problem, a 12th order expansion of the secular potential in powers of the eccentricities and showed that this analytical model can represent correctly the behavior of non-resonant planetary systems with surprisingly large eccentricities. More precisely, we introduced action–angle variables for the problem and by means of a Lie transformations perturbation technique obtained analytical expressions for the frequencies of the motion and for the secular evolution of the orbits.

The aim of the present contribution is to generalize this analytical approach to the non-coplanar case. The 12th order (in the eccentricities and the inclinations) expansion of the 3-D secular three-body problem has already been introduced in our previous paper (Libert and Henrard 2007b) where we study the dynamical features of the problem reduced to two degrees of freedom by the adoption of the Laplace plane. We were mainly interested by the position and stability of the equilibria, especially the fact that the stable equilibria related to the Kozai resonances are generated by bifurcation from a central equilibrium which becomes unstable for high mutual inclinations. Actually, the unstability of the central equilibrium and of the family of periodic orbits emanating from it is responsible for a rather large chaotic domain in the phase space.

The present work is based on the same 12th order expansion in the eccentricities and the inclinations, which is recalled in Sect. 2. Applying a Lie transform perturbation technique, we introduce, in Sect. 3, an action–angle formulation of the 3-D secular Hamiltonian and find an analytical expression for the four fundamental frequencies of the 3-D secular three-body problem. It also enables us to describe the long-term time evolution of the different Keplerian elements.

In Sect. 4, we apply this study to a particular 3-D system considered in two different reference frames, a general one and the Laplace plane, and compare the results. Among others, we identify, by frequency analysis, the main combinations of the fundamental frequencies influencing each long-term time evolution in order to find which variables are strongly related to each other. We also confirm our results by comparison with numerical integration.

Section 5 concerns the sensitivity of a 3-D system to its initial conditions. We explore the effects induced on the fundamental frequencies and on the behavior of the angles by a change of an angular initial condition.

In Sect. 6, we apply this analytical study to the exosystem υ Andromedae c-d and search whether the eccentricities evolutions and the aspidal configuration (libration of $\Delta\varpi$) observed in the coplanar case are maintained for increasing initial values of the mutual inclination of the two orbital planes. Finally, our results are summarized in Sect. 7.

2 Expansion of the Hamiltonian of the 3-D secular three-body problem

We consider a system consisting of a central star of mass m_0 and two planets of mass m_1 and m_2, with m_1 closest to the central star. The canonical variables we choose to work with are the classical modified Delaunay's elements (truncated at the first order in the mass ratios)

$$\begin{aligned} \lambda_i &= \text{mean longitude of } m_i & L_i &= m_i\sqrt{Gm_0 a_i} \\ p_i &= -\text{the longitude of the pericenter of } m_i & P_i &= L_i\left[1-\sqrt{1-{e_i}^2}\right] \\ q_i &= -\text{the longitude of the node of } m_i & Q_i &= L_i\sqrt{1-e_1^2}\,[1-\cos i_i]\,, \end{aligned} \tag{1}$$

where a_i, e_i and i_i are the semi-major axes, eccentricities and inclinations of the planets in a Jacobi reference frame (see, for instance, Brouwer and Clemence 1961 or Laskar 1990). The Hamiltonian which describes the dynamics of this system can be expanded in powers of the eccentricities and the inclinations (see, for instance, Murray and Dermott 1999). Actually we prefer to use the quantity $E_i = \sqrt{2P_i/L_i}$ close to e_i for small to moderate eccentricities and directly related to the canonical momenta. For the same reason we use $S_i = \sqrt{2Q_i/L_i}$ which is proportional to $\sin(i/2)$. With these notations, the Hamiltonian function reads:

$$\mathcal{H} = -\frac{Gm_0m_1}{2a_1} - \frac{Gm_0m_2}{2a_2} - \frac{Gm_1m_2}{a_2} \sum_{k,i_l,j_l,l\in\underline{4}} A_{i_l}^{k,j_l} E_1^{|j_1|+2i_1} E_2^{|j_2|+2i_2} S_1^{|j_3|+2i_3} S_2^{|j_4|+2i_4} \cos\Phi, \qquad (2)$$

with $\Phi = [(k + j_1 + j_3)\lambda_1 - (k + j_2 + j_4)\lambda_2 + j_1 p_1 - j_2 p_2 + j_3 q_1 - j_4 q_2]$. The indices $(k, i_l, l \in \underline{4})$ are positive integers and the sum $j_3 + j_4$ is even. The coefficients $A_{i_l}^{k,j_l}$ depend only on the ratio a_1/a_2 of the semi-major axes.

To the first order in the mass ratios planet/star, and assuming that the system is not close to a mean–motion resonance, the expansion of the secular perturbation is obtained by dropping the terms depending upon the mean anomalies of the planets from the Hamiltonian (2):

$$\mathcal{K} = -\frac{Gm_1m_2}{a_2} \sum_{k,j_1,j_2,i_l,l\in\underline{4}} B_{i_l}^{k,j_1,j_2} E_1^{|j_1|+2i_1} E_2^{|j_2|+2i_2} S_1^{|k+j_1|+2i_3} S_2^{|k+j_2|+2i_4} \cos\Phi, \qquad (3)$$

where $\Phi = [j_1 p_1 - j_2 p_2 - (k + j_1)q_1 + (k + j_2)q_2]$. This amounts to an "averaging by scissors". All a_i, E_i, S_i, p_i and q_i designate now values averaged over the fast variables λ_i. As the mean longitudes are ignorable, the associated momenta L_i are constant and so are the semi-major axes a_i. The first two terms can be dropped from the Hamiltonian as they depend only on L_1 and L_2 which are constant. So the secular Hamiltonian is a four degrees of freedom problem.

Actually the Hamiltonian depends on only three angular variables as we can rewrite the angle Φ in the following way :

$$\Phi = j_1(p_1 - q_1) - j_2(p_2 - q_2) - k(q_1 - q_2). \qquad (4)$$

This is due to the fact that the Hamiltonian is a sum of terms of the kind $\cos(i_1 p_1 + i_2 p_2 + i_3 q_1 + i_4 q_2)$ with $i_1 + i_2 + i_3 + i_4 = 0$.

The expansion was performed by computer using the formulae of Abu-el-Ata and Chapront (1975) and our own algebraic manipulator. We decided to keep in the Hamiltonian all the terms such that the sum of the exponents of E_1, E_2, S_1 and S_2 is lower or equal to 12.

As shown in Libert and Henrard (2007b), the numerical convergence of the series (3) is very good for a large set of parameters. We think that this good numerical behavior is an indication that the radius of convergence of the secular part of the perturbation is larger than the radius of convergence of the full perturbation.[1] But we do not have a formal proof of this. As in case of coplanar systems (see Libert and Henrard 2006), we quantify, in Table 1, this better numerical convergence—the *convergence au sens des astronomes* (Poincaré 1892)—by comparing, at each order, the largest (in absolute value) coefficient and the number of individual terms of the secular and the full expansions.

In the following sections, we always check the numerical convergence of the secular Hamiltonian of the considered exosystems. As already mentioned in Libert and Henrard (2007b), we will see that the higher the values of the eccentricities or of the mutual inclination, the weaker the numerical convergence. However, for small to moderate values, the

[1] To have further information on the precision of the truncation of the full expansion, see Lemaître and Henrard 1988 (for asteroid studies) and Veras 2007 (for studies on resonant exoplanetary systems).

Table 1 Comparison of the numerical convergence of the full expansion of the non-coplanar perturbation function and the expansion reduced to the secular terms (for the semi-major axes ratio $a_1/a_2 = 0.2$) in function of E_i and S_i

Order in e_i	Full expansion		Secular terms	
	Largest term	Number of terms	Largest term	Number of terms
0	1.01	18	1.01	1
1	1.03	74	0.	0
2	1.05	368	3.24×10^{-2}	6
3	1.20	1118	0.	0
4	1.65	3180	8.81×10^{-2}	31
5	3.15	7462	0.	0
6	5.65	16436	2.08×10^{-1}	110
7	9.74	32742	0.	0
8	2.11×10^{1}	62071	3.83×10^{-1}	326
9	4.40×10^{1}	110566	0.	0
10	8.95×10^{1}	189667	1.17	812
11	1.94×10^{2}	311594	0.	0
12	4.25×10^{2}	497348	5.33	1810

numerical convergence is enough to look for the frequencies of the problem and the long-term time evolutions of the Keplerian elements.

