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ORIGINAL ARTICLE

Periodic orbits as centers of stability in the secular 3D planetary three body problem

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Abstract In a previous paper, we have developed an analytical model of the secular 3D planetary problem by expanding the perturbation function up to the twelfth order in the eccentricities and the inclinations. Although the expansion is limited the model is able to describe with accuracy most of the observed systems of exoplanets. With the help of this model we were able to describe the geometry of the phase space of a typical system. The kernel of this description is a series of surfaces of section showing the chaotic and the regular domains of the phase space. We have observed in this previous paper that a family of unstable periodic orbits is responsible for the chaoticity, while we have hinted that the islands of stability are organized around stable periodic orbits. In this contribution we compute the main families of periodic orbits of the problem and show that indeed they are responsible for sculpting the phase space.

Keywords Exoplanetary systems · Three body problem · Periodic orbits · Homoclinic orbits · Stability · Bifurcations

1 Introduction

In the last decade, more and more exoplanetary systems were found and, probably due to observational bias, their orbital characteristics are quite different from the characteristics of our own planetary system. Observational bias or not, these systems exist, and the formation, evolution and stability of such systems are important questions which open new avenues of research in Celestial Mechanics. Due to the difficulties of observation, the basic parameters

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(such as the masses) and orbital elements of these systems are not known with much accuracy or even, for the inclinations, completely unknown.

With this in mind, we believe it is more important to try to understand the basic mechanisms explaining the long term dynamics of arbitrary but typical systems rather than to develop accurate theories of observed systems even if they are relatively well determined. Also, because most, if not all, of the numerical studies concern the mean motion resonances (see for instance Hadjidemetriou 2006; Voyatzis and Hadjidemetriou 2006; Callegari et al. 2006; Michtchenko et al. 2006b), we decided to focus on the general secular motion outside resonances. This was the aim of our previous contributions on the subject (Libert and Henrard 2005, 2007). In the last one, we observed that in the 3D problem (which has not been extensively investigated because of the lack of knowledge of the inclination of observed systems) the geometry of the phase space, i.e. the localization of regular and chaotic motions in the phase space, was apparently controlled by a few families of periodic orbits associated with the equilibria which were described by Jefferys and Moser (1966) for systems with very small ratio of semi-major axis and very small eccentricities and inclinations. We were able to extend this description to a very large range of parameters, including values typical for the observed exoplanetary systems. We believe that the description we have made in Libert and Henrard (2005, 2007) of the geometry of the phase space can be applied to many (non resonant) exoplanetary systems. Of course for much lower or much higher values of the angular momentum deficit, the topology could be quite different.

In this contribution we compute the main families of periodic orbits of the typical problem considered in Libert and Henrard (2007): mass ratio ($\mu = m_1/(m_1 + m_2)$) equal to 0.2, ratio of semi-major axis (α) equal to 0.3 and non-dimensional angular momentum deficit (Σ—see Eq. 8) equal to 0.03. This is to be compared for instance with the values attributed to the system υ Andromedae c and d in a recent catalog (http://www.exoplanet.eu/catalog-RV.php): mass ratio equal to 0.334, ratio of semi-major axis equal to 0.331 and non-dimensional angular momentum deficit equal to 0.026 *if the system is assumed to be coplanar*.

In Sect. 2, we describe briefly the reduction of the 3D secular planetary three body problem to a two degrees of freedom Hamiltonian system, and bring forward the symmetries which are quite useful in the study of this reduced system. In Sect. 3, we recall in a few words our findings concerning the geometry of the problem (Libert and Henrard 2007).

Section 4, the main contribution of this paper, is devoted to the description of the families of periodic orbits emanating from the three equilibria of the problem: the central unstable equilibrium and the two stable *Kozai equilibria*. The analogy between the development and bifurcations of these families with the unstable family of periodic orbits emanating from L_3, and the stable families of periodic orbits emanating from L_4 and L_5, is quite striking. In the final Sect. 5 we demonstrate step by step how the principal features of the dynamics of the system as described in our previous contribution (Libert and Henrard 2007) are connected with the evolution of these families.