3 Analytical frequencies of the 3-D secular three-body problem

3.1 Action-angle variables for the Laplace–Lagrange model

The quadratic terms in $\sqrt{P_i}$ and $\sqrt{Q_i}$ of the Hamiltonian (3) are of the form:

$$\begin{aligned}\mathcal{K}_0 &= -\frac{Gm_1m_2}{a_2}\left(2\delta\left[\frac{P_1}{L_1}+\frac{P_2}{L_2}-\frac{Q_1}{L_1}-\frac{Q_2}{L_2}+2\sqrt{\frac{Q_1Q_2}{L_1L_2}}\cos(q_1-q_2)\right]\right.\\ &\qquad\left.+2\gamma\sqrt{\frac{P_1P_2}{L_1L_2}}\cos(p_1-p_2)\right)\\ &= -\frac{Gm_1m_2}{a_2}\left(aP_1+bP_2+c\sqrt{4P_1P_2}\cos(p_1-p_2)+\tilde{a}Q_1+\tilde{b}Q_2\right.\\ &\qquad\left.+\tilde{c}\sqrt{4Q_1Q_2}\cos(q_1-q_2)\right) \qquad (5)\end{aligned}$$

where $\delta = B^{0,0,0}_{1,0,0,0} > 0$ and $\gamma = B^{1,-1,-1}_{0,0,0,0} < 0$. We note that the quadratic Hamiltonian is such that the momenta P_i and Q_i never appear simultaneously in a same term. Moreover, the two sums of the momenta $P_1 + P_2$ and $Q_1 + Q_2$ are constants of motion. These properties are no longer true at higher orders. As we have recalled in our previous paper (Libert and Henrard 2007b), only the angular momentum deficit $P_1 + P_2 + Q_1 + Q_2$ is a constant of the motion for the full problem.

The Hamiltonian $\mathcal{K}_0$ is the Hamiltonian of a linear problem in the Cartesian coordinates $Y_i = \sqrt{2P_i}\sin p_i$, $X_i = \sqrt{2P_i}\cos p_i$, $Z_i = \sqrt{2Q_i}\sin q_i$, $W_i = \sqrt{2Q_i}\cos q_i$, the so-called Laplace–Lagrange problem. In order to "untangle" the four degrees of freedom, we perform two similar "reducing transformations" (Henrard 1988; Henrard and Lemaître 2005):

$$
\begin{aligned}
Y_1 &= \ \ Y_1' \cos\beta_p + Y_2' \sin\beta_p & X_1 &= \ \ X_1' \cos\beta_p + X_2' \sin\beta_p \\
Y_2 &= -Y_1' \sin\beta_p + Y_2' \cos\beta_p & X_2 &= -X_1' \sin\beta_p + X_2' \cos\beta_p \\
Z_1 &= \ \ Z_1' \cos\beta_q + Z_2' \sin\beta_q & W_1 &= \ \ W_1' \cos\beta_q + W_2' \sin\beta_q \\
Z_2 &= -Z_1' \sin\beta_q + Z_2' \cos\beta_q & W_2 &= -W_1' \sin\beta_q + W_2' \cos\beta_q \ ,
\end{aligned} \tag{6}
$$

where the angles of the rotation are given by

$$
\tan 2\beta_p = \frac{\gamma\sqrt{L_1 L_2}}{\delta(L_1 - L_2)} \qquad \text{and} \qquad \tan 2\beta_q = -\frac{2\sqrt{L_1 L_2}}{L_1 - L_2}.
$$

A return to polar coordinates $Y_i' = \sqrt{2P_i'} \sin p_i'$, $X_i' = \sqrt{2P_i'} \cos p_i'$, $Z_i' = \sqrt{2Q_i'} \sin q_i'$ and $W_i' = \sqrt{2Q_i'} \cos q_i'$ introduces action–angle variables for the quadratic Hamiltonian which becomes

$$
\mathcal{K}_0' = -\frac{n_2(m_1 + m_2)}{m_0} \left[\nu_1 P_1' + \nu_2 P_2' + \nu_3 Q_1' + \nu_4 Q_2'\right], \tag{7}
$$

where $n_2 = \sqrt{Gm_0/a_2^3}$ approximates the mean–motion of m_2. We insist on the fact that, after the reducing transformations (6), the indices 1 and 2 are no longer attached to the planets 1 and 2, respectively (for instance, the quantity P_1' is a function of both planets m_1 and m_2). Note that the reducing rotations (6) leave unchanged the two sums of the momenta: $P_1 + P_2 = P_1' + P_2'$ and $Q_1 + Q_2 = Q_1' + Q_2'$. As all the angles are ignorable, the actions P_i', Q_i' are constant for the Laplace–Lagrange problem.

The frequencies ν_i are function of the semi-major axes ratio α (by means of γ and δ) and the mass ratio of the planets $\mu = m_1/(m_1 + m_2)$ and have the following values:

$$
\begin{aligned}
\nu_1 &= \mu[\delta(L_1 + L_2) + \sqrt{\delta^2(L_1 - L_2)^2 + \gamma^2 L_1 L_2}]/L_1 \\
\nu_2 &= \mu[\delta(L_1 + L_2) - \sqrt{\delta^2(L_1 - L_2)^2 + \gamma^2 L_1 L_2}]/L_1 \\
\nu_3 &= -2\mu\delta(L_1 + L_2)/L_1 \\
\nu_4 &= 0.
\end{aligned} \tag{8}
$$

The last frequency is zero and the sum of the others $\nu_1 + \nu_2 + \nu_3$ is also equal to zero. This relation between the frequencies of the linear Laplace-Lagrange problem has already been pointed out by Murray and Dermott (1999) and Abdullah and Albouy (2001) where it is called the Herman's resonance.

3.2 Action–angle variables for the non-linear problem

In the action–angle variables of the Laplace–Lagrange problem, the non-linear Hamiltonian (3) (truncated at order 12 in the eccentricities and inclinations) becomes:

$$
\mathcal{K}' = \mathcal{K}_0' - \frac{Gm_1 m_2}{a_2} \sum_{k, j_1, j_2, i_l, l \in \underline{4}} B'^{k, j_1, j_2}_{i_l} E'^{|j_1| + 2i_1}_1 E'^{|j_2| + 2i_2}_2 S'^{|k + j_1| + 2i_3}_1 S'^{|k + j_2| + 2i_4}_2 \cos\Phi', \tag{9}
$$

where $\Phi' = [j_1 p_1' - j_2 p_2' - (k + j_1) q_1' + (k + j_2) q_2']$. E_i' and S_i' have the same relation to the actions P_i', Q_i' as E_i, S_i to the momenta P_i, Q_i. Once again we see that the sum of the coefficients of the angles is null. In fact, it is equivalent to say that the function remains the same after a translation of each of the angles by an arbitrary quantity. By inspecting the formulas of the reducing transformations (6), we see that a translation by some constant of the angles (p_1, p_2, q_1, q_2) is equivalent to the translation by the same constant of the angles (p_1', p_2', q_1', q_2'). So the Hamiltonian after translation is such as the sum of the coefficients of the variables p_1', p_2', q_1' and q_2' is null again. Furthermore, as highlighted in the previous section, the relations (6) also show that the sum of the momenta is unchanged after the reducing rotations: $P_1 + P_2 + Q_1 + Q_2 = P_1' + P_2' + Q_1' + Q_2'$.

In order to develop an analytical non-linear theory, we use the Lie transform perturbation scheme (Hori 1966; Deprit 1969) to average the Hamiltonian (9) of the secular three-body problem over the angular variables p_i' and q_i'. For practical purposes, at each order, the homological equation is solved in such a way that all the terms are integrated to form the generator of the transformation, excepted those inducing small denominator problems (i.e. those containing angular combinations for which the first order frequency is nearly zero). These last terms are kept in the averaged Hamiltonian. As the sum of the linear frequencies $\nu_1 + \nu_2 + \nu_3$ and the last frequency ν_4 are both equal to zero, the averaged Hamiltonian not only contains terms without angular variable, but also terms of the form $\cos k(p_1' + p_2' + q_1' - 3q_2')$. Indeed they are the only combination of angular variables which is of zero frequency and which respects the symmetry described in the previous paragraph. These last terms could in principle appear at order 6 and above due to the d'Alembert characteristic, but, actually, they are present only at order 10 and 12 with $k = 1$. So the averaged Hamitonian is of the form (hereafter we omit the factor $-Gm_1m_2/a_2$):

$$\bar{\mathcal{K}}' = \sum_{l_1+l_2+l_3 \le 6} C_{l_1,l_2,l_3} \bar{E}_1'^{\,2l_1} \bar{E}_2'^{\,2l_2} \bar{S}_1'^{\,2l_3} + \sum_{m_l, l \in \underline{4}} D_{m_l} \bar{E}_1'^{\,m_1} \bar{E}_2'^{\,m_2} \bar{S}_1'^{\,m_3} \bar{S}_2'^{\,m_4} \cos(\bar{p}_1' + \bar{p}_2' + \bar{q}_1' - 3\bar{q}_2'). \tag{10}$$

We found that the first sum in this expression does not depend on the variable $\bar{Q}_2'$, as it is the case in the Laplace–Lagrange model; we do not have an explanation of this *hidden symmetry*, but it is very clear from our computations. The constants $\bar{E}_i'$ and $\bar{S}_i'$ designate values averaged over the secular motion. These averaged values can be computed, on the basis of the initial Keplerian values, by implementing the inverse of the averaging Lie transformation with the algorithm of the inverse (see Henrard 1973).

To obtain an action–angle formulation of the secular Hamiltonian, it is still necessary to average it over the remaining combination of the angular variables. First we perform a canonical transformation in order to reduce the averaged Hamiltonian to one degree of freedom, namely the angle $\bar{p}_1' + \bar{p}_2' + \bar{q}_1' - 3\bar{q}_2'$:

$$\begin{array}{ll} u_1 = \bar{p}_1' - \bar{q}_1' & U_1 = (\bar{P}_1' - \bar{Q}_1')/2 \\ u_2 = \bar{p}_2' - \bar{q}_2' & U_2 = \bar{P}_2' - (\bar{P}_1' + \bar{Q}_1')/2 \\ v_1 = \bar{p}_1' + \bar{p}_2' + \bar{q}_1' - 3\bar{q}_2' & V_1 = (\bar{P}_1' + \bar{Q}_1')/2 \\ v_2 = \bar{q}_2' & V_2 = \bar{P}_1' + \bar{P}_2' + \bar{Q}_1' + \bar{Q}_2'. \end{array} \tag{11}$$

The angles u_1, u_2, v_2 are ignorable, which means that U_1, U_2 and V_2 are first integrals of the averaged problem of one degree of freedom (v_1, V_1). So the objective is the averaging over the angular variable v_1. In the variables (11), the quadratic averaged Hamiltonian writes

$$\bar{\mathcal{K}}_0' = (\nu_1 - \nu_3)U_1 + \nu_2 U_2. \tag{12}$$

As the quadratic terms are constant and independent of the moment V_1, the kernel of the elimination of v_1 is composed by the fourth degree (in the eccentricities and the inclinations) terms of the Hamiltonian, which contain V_1, namely

$$\mathcal{F}_0 = a_1 V_1^2 + a_2 V_1 U_1 + a_3 V_1 U_2. \tag{13}$$

Let us define $\mathcal{N}$ its derivative with respect to V_1

$$\mathcal{N} = 2a_1 V_1 + a_2 U_1 + a_3 U_2, \tag{14}$$

where the a_i are all negative, preventing that $\mathcal{N}$ vanishes except, of course, for $V_1 = U_1 = U_2 = 0$.

The homological equation defining the generator $\mathcal{V}_1$ of the transformation, which eliminates the angle v_1 from the averaged secular Hamiltonian $\bar{\mathcal{K}}' = \mathcal{F}_0 + \mathcal{F}_1$, is

$$\mathcal{N}\frac{\partial \mathcal{V}_1}{\partial v_1} = \mathcal{F}_1. \tag{15}$$

We need only a first order elimination, as the small quantity $\mathcal{F}_1/\mathcal{N}$ is already of order 8 in the Lie triangle (i.e. order 10 in the eccentricities and the inclinations) and it is enough to produce the following formulation in action–angle variables of the secular non-linear Hamiltonian

$$\bar{\bar{\mathcal{K}}}' = \sum_{l_1+l_2+l_3\leq 12} C_{l_1,l_2,l_3}\, \bar{\bar{E}}_1'^{\,2l_1}\, \bar{\bar{E}}_2'^{\,2l_2}\, \bar{\bar{S}}_1'^{\,2l_3}. \tag{16}$$

These two averagings leave the sum of the momenta invariant ($\bar{\bar{P}}_1' + \bar{\bar{P}}_2' + \bar{\bar{Q}}_1' + \bar{\bar{Q}}_2' = \bar{P}_1' + \bar{P}_2' + \bar{Q}_1' + \bar{Q}_2' = P_1' + P_2' + Q_1' + Q_2'$). Indeed the sums of the coefficients of the angular variables p_1', p_2', q_1', q_2' and $\bar{p}_1', \bar{p}_2', \bar{q}_1', \bar{q}_2'$ in the sine functions of the two generators vanish.

Actually this second averaging, while theoretically necessary, is practically insignificant. Indeed, all the terms of the second part of the Hamiltonian (10) have a power of $\bar{S}_2'$ in factor, and, as we shall point out in Sect. 4.2, this quantity vanishes in the frame based on the Laplace plane. Hence the corrections brought by this second averaging are very small and even zero in the frame based on the Laplace plane. So the constants $\bar{\bar{E}}_i'$ and $\bar{\bar{S}}_i'$ can be considered well approximated by their previous values $\bar{E}_i'$ and $\bar{S}_i'$. In the following we will work with these approximated values and therefore we adopt the ($^-$) notation for the double averaged Hamiltonian.

The equations of motion derived from $\bar{\mathcal{K}}'$ are:

$$\dot{\bar{p}}_i{}' = \frac{\partial \bar{\mathcal{K}}'}{\partial \bar{P}_i'} \quad \text{and} \quad \dot{\bar{q}}_i{}' = \frac{\partial \bar{\mathcal{K}}'}{\partial \bar{Q}_i'}. \tag{17}$$

They lead to the expression of the four frequencies—hereafter we will call them the *fundamental* frequencies:

$$\begin{aligned}
\dot{\bar{p}}_1{}' &= -\tfrac{(1-\mu)}{\sqrt{\alpha}} \sum_{l_i, i\in \underline{3}} 2l_1 C_{l_1,l_2,l_3} \bar{E}_1'^{\,2(l_1-1)} \bar{E}_2'^{\,2l_2} \bar{S}_1'^{\,2l_3}\\
\dot{\bar{p}}_2{}' &= -\mu \sum_{l_i, i\in \underline{3}} 2l_2 C_{l_1,l_2,l_3} \bar{E}_1'^{\,2l_1} \bar{E}_2'^{\,2(l_2-1)} \bar{S}_1'^{\,2l_3}\\
\dot{\bar{q}}_1{}' &= -\tfrac{(1-\mu)}{\sqrt{\alpha}} \sum_{l_i, i\in \underline{3}} 2l_3 C_{l_1,l_2,l_3} \bar{E}_1'^{\,2l_1} \bar{E}_2'^{\,2l_2} \bar{S}_1'^{\,2(l_3-1)}\\
\dot{\bar{q}}_2{}' &= 0.
\end{aligned} \tag{18}$$

The unit of frequency is the Keplerian frequency $n_2 = \sqrt{Gm_0/a_2^3}$ of the mass m_2 multiplied by the mass ratio $(m_1 + m_2)\,/\,m_0$. The periods associated to these frequencies are inversely proportional to the values of the real masses. For exoplanetary systems, Eq. 18 are associated with maximal values of the periods, since radial velocity measurements give only minimal values of the masses of the exoplanets.

Finally, the Lie transform algorithm is also used to compute the expressions of the eccentricities e_i, the inclinations i_i, the arguments of the pericenters $\omega_i = -(p_i - q_i)$, the difference of the longitudes of the pericenters $\Delta\varpi = \varpi_1 - \varpi_2$ and the difference of the longitudes of the nodes $\Delta\Omega = \Omega_1 - \Omega_2 = q_2 - q_1$, as temporal functions of the averaged elements.

4 Comparison of the results for two different reference frames

Researches on the 3-D three-body problem are generally developped with the Laplace plane as reference frame (for instance Michtchenko et al. 2006; Libert and Henrard 2007b). This choice is based upon the invariance of the total angular momentum in norm and in direction, and allows to reduce the secular Hamiltonian function to a two degrees of freedom function only. In this section, we apply our analytical theory to a particular system considered in two different reference frames, a general one and the Laplace plane, and discuss the differences.

To be more specific, we consider a system defined by the following osculating orbital elements: the masses $m_0 = M_{Sun}$, $m_1 = 2M_{Jup}$ and $m_2 = 4M_{Jup}$, the semi-major axes $a_1 = 0.5$ AU and $a_2 = 2.5$ AU, the eccentricities $e_1 = 0.1$ and $e_2 = 0.2$. In a general reference frame, we also consider the arbitrary values: $i_1 = 21.739°$, $i_2 = 3.369°$, $\omega_1 = 270°$, $\omega_2 = 90°$, $\Omega_1 = 0°$ and $\Omega_2 = 0°$. This system is represented on Fig. 1 left.

4.1 General reference frame

First of all, it is useful to check the numerical convergence of the expansions (3) and (16) applied to this system. The contributions from order 2 to order 12 in E_i and S_i (or $\bar{E}'_i$ and $\bar{S}'_i$) are reported in Table 2. We see that the numerical convergence of the Hamiltonian $\mathcal{K}$ is excellent. The numerical convergence of the averaged Hamiltonian $\bar{\mathcal{K}}'$ is weaker but, as we will see, is enough to represent the orbits with accuracy.