2 Analytical modelization of the problem

We consider a system consisting of a central star of mass m_0 and two planets of mass m_1 and m_2, with m_1 the one closest to the central star. The Hamiltonian of the dynamics of this system, in the usual Jacobi coordinates and limited to the second degree in the mass ratios m_1/m_0 and m_2/m_0, is:

$$\mathcal{H} = -\frac{Gm_0m_1}{2a_1} - \frac{Gm_0m_2}{2a_2} - Gm_1m_2\left[\frac{1}{|\vec{r_1} - \vec{r_2}|} - \frac{(\vec{r_1}|\vec{r_2})}{r_2^3}\right], \tag{1}$$

where a_i, $\overrightarrow{r_i}$ and r_i are, respectively, the osculating semi-major axis, the position vector and the norm of the position vector of the mass m_i (see for instance Laskar 1990).

A set of canonical variables is formed by the classical modified Delaunay's elements (to the second degree in the mass ratios):

$$\begin{aligned}
\lambda_i &= \text{mean longitudes}, & L_i &= m_i\sqrt{Gm_0 a_i} \\
p_i &= \text{longitudes of the pericenter}, & P_i &= L_i\left[1-\sqrt{1-{e_i}^2}\right] \\
q_i &= \text{longitudes of the node}, & Q_i &= L_i\sqrt{1-e_i^2}\,[1-\cos i_i]\,,
\end{aligned}$$

where e_i and i_i are the eccentricities and inclinations of the planets. In order to expand the last term of Eq. 1 in powers of the eccentricities and the inclinations, we prefer to use the (non-dimensional) expressions $E_i = \sqrt{2P_i/L_i}$ instead of the eccentricities e_i and $S_i = \sqrt{2Q_i/L_i}$ instead of the inclinations i_i; they are immediately related to the Delaunay's canonical variables, and, at least for small to moderate eccentricities and inclinations, they have similar meanings and values.

We assume that the system is not close to a mean motion resonance and we average the Hamiltonian function over the "fast variables" λ_i, and obtain an averaged Hamiltonian

$$\mathcal{K} = \sum_{k,j_1,j_2,i_l,l\in\underline{4}} B_{i_l}^{k,j_1,j_2} E_1^{|j_1|+2i_1} E_2^{|j_2|+2i_2} S_1^{|k+j_1|+2i_3} S_2^{|k+j_2|+2i_4} \cos\Phi, \tag{2}$$

where $\Phi = j_1(p_1-q_1) - j_2(p_2-q_2) - k(q_1-q_2)$. All variables $a_i,\ E_i,\ S_i,\ p_i$ and q_i now designate values averaged over the fast variables λ_i. Actually, we have implemented the averaging by simply removing from the expanded Hamiltonian the terms depending upon the mean anomalies of the planets. This amounts to a first order (in the mass ratios) averaging. As the mean longitudes are ignorable, the associated moments L_i are constant and so are the semi-major axes a_i. The first two terms and the factor $-Gm_1m_2/a_2$, which can be absorbed by redefining the time scale, have been dropped from the Hamiltonian as they depend only on L_1 and L_2 which are constant. So the secular Hamiltonian is a four degrees of freedom problem.

These expansions were performed by computer using our own algebraic manipulator. We decided to keep in the Hamiltonian all the terms such as the sum of the exponents of $E_1,\ E_2,\ S_1$ and S_2 is lower or equal to 12.

In the case of the three-body problem, the Jacobi's reduction, also called elimination of the nodes (Jacobi 1842), allows us to reduce the Hamiltonian function (2) to a two degrees of freedom function only. It is based on the invariance of the total angular momentum, $\vec{C}$, in norm and in direction. The constant direction of the vector $\vec{C}$ defines an invariant plane perpendicular to this vector. This plane is known as the invariant Laplace plane. The choice of this plane as reference plane implies the following relations

$$q_1 - q_2 = \pm 180^\circ \tag{3}$$

$$(L_1 - P_1)\cos i_1 + (L_2 - P_2)\cos i_2 = C \tag{4}$$

$$(L_1 - P_1)\sin i_1 + (L_2 - P_2)\sin i_2 = 0 \tag{5}$$

with C the norm of the total angular momentum.

In order to take advantage of these relations, we introduce the following canonical transformation:

$$\begin{array}{ll} w_1 = p_1 - q_1 & W_1 = P_1 \\ w_2 = p_2 - q_2 & W_2 = P_2 \\ r_1 = q_1 - q_2 & R_1 = P_1 + Q_1 \\ r_2 = q_2 & R_2 = P_1 + P_2 + Q_1 + Q_2. \end{array} \tag{6}$$

The angle w_i corresponds, in this definition, to the opposite of the (averaged) argument of the pericenter. Only the three first angles are present in the Hamiltonian (r_2 is ignorable), which means that R_2 is a first integral of the problem. This constant is related with the constant norm of the total angular momentum and is often introduced in the literature (see for instance Laskar 1997) as the *angular momentum deficit*:

$$AMD = \sum_{i=1}^{2} L_i \left(1 - \sqrt{1 - e_i^2} \cos i_i\right) = L_1 + L_2 - C, \tag{7}$$

where the last equality is due to the definition of the Laplace plane (4). In the following sections, we prefer to refer to a non-dimensional angular momentum deficit, denoted by the symbol Σ :

$$\Sigma = \frac{AMD}{L_2}(1 - \mu) \tag{8}$$

where $\mu = m_1/(m_1 + m_2)$.

Also, the angle r_1 is constant (see (3)) and the problem is reduced to two degrees of freedom (w_1, w_2, W_1, W_2). Then, for a fixed value of the total angular momentum C or equivalently for a fixed value of the angular momentum deficit Σ, the relations (4) and (5) allow us to calculate the values of the inclinations as functions of the eccentricities.

We execute a last transformation of coordinates to the usual Poincaré like canonical variables

$$x_i = \sqrt{2P_i} \cos w_i \quad \text{and} \quad y_i = \sqrt{2P_i} \sin w_i, \tag{9}$$

and the Hamiltonian becomes

$$\mathcal{K} = \sum_{n_l, l \in \underline{5}} E_{n_l} x_1^{n_1} y_1^{n_2} x_2^{n_3} y_2^{n_4} \chi^{n_5}, \tag{10}$$

where χ is the part of the angular momentum deficit due to the mutual inclination of the orbits, i.e.

$$\chi = AMD - P_1 - P_2 = AMD - (x_1^2 + y_1^2 + x_2^2 + y_2^2)/2. \tag{11}$$

The coefficients E_{n_l} depend on the three parameters of the problem: the ratio of the semi-major axes α, the mass ratio μ and the value of Σ. We refer to Libert and Henrard (2007) for more details.

The fact that the expansion (2) is an even function of the angular variables implies that in the expansion (10) the sum $n_1 + n_3$ and $n_2 + n_4$ are always even. From this observation it follows that:

$$\begin{aligned} \dot{x}_i &= -\frac{\partial \mathcal{K}}{\partial y_i} = F_i(x_i, y_i) = F_i(-x_i, y_i) = -F_i(x_i, -y_i) = -F_i(-x_i, -y_i), \\ \dot{y}_i &= -\frac{\partial \mathcal{K}}{\partial x_i} = G_i(x_i, y_i) = G_i(x_i, -y_i) = -G_i(-x_i, y_i) = -G_i(-x_i, -y_i). \end{aligned} \tag{12}$$

Hence, for each solution $(x_i(t), y_i(t))$ of the differential equations, we can define three other solutions:

$$(-x_i(-t),\ y_i(-t)),\quad (x_i(-t),\ -y_i(-t)),\quad (-x_i(-t),\ -y_i(-t)). \tag{13}$$

These symmetries are similar to the symmetries of the restricted problem with equal masses and are quite handy in the computation of surfaces of section or of periodic orbits. Indeed, as in the restricted problem with equal masses, there exist three types of symmetric periodic orbits; orbits symmetric with respect to the y_i axis, orbits symmetric with respect to the x_i axis and orbits symmetric with respect to all axes (see also Muñoz-Almaraz et al. 2007).

As a further consequence of the evenness of $n_1 + n_3$ and of $n_2 + n_4$ in the expansion (10), we also derive the following useful observation:

$$\begin{aligned} \dot{y}_i &= \frac{d\mathcal{K}}{dx_i} = x_1 A_1(x_i, y_i) + x_2 A_2(x_i, y_i) \\ \dot{x}_i &= -\frac{d\mathcal{K}}{dy_i} = y_1 B_1(x_i, y_i) + y_2 B_2(x_i, y_i). \end{aligned} \tag{14}$$

Hence whenever $x_1 = x_2 = 0$, the velocities $\dot{y}_i$ vanish, and similarly the velocities $\dot{x}_i$ vanish whenever $y_1 = y_2 = 0$. A first consequence is that the origin is always an equilibrium.