Computing the inverse of the Lie transform, we obtain the following averaged elements: $\bar{e}'_1 = 0.152$, $\bar{e}'_2 = 0.191$, $\bar{i}'_1 = 16.86°$ and $\bar{i}'_2 = 7.43°$. They are related to $\bar{E}'_i$ and $\bar{S}'_i$ by the same relation than e_i, i_i to E_i, S_i. Using Eq. 18, we compute the values of the non-zero fundamental frequencies of the problem: $\dot{\bar{p}}'_1 = -3.953 \times 10^{-2}$, $\dot{\bar{p}}'_2 = -9.130 \times 10^{-3}$ and $\dot{\bar{q}}'_1 = 6.149 \times 10^{-2}$. They correspond to periods of $T_{\bar{p}'_1} = 17{,}456$ years, $T_{\bar{p}'_2} = 75{,}578$ years

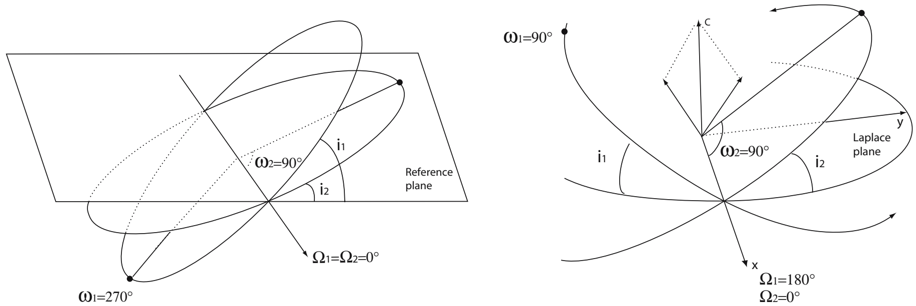

Fig. 1 Same planetary system represented in two different reference frames: a general one (left) and the Laplace plane (right)

Table 2 Numerical convergence of the expansions $\mathcal{K}$ (3) and $\bar{\mathcal{K}}'$ (16) for the application of Sect. 4.1. The last three columns show calculations of the periods for truncations at different orders

	$\mathcal{K}$	$\bar{\mathcal{K}}'$	$T_{\bar{p}'_1}$	$T_{\bar{p}'_2}$	$T_{\bar{q}'_1}$
Order 2	-6.6×10^{-4}	-7.9×10^{-3}	14 046.32	69 313.98	11 679.50
Order 4	-1.7×10^{-4}	-2.0×10^{-3}	17 213.12	74 653.89	11 269.42
Order 6	-6.7×10^{-6}	-6.6×10^{-4}	17 467.37	75 526.56	11 226.50
Order 8	2.7×10^{-7}	-2.9×10^{-4}	17 460.73	75 577.26	11 220.90
Order 10	3.9×10^{-8}	3.6×10^{-5}	17 456.92	75 577.72	11 220.51
Order 12	1.0×10^{-9}	2.1×10^{-4}	17 456.48	75 577.55	11 220.55

and $T_{\bar{q}'_1} = 11{,}221$ years. An interesting remark can be made regarding the first-order Laplace–Lagrange theory. The first-order values of the periods are quite different from the non-linear ones: $T_{\bar{p}'_1} = 14{,}046$ years, $T_{\bar{p}'_2} = 69{,}314$ years and $T_{\bar{q}'_1} = 11{,}679$ years, namely a difference of nearly 20% for the first frequency. These differences illustrate the well-known limitations of the Laplace–Lagrange theory to predict the secular motion of exoplanetary systems. To better illustrate the gap between order 2 and order 12 results, we list in Table 2 the values of the three non-zero fundamental periods from order 2 to order 12 in the eccentricities and the inclinations. The numerical convergence of the results is obvious and explains the limitation to order 12 expansion only. In case of good numerical convergence, approximation limited to order 8 or 10 seems already satisfactory.

We can use the Lie transform algorithm to compute the long-term behavior of the eccentricities e_i, the inclinations i_i, the arguments of the pericenters $\omega_i = -(p_i - q_i)$, the difference of the longitudes of the pericenters $\Delta\varpi = \varpi_1 - \varpi_2$ and the difference of the longitudes of the nodes $\Delta\Omega = \Omega_1 - \Omega_2 = q_2 - q_1$. These time variations (over 1.75×10^5 years) are displayed in Fig. 2, from top to bottom respectively. The long-term behavior of each element is influenced by different linear combinations of the fundamental frequencies. Our analytical theory lets us know the main frequencies acting on the variables (E'_i, S'_i, p'_i, q'_i), but the reducing transformation makes it difficult to do the same for the initial variables (E_i, S_i, p_i, q_i). In order to avoid lengthy analytical transformation, we performed a frequency analysis on the data sets obtained analytically and represented on Fig. 2, the algorithm of which was first introduced by Champenois (1998). This algorithm is based on the original work of Laskar (1993). For more details we refer to Lainey et al. (2006).

As suggested by Fig. 2, the eccentricities e_1 and e_2 are influenced by the same combinations of fundamental frequencies. The same is true for the inclinations i_1 and i_2. The frequency analysis enables us to approximate the inclinations (expressed in radians) by the following trigonometric functions:

$$
\begin{aligned}
i_1 \approx\ & 0.2789 + 0.1147\cos(0.00055997t) - 0.01204\cos(0.0011199t) \\
& - 0.004413\cos(0.0018398t) + 0.002619\cos(0.0016799t) \\
i_2 \approx\ & 0.1251 - 0.05719\cos(0.00055997t) - 0.006908\cos(0.0011199t) \\
& - 0.000259\cos(0.0018398t) - 0.001719\cos(0.0016799t).
\end{aligned}
\tag{19}
$$

where the frequencies are this time expressed in radians per year. In this work, we limit the approximation to the four trigonometric terms with the largest amplitudes. It is already a very good approximation since the maximal error is less than 0.5° for the inner inclination and less than 0.2° for the outer inclination. The detected frequencies can be recognized as integer combinations of the fundamental frequencies, i.e. respectively $\dot{\bar{q}}'_1 - \dot{\bar{q}}'_2$, $2\dot{\bar{q}}'_1 - 2\dot{\bar{q}}'_2$,

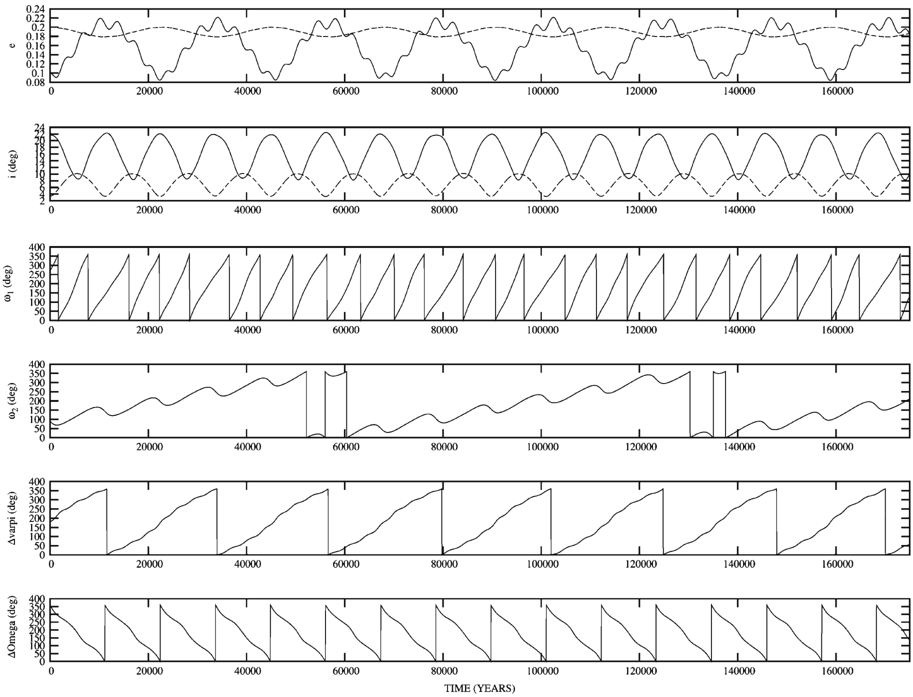

Fig. 2 Time variations on 1.75×10^5 years of the system of Sect. 4.1: from top to bottom, the eccentricities e_i, the inclinations i_i, the arguments of the pericenters ω_1 and ω_2, the difference of the longitudes of the pericenters $\Delta\varpi$ and the difference of the longitudes of the nodes $\Delta\Omega$. In the two upper graphs, the dotted lines stand for the outer body m_2

$-2\dot{\bar{p}}_1' + 2\dot{\bar{q}}_1'$ and $3\dot{\bar{q}}_1' - 3\dot{\bar{q}}_2'$. Similar trigonometric approximations can be found for the eccentricities.

Concerning the angles represented on Fig. 2, we see that they all circulate with precession rates which can be detected by the frequency analysis. Then this linear contribution is removed from the graphic and the resulting oscillation can in turn be analyzed. For instance, we find the following approximation of $\Delta\Omega$ (expressed in radians):

$$\begin{aligned}\Delta\Omega \approx\ & -0.00055997t - 0.2228\sin(0.0011199t) \\ & - 0.07188\sin(0.00055997t) - 0.02524\sin(0.0022399t) \\ & - 0.02032\sin(0.0018399t). \end{aligned} \tag{20}$$

For this order of truncation, the maximal error is less than 3.5°. The precession rate of $\Delta\Omega$ is $-\dot{\bar{q}}_1' + \dot{\bar{q}}_2'$ and the frequencies of the expression (20) may be easily identified as: $2\dot{\bar{q}}_1' - 2\dot{\bar{q}}_2'$, $\dot{\bar{q}}_1' - \dot{\bar{q}}_2'$, $4\dot{\bar{q}}_1' - 4\dot{\bar{q}}_2'$ and $-2\dot{\bar{p}}_1' + 2\dot{\bar{q}}_1'$.