3 The geometry of the phase space

The equilibrium at $x_i = y_i = 0$, corresponds to circular orbits, and (according to (7)) to the maximal mutual inclination compatible with a fixed value of Σ. Let us remark that this equilibrium of the reduced secular problem corresponds in the physical space to a quasi periodic orbit. When the angular momentum deficit is small, the equilibrium is stable. To keep the same terminology as in our previous work we refer to it as the *central* equilibrium. It is this equilibrium which Poincaré (1892) used as a stepping stone in order to show the existence of the periodic orbits *de la troisième sorte* in the case of planets with vanishingly small masses. Jefferys and Moser (1966) and Robutel (1995) have shown that for vanishingly small ratios of semi-major axis this equilibrium becomes unstable at a large value of Σ (corresponding to a mutual inclination of 39°, 23°) and generates by bifurcation two stable equilibria.

In a previous contribution (Libert and Henrard 2007) and with the help of the expansions described in the previous section, we have computed the critical value of the mutual inclination (which corresponds to the change of stability of the central equilibrium) for different values of the ratio of semi-major axis and of the mass ratio compatible with exoplanetary systems.

We have found that the zone of chaotic motion, described by Michtchenko et al. (2006a) for the case of the c–d υ Andromedae system, is due to the tangles of stable and unstable manifolds associated with the unstable family of periodic orbits emanating from the central equilibrium. For values of the parameters close to the υ Andromedae system ($\alpha = 0.3$, $\mu = 0.2$ and $\Sigma = 0.03$) we have explored in details the dynamics of the system by means of a series of surfaces of section for different values of the energy. We found that in the vicinity of the stable equilibria created by bifurcation from the central equilibrium, regions of regular motions are centered around the Lyapunov families of periodic orbits emanating from them. These regions have been called *Kozai resonances* (Michtchenko et al. 2006a), by analogy with the restricted problem (see Kozai 1962; Kinoshita and Nakai 2007).

4 The web of periodic orbits emanating from the central equilibrium and the Kozai equilibria

The family of unstable periodic orbits emanating from the central equilibrium is doubly symmetric; the four orbits obtained by the symmetry described in Eq. 12 form one and only one orbit. On the other hand, the Lyapunov families emanating from the Kozai equilibria are simply symmetric with respect to the y_i axes and the families associated with each of the two Kozai equilibria are mirror images of each other traveled in an opposite direction. This is analogous to the relation between the Lyapunov families emanating from the equilateral equilibria of the restricted problem (for small mass ratio). As we shall see the analogy is even more profound as the family of unstable periodic orbits emanating from the central equilibrium plays, with respect to them, a role very similar to the role played by the family emanating from the L_3 equilibrium with respect to the families emanating from L_4 and L_5.

The fact that they are symmetric is quite helpful in computing them. We set x_1 and x_2 to zero, pick a value of y_1 and adjust y_2 so that the orbit lies on a given energy level. We compute the orbit until it reaches again $x_2 = 0$, and redefine the value of y_1 so that at the same time x_1 vanishes also. Once the symmetric orbit is reached we integrate the variational equations in order to compute its stability. We use a Runge–Kutta fourth order integrator with a time step adjusted so that the local error is less than 10^{-12}. The constancy of the energy is monitored as well as the fact that the velocity is a solution of the variational equations.

We plot in Fig. 1, the initial values of periodic orbits associated with the equilibria of the secular 3D planetary three body problem with $\alpha = 0.3$, $\mu = 0.2$ and $\Sigma = 0.03$. We plot the energy versus e_1 affected with the sign of x_1 (i.e. positive when $\omega_1 = 90°$ and negative when $\omega_1 = -90°$). The initial values correspond to the point where the orbits cross the surface of section $x_2 = 0$ with a negative velocity. As the orbits around the two Kozai equilibria are traveled in opposite direction, these points are not the same for both orbits although the orbits are mirror images of each other (see the little sketches in Fig. 2). This is why the graph is not symmetric.

We show in Fig. 3 a sample of the periodic orbits under consideration. Their place on Fig. 1 is identified by the letters A, B, C, ..., and f, g, h, ...