Table 3 summarizes the main frequencies which dominate the long-term behavior of each graphic of Fig. 2. They are listed by decreasing amplitude of the trigonometric term and noted c_1 (largest amplitude) to c_5. Bold type c_1 indicates the precession rate of an angular variable in circulation. Last column displays the identification of the different combinations of the fundamental frequencies which all respect the symmetry resulting from the invariance by rotation. We remark that the decomposition in frequencies of e_i is strongly related to the $\Delta\varpi$'s one. Among others, the main frequency of the eccentricities is the precession rate of

Table 3 Main results concerning the long-term behavior of the system of Sect. 4.1, obtained by decompositions in frequencies of the elements reproduced on Fig. 2

Periods	e	i	ω_1	ω_2	$\Delta\varpi$	$\Delta\Omega$	$2\omega_1$	Identification
22699	c_1		c_2	c_4	$\mathbf{c_1}, c_2$		c_2	$-\dot{p}'_1+\dot{p}'_2$
3415	c_2	c_3			c_4	c_5	$\mathbf{c_1}, c_5$	$-2\dot{p}'_1+2\dot{q}'_1$
11350	c_3		c_5		c_3			$-2\dot{p}'_1+2\dot{p}'_2$
4020	c_4							$-\dot{p}'_1-\dot{p}'_2+2\dot{q}'_1$
2969					c_5			$-3\dot{p}'_1+\dot{p}'_2+2\dot{q}'_1$
6830			$\mathbf{c_1}$					$-\dot{p}'_1+\dot{q}'_1$
75578				$\mathbf{c_1}$				$-\dot{p}'_2+\dot{q}'_2$
11221		c_1	c_3	c_2		$-\mathbf{c_1}, c_3$	c_3	$\dot{q}'_1-\dot{q}'_2$
5610		c_2	c_4	c_3		c_2	c_4	$2\dot{q}'_1-2\dot{q}'_2$
3740		c_4		c_5				$3\dot{q}'_1-3\dot{q}'_2$
2805						c_4		$4\dot{q}'_1-4\dot{q}'_2$

The periods are expressed in years

the difference of the longitudes of the pericenters. It explains that, abstraction made of the shorter periods, the extrema of e_i are reached when $\sin(\Delta\varpi)=0$. We observe the same link between the decompositions of i_i and $\Delta\Omega$.

4.2 Laplace plane

In this section, we keep the same exosystem but adopt another reference frame: the Laplace plane. To explore the 3-D three-body problem, one usually performs the Jacobi's reduction, also called elimination of the nodes (Jacobi 1842). Thereby the secular Hamiltonian function (3) is reduced to a two degrees of freedom function and so is also the parameter space to study.

The invariance of the total angular momentum $\overrightarrow{C}$ in norm and in direction defines an invariant plane perpendicular to this vector. This plane is known as the invariant Laplace plane. The choice of this plane as reference plane implies the following relations (see for instance Laskar 1990):

$$q_1 - q_2 = \pm 180° \tag{21}$$

$$(L_1 - P_1)\cos i_1 + (L_2 - P_2)\cos i_2 = C \tag{22}$$

$$(L_1 - P_1)\sin i_1 + (L_2 - P_2)\sin i_2 = 0 \tag{23}$$

with C the norm of the total angular momentum. Generally another quantity related to the total angular momentum is used, the *angular momentum deficit*. It is defined (see Laskar 1997) as

$$\text{AMD} = \sum_{i=1}^{2} L_i\left(1-\sqrt{1-e_i^2}\cos i_i\right) = L_1 + L_2 - C. \tag{24}$$

This angular momentum deficit is precisely the sum of the momenta $P_1+P_2+Q_1+Q_2$ and we have shown in the previous section that this sum is invariant by the reducing rotations and the averaging process. For a fixed value of the total angular momentum C or equivalently for a fixed value of the angular momentum deficit, the relations (22) and (23) allow us to calculate the values of the inclinations as functions of the eccentricities.

In this reference frame, we have (for the system of Sect. 4.1) $i_1 = 15.0006°$, $i_2 = 3.3694°$, $\omega_1 = 90°$, $\omega_2 = 90°$, $\Omega_1 = 180°$ and $\Omega_2 = 0°$ (see Fig. 1 right). The numerical convergence of the Hamiltonian $\mathcal{K}$ is once again excellent; contributions from order 2 to order 12 are -6.6×10^{-4}, -1.7×10^{-4}, -5.8×10^{-6}, 2.7×10^{-7}, 3.3×10^{-8} and 6.0×10^{-10}. The inverse of the Lie transform gives the following averaged elements: $\bar{e}'_1 = 0.152$, $\bar{e}'_2 = 0.191$, $\bar{i}'_1 = 16.86°$ and $\bar{i}'_2 = 0°$. They are the same as in the other reference frame except for the last one $\bar{i}'_2$ which vanishes. This does not mean of course that the averaged inclination of the second planet is zero. We recall that after the rotations (6) the indices 1 and 2 are no longer attached to the planets 1 and 2, respectively. The quantity i'_2 is a function of both planets, and we have found, in all the examples we have run, that, in the Laplace plane reference frame, it is quite small (at most of the order of 10^{-7}). The surprizing fact is that, in all the examples, the averaged value $\bar{i}'_2$ vanishes down to the level of accuracy we can claim (less than 10^{-10}). This is a puzzling fact to be considered on the same level as the hidden symmetry we have already mentioned. This second *hidden symmetry* has also the very interesting consequence that the second averaging transformation (see Eq. 15) is no longer necessary. All the terms in the second sum of (10) have $\bar{S}'_2$ as factor and thus vanish.

As the averaged Hamiltonian $\bar{\mathcal{K}}'$ is evaluated for the same values of the momenta, we find the same numerical convergence of $\bar{\mathcal{K}}'$ and the same fundamental frequencies as previously.

The time variations of the different variables (on 1.75×10^5 years) are displayed in Fig. 3. We directly recognize the same long-term behavior of the eccentricities and of the difference of the longitudes of the pericenters as in Sect. 4.1. The difference of the longitudes of the nodes is constant to 180°, as requested by Eq. 21. The change of reference frame is also

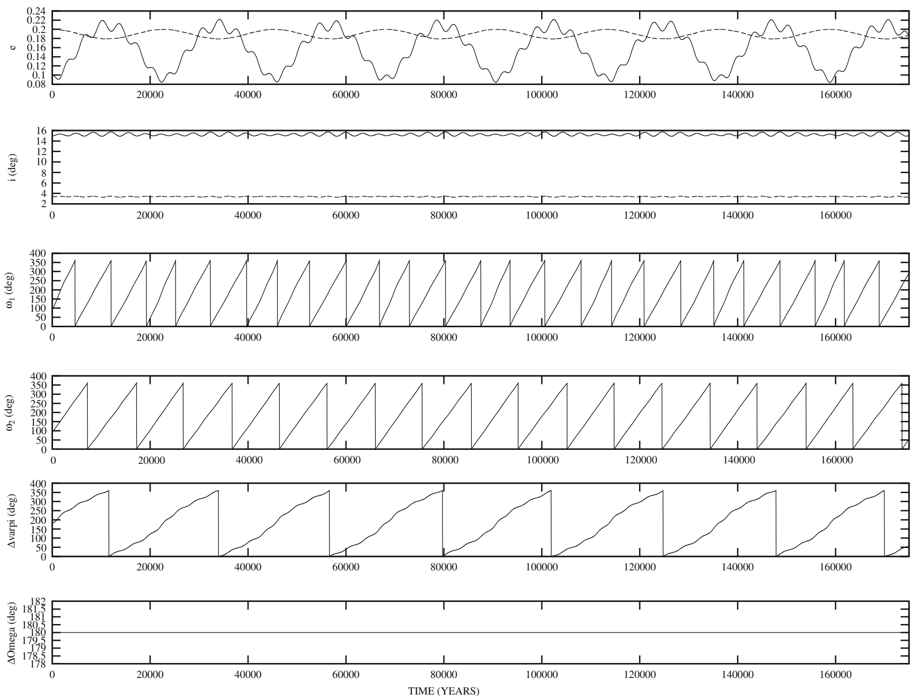

Fig. 3 Same representation as Fig. 2 for the same system situated, this time, in the Laplace plane (see Sect. 4.2)

Table 4 Main results concerning the long-term behavior of the system of Sect. 4.2, obtained by decompositions in frequencies of the elements reproduced on Fig. 3

Periods	e	i	ω_1	ω_2	$\Delta\varpi$	$2\omega_1$	Identification
22699	c_1	c_3	c_2	c_2	$\mathbf{c_1}, c_2$	c_2	$-\dot{\bar{p}}_1' + \dot{\bar{p}}_2'$
3415	c_2	c_1	c_4	c_3	c_4	$\mathbf{c_1}, c_4$	$-2\dot{\bar{p}}_1' + 2\dot{\bar{q}}_1'$
11350	c_3		c_3		c_3	c_3	$-2\dot{\bar{p}}_1' + 2\dot{\bar{p}}_2'$
4020	c_4	c_2		c_4			$-\dot{\bar{p}}_1' - \dot{\bar{p}}_2' + 2\dot{\bar{q}}_1'$
2969			c_5		c_5	c_5	$-3\dot{\bar{p}}_1' + \dot{\bar{p}}_2' + 2\dot{\bar{q}}_1'$
6830			$\mathbf{c_1}$				$-\dot{\bar{p}}_1' + \dot{\bar{q}}_1'$
4885		c_4		c_5			$-2\dot{\bar{p}}_2' + 2\dot{\bar{q}}_1'$
9770				$\mathbf{c_1}$			$-\dot{\bar{p}}_2' + \dot{\bar{q}}_1'$

The periods are expressed in years

clearly visible on the graphs of the arguments of the pericenters and on the graph of the inclinations where the amplitudes of the oscillations are very much reduced.