As in the family of periodic orbits emanating from the L_3 equilibrium in the restricted problem, the family $\mathcal{C}$ emanating from the central equilibrium is first unstable (with a stability index, $\mathcal{U} = 2\cosh(\lambda T)$) large and positive—see for instance Deprit and Henrard (1968) for a definition of the stability index. The value of the index goes down rapidly, reaches the critical value $\mathcal{U} = 2$ at $\mathcal{K} = -0.0139224396$. At this point it serves as the termination of the two *short period Lyapunov families*, $\mathcal{S}$ and $\mathcal{S}'$ emanating from the Kozai equilibria. It is the same "non-generic" bifurcation by which the short period families of periodic orbits of L_4 and L_5 in the restricted problem end on a symmetric orbit emanating from L_3. It is non-generic because of the symmetry which makes two simply symmetric orbits, symmetric to each other, bifurcate from a doubly symmetric orbit (see Henrard 2002).

After the bifurcation, the family $\mathcal{C}$ becomes stable, reaches a maximum value of the energy, where it changes again its stability. Later on, it gets stable again; its period increases until it approaches a type of termination which has already been described in the context of the restricted problem (Henrard 1965, 1983; see also Henrard 2002). This termination is a generalization of the termination conjectured by Strömgren (1933) and investigated by Henrard (1973) and Devaney (1977). The terminating feature is no longer a single homoclinic orbit to an unstable equilibrium but a family of homoclinic orbits to a family of unstable periodic orbits. In the case of the web of long periodic orbits emanating from L_4, investigated earlier (Henrard 1983), we have conjectured that the members of the web wind up

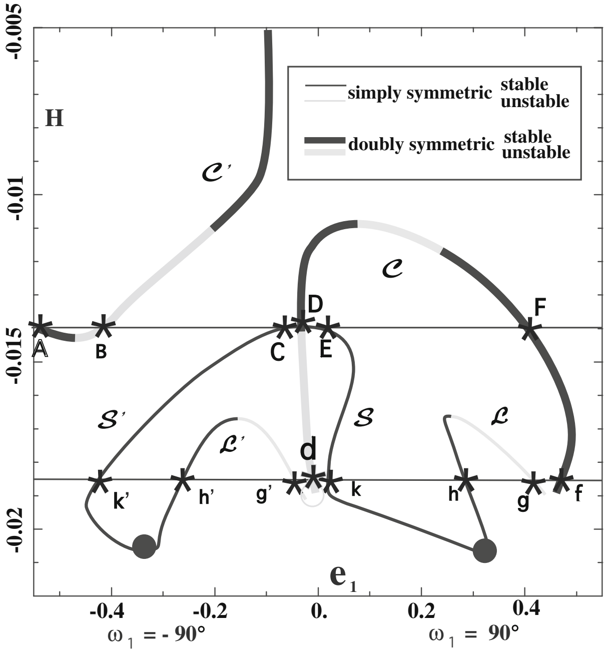

Fig. 1 Graphs of energy versus initial e_1 of periodic orbits associated with the equilibria of the secular 3D planetary three body problem with $\alpha = 0.3$, $\mu = 0.2$ and $\Sigma = 0.03$. The gray big dot stands for the central equilibrium, the black ones for the Kozai equilibria. The letters refer to the drawing of individual periodic orbits in Fig. 3

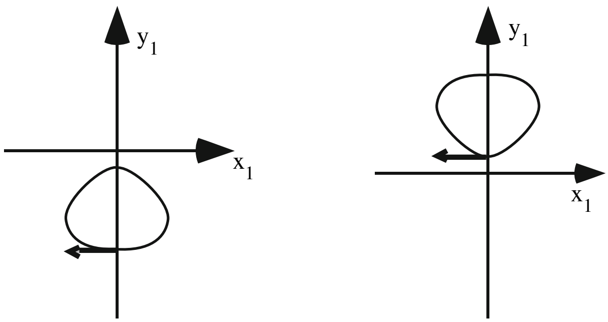

Fig. 2 Sketches explaining why the Fig. 1 is not symmetric. The orbits are mirror images of each other, but they are traveled in opposite direction; which means that the initial values (the point where $\dot{x}_1 < 0$) are not mirror images of each other

around a family of homoclinic orbits emanating from L_3, getting closer to it while the period increases.

In our case here, the family of homoclinic orbits which serves as the focus of the terminating process is actually formed by a family of pairs of orbits homoclinic to members of the family $\mathcal{C}$ of periodic orbits. We have plotted in Fig. 4 (left panel) such a homoclinic orbit. By varying the energy of the $\mathcal{C}$-orbit, two families of doubly-asymptotic orbits, symmetric to each other, are generated. We call them Ω (the one corresponding to Fig. 4) and Ω' the family symmetric to it. The termination under investigation is formed by the composition of two homoclinic orbits of the same energy, one belonging to Ω and the other one to Ω', as shown in the right panel of Fig. 4.