In order to quantify these differences, we resort to the frequency analysis and produce the results on Table 4. The choice of the Laplace plane induces no change on the coupling between the eccentricities and the difference of the longitudes of the pericenters. On the other hand, the inclinations are no longer related to the difference of the longitudes of the nodes (which is constant in the Laplace plane) but to two angles: $2\omega_1$ and $\Delta\varpi$ (see Fig. 4). In fact, the main frequency of the inclinations is the precession rate of $2\omega_1$, so that, abstraction made of the other periods, the *local* extrema of i_i are reached when $\sin(2\omega_1) = 0$. Furthermore, contrary to the previous reference frame, the inclinations i_i are mainly influenced by the main frequency of e_i (or equivalently the precession rate of $\Delta\varpi$). Then, abstraction made of the other periods, the *global* extrema of i_i are reached when $\sin(\Delta\varpi) = 0$. So the relationships between the motions of the eccentricities and of the inclinations are more clear in the Laplace plane than in the previous reference frame.

Finally, concerning the identification of the frequencies of Table 4, we remark that all the frequencies in the Laplace plane are linear combinations of only two frequencies: $-\dot{\bar{p}}_1' + \dot{\bar{p}}_2'$ and $-\dot{\bar{p}}_1' + \dot{\bar{q}}_1'$. In the previous reference frame, combinations including the fundamental frequency $\dot{\bar{q}}_2'$, like $-\dot{\bar{p}}_2' + \dot{\bar{q}}_2'$ and $\dot{\bar{q}}_1' - \dot{\bar{q}}_2'$, were also present.

4.3 Accuracy of the analytical approach

We have already considered, in Table 2, the error induced by the limitation of our analytical development to different orders in the eccentricities. In this section, we are concerned with the comparison of the long-term time behavior of the system of Sect. 4, given by our analytical approach (limited to order 12) and by a numerical integration of the full three-body problem. We have performed the numerical integration with the SWIFT software package developed by Duncan and Levison (Wisdom and Holman 1991). In Fig. 5, we have plotted the time evolutions of the inner eccentricity and of the difference of the longitudes of the pericenters obtained with the numerical integration (solid lines) together with the results of our analytical theory (dotted lines). The agreement is good, considering the fact that our analytical approach neglects the non-secular perturbations which, for a system not too close to mean–motion resonances, are the source of a small deviation like the one observed in Fig. 5. Let us also point out that the analytically averaged evolution over time is computed on the basis of osculating initial conditions. We have shown in (Libert and Henrard 2007a),

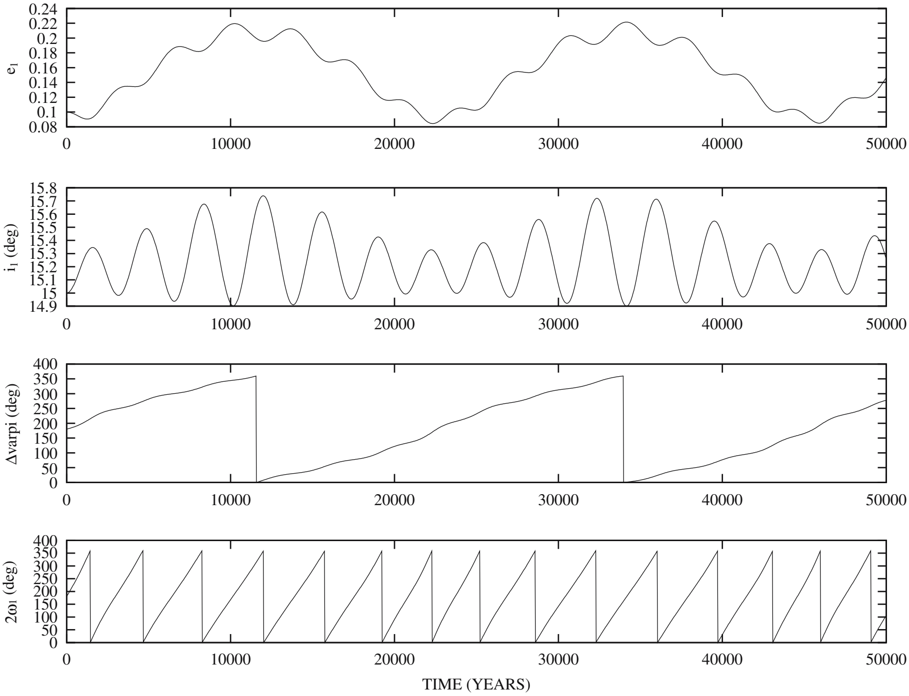

Fig. 4 Long-term behavior of, from top to bottom, the inner eccentricity, the inner inclination, the difference of the longitudes of the pericenters and the angle $2\omega_1$ of Fig. 3 limited at 5×10^4 years. The influence of the precession rate of $\Delta\varpi$ on e_1 and i_1 is obvious (long period of 22,699 years), as well as the one of $2\omega_1$ (short period of 3,415 years). These interrelations between the eccentricities and the inclinations are specific features of the Laplace plane

that the difference between averaged values of the eccentricities and osculating ones is of the order of 10^{-2} which also may explain the small deviation observed in Fig. 5.

Finally, it is also interesting to remark that our perturbation technique cannot cover secular resonance regimes. So far, no exoplanetary systems are located in such a resonance. Remember that the libration of the difference of the periapses of exosystems (e.g. υ Andromedae system) is only a kinematical feature and not a secular resonance.

5 Variation of the secular frequencies with initial conditions

As the uncertainties in the observational data are large, it may be useful to see the long-term effect of a change of the initial parameters values. Such a study of sensitivity to initial conditions can be easily performed with our analytical theory. In this section, we observe the sensitivity of the system of Sect. 4.1 to the initial values of the arguments of the pericenters (ω_1 and ω_2) and the longitudes of the nodes (Ω_1 and Ω_2). For each set of initial parameters, we always check that the 12th order expansion and the averaged Hamiltonian both converge numerically. As the variations of initial ω_1 and ω_2 (or Ω_1 and Ω_2) values lead to similar results, we only discuss the variation of the angles of the mass m_1. Since the results in Table 2 obtained with the order 10 expansion are accurate enough, we limit here the calculation to this order.

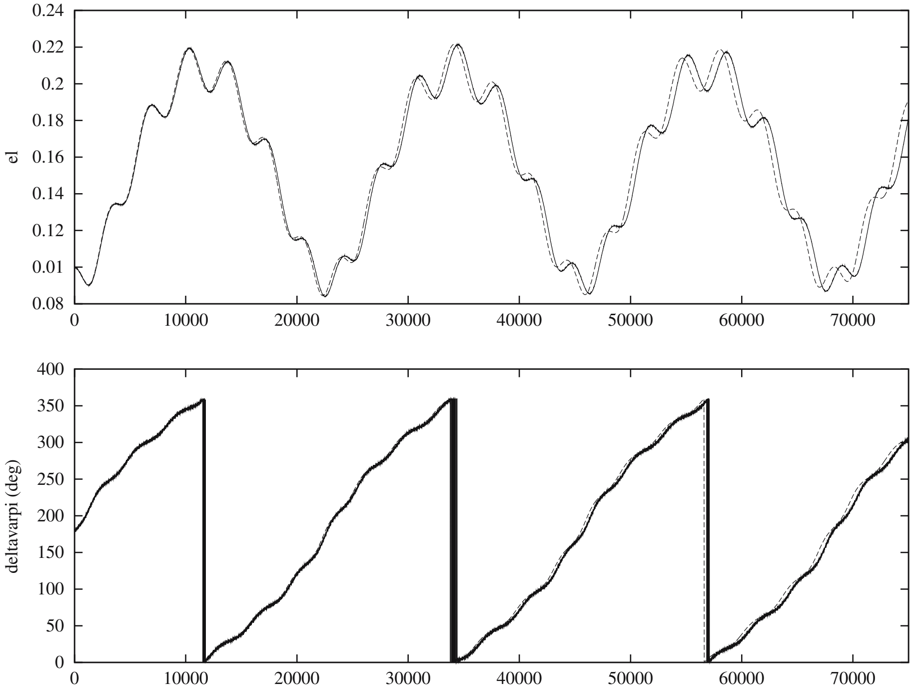

Fig. 5 Comparison between the time evolution of e_1 (top) and $\Delta\varpi$ (bottom) of the system of Sect. 4, as given by our analytical theory (dotted lines) and the numerical integration of the full three-body problem computed using SWIFT (with $M_1 = 50°$ and $M_2 = 0°$) (solid lines). The agreement is very good

First we pay attention to the changes produced on the fundamental frequencies. Figure 6 shows the calculation of the three periods $T_{\bar{p}_1'}$, $T_{\bar{p}_2'}$ and $T_{\bar{q}_1'}$ when initial ω_1 varies from 0° to 360° (solid lines). All the other parameters are fixed to the same initial values as those of Sect. 4.1. We do the same for different initial values of Ω_1 (dotted lines). We see that the changes induced on the values of the periods are weaker for different initial ω_1 values (changes of a hundred years) than for different initial Ω_1 values (changes of a thousand years). So the influence of the longitudes of the nodes on the fundamental frequencies is larger than the one of the arguments of the pericenters.

Moreover, we look at the changes produced on the difference of the longitudes of the pericenters $\Delta\varpi$ and on the difference of the longitudes of the nodes $\Delta\Omega$ by a variation in the initial ω_1 value. Figure 7 gives the oscillation amplitude evolution of $\Delta\varpi$ (left) and of $\Delta\Omega$ (right) when initial ω_1 varies from 0° to 360°. These values are calculated by means of the analytical expressions of the long-term time evolutions of these two Keplerian elements. Libration angle set to 180° represents a circulating case. It is the case of the system of Sect. 4.1 which is fully inside the circulating region of Fig. 7. However we observe that, for some initial values of ω_1, oscillation around $\Delta\varpi = 0°$ occurs. These values are centered around initial $\omega_1 = 90°$ for which the oscillation amplitude of $\Delta\varpi$ is the smallest and which corresponds to an initial value of $\Delta\varpi$ of 0°. Concerning the difference of the longitudes of the nodes, only circulation of $\Delta\Omega$ is possible for all the initial values of ω_1.

Figure 8 shows that different initial Ω_1 values produce, like previously, either a circulation or a libration (around 0°) of the difference of the longitudes of the pericenters $\Delta\varpi$ (left).

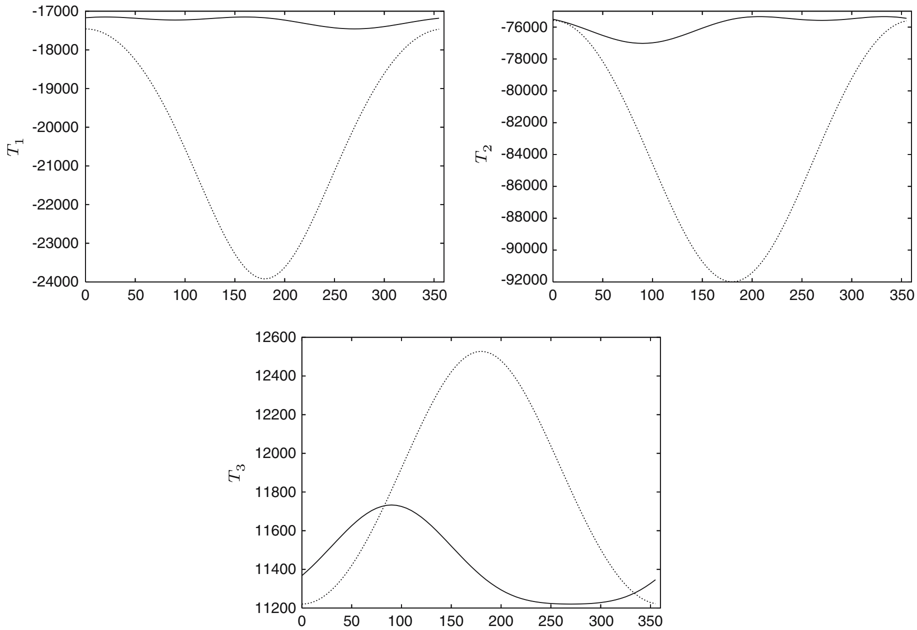

Fig. 6 Sensitivity of the three fundamental periods ($T_1 = T_{\bar{p}'_1}$, $T_2 = T_{\bar{p}'_2}$ and $T_3 = T_{\bar{q}'_1}$) to the initial argument of the pericenter (solid lines) and to the longitude of the node (dotted lines) of the mass m_1. All the computed initial conditions are spaced by 5°

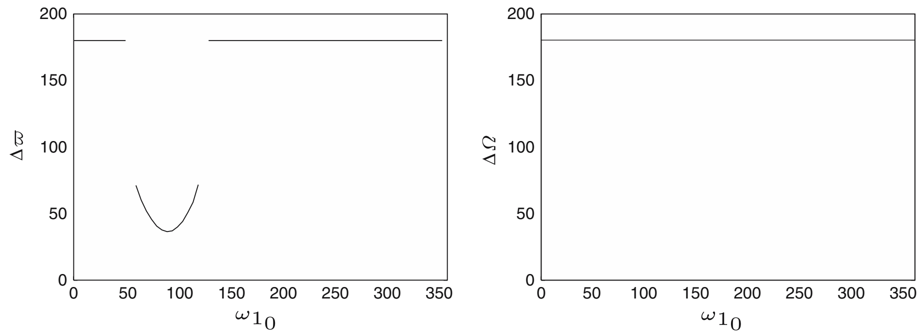

Fig. 7 Sensitivity of the difference of the longitudes of the pericenters $\Delta\varpi$ (left) and of the difference of the longitudes of the nodes $\Delta\Omega$ (right) to the initial ω_1 value. Oscillation amplitudes are given in degrees. An amplitude of 180° stands for circulation

The new feature is the possibility of libration of the longitudes of the nodes $\Delta\Omega$ around 180° (right).

Eventually, graphics of this section show that a little variation of the initial conditions can produce very different long-term results. Oscillation period and amplitude are quite sensitive to the initial configuration. It gives an idea of the consequences on the secular behavior which may be produced by imprecisions in the observations.

A last interesting remark can be made regarding the mutual inclination. To vary the inner longitude of the node (or equivalently the difference of the longitudes of the nodes $\Delta\Omega$)

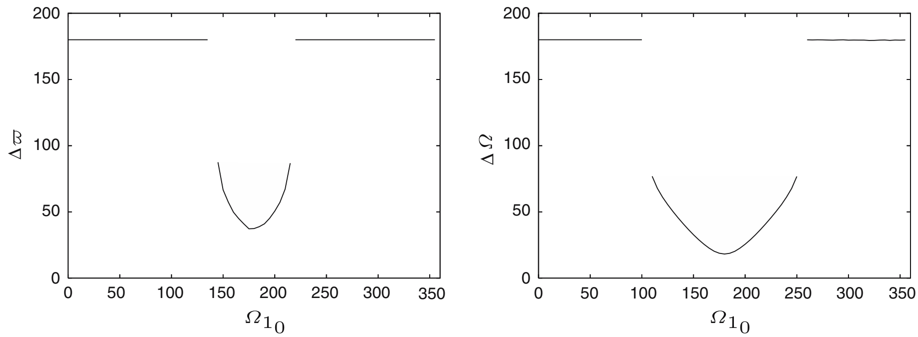

Fig. 8 Sensitivity of the difference of the longitudes of the pericenters $\Delta\varpi$ (left) and of the difference of the longitudes of the nodes $\Delta\Omega$ (right) to the initial Ω_1 value

amounts to vary the mutual inclination I_{mut}. In fact, if we consider the spherical triangle formed by the two orbital planes (of the masses m_1 and m_2) and the basic plane, we find the relation:

$$\cos I_{mut} = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos \Delta\Omega. \quad (25)$$

Thus, $\Delta\Omega = 0°$ corresponds to the minimal value of I_{mut}, namely $i_1 - i_2$. The maximal value of I_{mut}, $i_1 + i_2$, is given by $\Delta\Omega = 180°$. For the system of Sect. 4.1, the increase of initial Ω_1 from 0° to 180° is equivalent to an increase of the initial mutual inclination from 18.37° to 25.11°. In the next section, we consider the υ Andromedae exoplanetary system and observe the changes induced on the dynamics by an increase of the initial mutual inclination.

6 Increasing values of mutual inclination: application to the exosystem υ Andromedae

In this section, we study the long-term behavior of the exoplanetary system υ Andromedae c-d. In the coplanar case, the angular difference of the apsidal lines, $\Delta\varpi$, is in libration around 0°. Here, we determine whether this apsidal configuration is maintained for increasing values of the initial mutual inclination of the two orbital planes.

As seen previously, the secular frequencies and behavior of the angles are very sensitive to the initial configuration of the system. Currently, the observations of exosystems do not give any information on the inclination of the orbits and thus provide only a part of the Keplerian elements. The lack of knowledge leads also to a poor determination of the masses: only minimal masses can be inferred, which correspond to the real ones only in the case of an inclination of 90° of the two orbital planes to the plane of the sky (orbits seen on the edge). So there is a great space of parameters (inclinations and longitudes of the nodes) to study to characterize the possible dynamics.