We have already mentioned the fate of the short period families $\mathcal{S}$ and $\mathcal{S}'$ emanating from the Kozai equilibria. The *long period families* $\mathcal{L}$ and $\mathcal{L}'$ reach a maximum of the energy and then, similarly to the web of long period orbits associated to L_4 and L_5

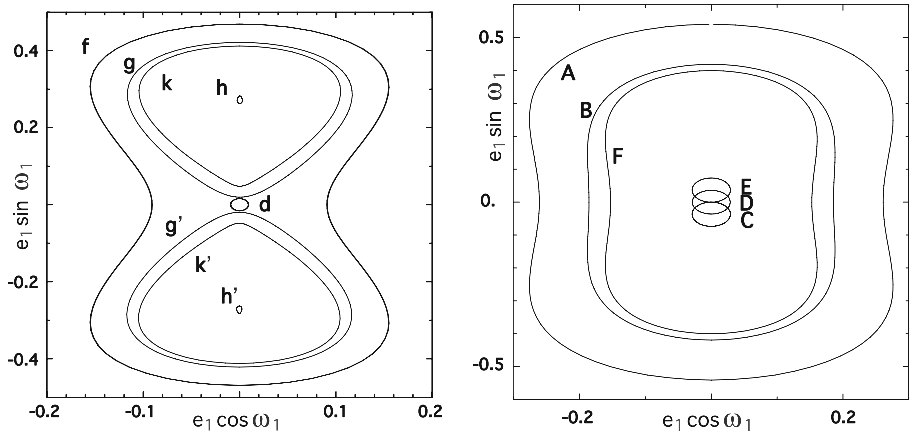

Fig. 3 Graphs of periodic orbits at the level of energy $\mathcal{K} = -0.0185$ (left) and $\mathcal{K} = -0.014$ (right). The orbits are identified on Fig. 1 by the letters f, g, h, ..., A, B, C, ...

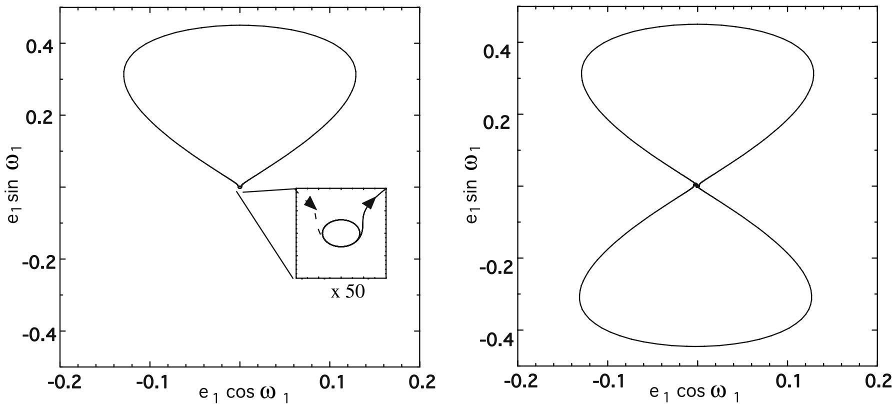

Fig. 4 Homoclinic orbit to a small periodic orbit (for $\mathcal{K} = -0.01916$)

in the restricted problem, proceed toward their terminations by increasing their periods and winding around the family Ω (a specimen of which is depicted on the left panel of Fig. 4) or Ω'.

We have also plotted in Fig. 1, the family $\mathcal{C}'$ of doubly symmetric periodic orbits, which takes the relay of $\mathcal{C}$ as the backbone of the phase space for larger value of the energy. It goes up to $\mathcal{K} = 0.002725$ at which point the two planets are coplanar and the eccentricities are constant ($e_1 \approx -0.11977$ and $e_2 \approx 0.26776$).