To explore this space of initial conditions, we choose to work in the same way as Stepinski et al. (2000): considering fixed values of the inclinations of the orbital planes to the plane of the sky, we observe the changes of the dynamics induced by a variation of the initial longitude of the node of the outer body (or equivalently by a variation of the initial difference of the longitudes of the nodes). It amounts to look at the changes produced by a variation of the mutual inclination I_{mut} (see Eq. 25). In order to compare the results, we adopt the same parameters of υ Andromedae c-d system as the "Lick" ones used in Stepinski et al. (2000) ($m_0 = 1.3 M_{Sun}$, $m_1 \sin i_1 = 1.88942 M_{Jup}$, $m_2 \sin i_2 = 3.90870 M_{Jup}$, $a_1 = 0.8282$AU,

$a_2 = 2.5334$ AU, $e_1 = 0.3478$, $e_2 = 0.2906$, $\omega_1 = 248.21°$ and $\omega_2 = 242.99°$). The reference frame we consider is such that the angle i_i corresponds to the inclination of the orbital plane of the mass m_i to the plane of the sky.

Figure 9 represents the case of an initial inclination of 30° of both orbital planes to the plane of the sky ($\sin i_1 = \sin i_2 = 0.5$). Consequently the masses of the planets given above have to be doubled. We show the evolution of the dynamics for some values of the initial mutual inclination I_{mut}: 0° ($\Omega_2 - \Omega_1 = 0°$ first column), 7.5° ($\Omega_2 - \Omega_1 = 15°$ second column) and 15° ($\Omega_2 - \Omega_1 = 30°$ third column). The time variations on 2×10^4 years represented on Fig. 9 are, from top to bottom, the eccentricities, the difference of the longitudes of the pericenters $\Delta\varpi$, the difference of the longitudes of the nodes $\Delta\Omega$ and the mutual inclination I_{mut}. The graphics of the first column correspond to the coplanar case of υ Andromedae system (with double masses) and are obtained with our previous coplanar three-body study (see Libert and Henrard 2006). We see that the inner body suffers from great variations in eccentricities such that its orbit is nearly circular every 3, 718 years. The angular difference of the apsidal lines $\Delta\varpi$ librates around 0° with an amplitude of 61° and the same frequency $(-\dot{p}'_1 + \dot{p}'_2)$ which is the only one characterizing the motion in the coplanar case.

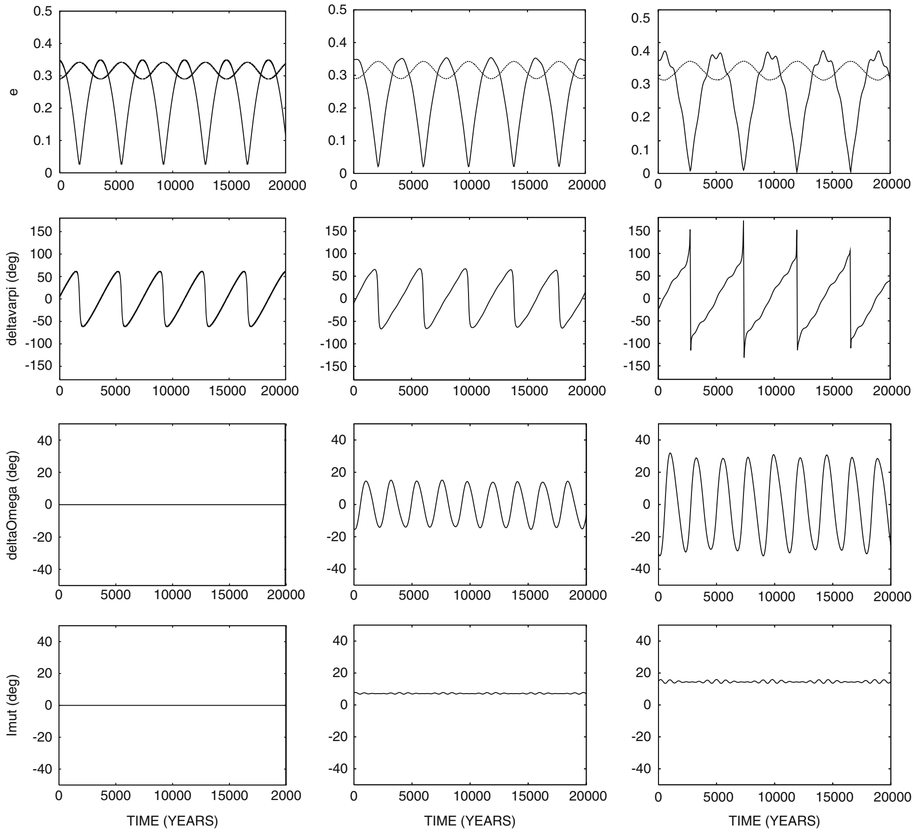

Fig. 9 Time evolutions (on 2×10^4 years) of the $\sin i_1 = \sin i_2 = 0.5$ υ Andromedae system with different initial mutual inclinations of the orbital planes to the plane of the sky: 0° (first column), 7.5° (second column) and 15° (third column). Are represented, from top to bottom, the eccentricities, the difference of the longitudes of the pericenters $\Delta\varpi$, the difference of the longitudes of the nodes $\Delta\Omega$ and the mutual inclination I_{mut}

At first sight, the second initial mutual inclination, 7.5° (second column), presents eccentricities and $\Delta\varpi$ evolutions very similar to those of the coplanar case. In fact, as seen previously, three fundamental frequencies are associated to this non-coplanar secular three-body problem but the main combination of frequencies acting on the eccentricities and the angle $\Delta\varpi$ is $-\dot{p}_1' + \dot{p}_2'$, as in the coplanar case. We see that its value decreases with increasing initial mutual inclination (period of 4, 234 years for initial $I_{mut} = 7.5°$). Furthermore, the difference of the longitudes of the nodes also oscillates around 0° with an amplitude close to the initial difference $\Omega_2 - \Omega_1$. The mutual inclination always stays close to its initial value. The large secular variations observed on the eccentricities were also shown by Stepinski et al. (2000) in their numerical study. The good agreement between the two works gives an idea of the accuracy provided by our analytical method, despite the large eccentricities and the introduction of non-negligible mutual inclination of the two orbital planes.

For higher mutual inclinations (see for instance last column of Fig. 9), larger secular variations of the variables are observed. Minimal values of the inner body approach values as close as 0.0001 and it explains the local singularities in the graphic of $\Delta\varpi$. However we still note an increase in the amplitude of the $\Delta\varpi$ motion. These features are also pointed out by the numerical investigations of Chiang et al. (2001).

Finally, in order to see the influence of the masses of the planets on the dynamics of the system, we consider different values of the initial inclination of the orbital planes to the plane of the sky ($\sin i_1 = \sin i_2 = 0.25$, 0.75 and 1). For each value, we find the same kind of results, namely larger secular variations for increasing initial mutual inclination I_{mut} and close proximity to zero of the minimum eccentricity of the inner body. The higher the value of the initial inclination of the orbital planes to the plane of the sky ($i_1 = i_2$), or equivalently the smaller planetary masses, the higher the value of initial I_{mut} for which a very close relation of the inner eccentricity to zero is observed (about 8° for $\sin i_1 = \sin i_2 = 0.25$ and about 16° for $\sin i_1 = \sin i_2 = 1$).

7 Conclusion

Our study of the secular evolution of two non-coplanar planets (which are not in mean–motion resonance) is based on an analytical 12th order expansion in the eccentricities and the inclinations of the perturbative potential. We apply a Lie averaging in order to introduce an action-angle formulation of the 3-D secular Hamiltonian and in order to find analytical expressions of the fundamental frequencies and of the secular evolution of the elliptic elements of the two planets.

This approach can be carried out in any inertial frame or in the frame based on the Laplace plane. We have compared the results in both frames pointing out in this way the advantages of using the special frame based upon the Laplace plane. The variations of the orbital elements are much simpler and the relations between the evolutions of the eccentricities and the inclinations made clearer. In the general frame, the strong relations, on one hand, between the eccentricities and the difference of the longitudes of the pericenters $\Delta\varpi$ and, on the other hand, between the inclinations and the difference of the longitudes of the nodes $\Delta\Omega$ are already present. But, in the Laplace frame, we observe a closer relationship between eccentricities, inclinations and the angles $\Delta\varpi$ and $2\omega_1$. The amplitudes of variation of the inclinations are also much smaller.

We found in all the cases we have investigated (many more than the ones reported here) two *hidden symmetries* of the averaged Hamiltonian: one valid in any inertial frame (see the remark after Eq. 10), the other one valid only in the Laplace reference frame (annulation of the $\bar{i}_2'$ quantity which makes unnecessary the second averaging—see Eq. 15). On the same level, we can point also to the fact that the special terms with zero first order frequency (see the remark before Eq. 10) appear only at much higher order than expected. We do not have a theoretical justification for these symmetries, but they appear to be real and useful.

As the uncertainties in the observational data are large, we have studied the sensitivity of the frequencies and the evolution to the angular initial conditions and we have shown that some little variations of initial values can produce very different long-term behaviors.

Finally, we have applied our analytical model to the well-known υ Andromedae c-d exoplanetary system. We have studied the dynamics for different initial mutual inclinations (I_{mut} or equivalently different $\Delta\Omega$) and different initial orbital inclinations to the plane of the sky ($\sin i_1 = \sin i_2$). Our results are in agreement with numerical results of previous works and give an idea of the accuracy provided by our analytical model, despite the large eccentricities and the possibly non-negligible mutual inclination of the two orbital planes of an exoplanetary system.

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