The projections of the orbit in the planes $(e_1 \cos\omega_1, e_1 \sin\omega_1)$ and $(e_2 \cos\omega_2, e_2 \sin\omega_2)$ are two perfect circles, in conformity with the analysis done in Libert and Henrard (2005). They correspond to one of the equilibria we called *dynamical poles* of the problem reduced to one degree of freedom. At these equilibria the values of e_1 and e_2 are constant; hence the fact that the projections are circles. This should be considered as the end of the family of periodic orbits as far as the planetary problem is concerned, but it is not the end for the

Table 1 How the families $\mathcal{C}$ and $\mathcal{C}'$ avoid to enter the unstability domain $\mathcal{U} < -2$

Family $\mathcal{C}'$		Family $\mathcal{C}$	
$\mathcal{K}$	$\mathcal{U}$	$\mathcal{K}$	$\mathcal{U}$
−0.003	−1.97	−0.013	−1.89
−0.0034	−1.996	−0.01315	−1.997
−0.0036275	−1.9999993	−0.0131725	−1.99993
−0.00363	−1.99999997	−0.0131775	−1.999997
−0.0036325	−1.9999997	−0.01318	−1.99996
−0.0039	−1.994	−0.0132	−1.998
−0.004	−1.98	−0.0133	−1.93

Hamiltonian problem described by (10). Indeed solutions of this Hamiltonian problem can be found for negative values of the function χ, i.e. for imaginary values of the inclinations.

The other part of the family $\mathcal{C}'$ run outside Fig. 1. We did not plot it because its evolution is uneventful. While remaining stable, it goes up in eccentricity and in energy until it eventually ends up on the second periodic orbit of the coplanar problem at $p_1 - p_2 = \pi$, $e_1 = 0.65563792$ and $e_2 = 0.0889325589$ for an energy level of $\mathcal{K} = 0.016023520$.

It may be worth mentioning a puzzling fact concerning the evolution of the stability index $\mathcal{U}$ along the families $\mathcal{C}$ and $\mathcal{C}'$. In both cases, the value of the stability index goes down from the unstable region $\mathcal{U} > 2$, to the stable region $-2 > \mathcal{U} > 2$, but instead of going down to the unstable region $\mathcal{U} < -2$, it makes a sharp turn at precisely the critical value $\mathcal{U} = -2$, to return in the stable region. Table 1 illustrates this puzzling behavior which may be due to the symmetries of the problem.

5 Periodic orbits as the backbones of the surfaces of section

In a previous paper (Libert and Henrard 2007), we have computed a sequence of surfaces of section of the problem and we have hinted that their principal features are due to the existence of the periodic orbits we have just described. We proceed now to clarify this point by producing six portraits. Each one corresponds to a particular value of the energy and is composed of two panels. The lower one is a simplified reproduction of Fig. 1, in order to show what are the periodic orbits belonging to this level of energy; the upper one is a reproduction of the surfaces of section published in Libert and Henrard (2007). The correspondence between the two panels is sometimes a little distorted as the scales of the x axis of the panels are slightly different.

For a value of the energy just above the value corresponding to the Kozai equilibria (see the left portrait of Fig. 5), the surface of section is composed of two separate regular domains, formed by quasi-periodic motions centered around a periodic orbit belonging to one of the Lyapunov families emanating from the equilibria, the family $\mathcal{L}$ and $\mathcal{L}'$ respectively. They are marked by X on the lower panel. This picture is a little deceptive. Actually the families $\mathcal{S}$ and $\mathcal{S}'$ are also fixed points of the surface of section surrounded by quasi-periodic motions. But the area covered by these motions is so small that it cannot be seen on the surface of section pictured in the left panel of Fig. 5. We have shown in Libert and Henrard (2007), with the help of a simplified model, how these two domains of quasi-periodic motions can coexist without being separated by a critical orbit as it would be the case if we were confronted with a resonance.

For a value of the energy slightly above the value corresponding to the central equilibrium (see the right portrait of Fig. 5), the tangle of the stable and unstable manifolds of the unstable

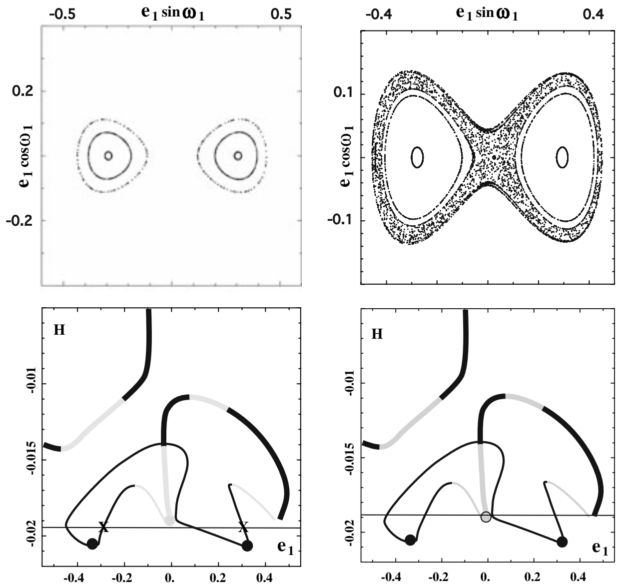

Fig. 5 Levels of energy $\mathcal{K} = -0.0195$ (to the left) and $\mathcal{K} = -0.019$ (to the right). See text for comments

periodic orbit belonging to the family $\mathcal{C}$, generates a chaotic domain which surrounds the regular domains around the orbits of the families $\mathcal{L}$ and $\mathcal{L}'$. The small regular domains around the orbits of the families $\mathcal{S}$ and $\mathcal{S}'$ are engulfed in the chaotic sea and can be barely seen; one as a small dot close to the center of the graph, the other as an elongated island to the left of the graph. We do not think that these regular domains can be described as resonances separated from the main regular domain by a chaotic layer.

When we reach the level $\mathcal{K} = -0.017$ (see the left portrait of Fig. 6), the regular regions centered around $\mathcal{L}$ and $\mathcal{L}'$ shrink as the stable segments of the families of periodic orbits will soon disappear. The regular region around members of the $\mathcal{S}$ and $\mathcal{S}'$ families are still hardly visible. But what becomes an important feature is the island to the right, surrounding a periodic orbit member of the return segment of the family $\mathcal{C}$. This island grows in importance at the level $\mathcal{K} = -0.015$ (see the right portrait of Fig. 6), while the chaotic domain shrinks, reflecting the fact that the unstable member of the family $\mathcal{C}$ becomes less unstable ($\mathcal{U}$ approaches the stability limit $\mathcal{U} = 2$). Also, the islands surrounding the families $\mathcal{L}$ and $\mathcal{L}'$ have disappeared, while smaller islands surrounding this time the families $\mathcal{S}$ and $\mathcal{S}'$ are now visible. The distortions in the curves to the left of the surface of section announce the future intrusion of the family $\mathcal{C}'$.

At the level $\mathcal{K} = -0.013$ (see the left portrait of Fig. 7), the central member of the family $\mathcal{C}$ is stable and the families $\mathcal{S}$ and $\mathcal{S}'$ have disappeared. The surface of section looks like the surface of section of a resonance problem, with the largest member of the family $\mathcal{C}$, playing the role of the center of the resonance and a member of the family $\mathcal{C}'$, playing the role of the unstable orbit generating the separatrix. But this may be misleading. There is no indication of commensurability between frequencies as it would be the case in a genuine resonance.

Indeed at the level $\mathcal{K} = -0.0115$, the member of the family $\mathcal{C}'$ is stable destroying this picture, while at a level a little higher (see the left panel of Fig. 8) the pattern is reversed, it

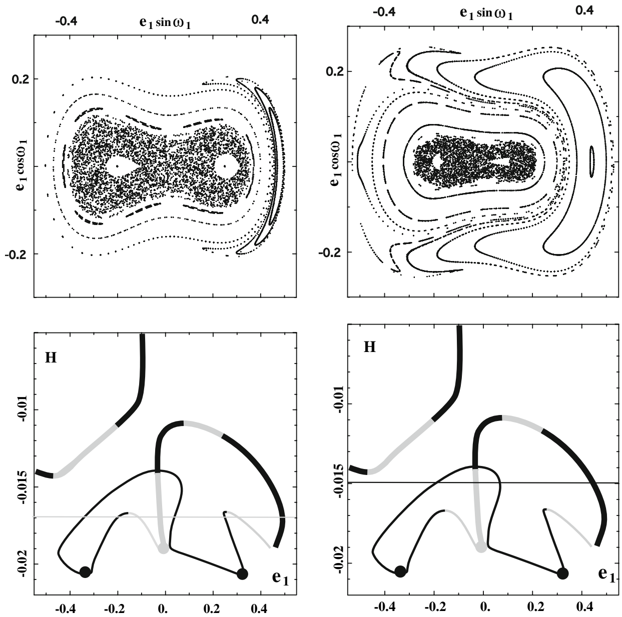

Fig. 6 Levels of energy $\mathcal{K} = -0.017$ (to the left) and $\mathcal{K} = -0.015$ (to the right). See text for comments

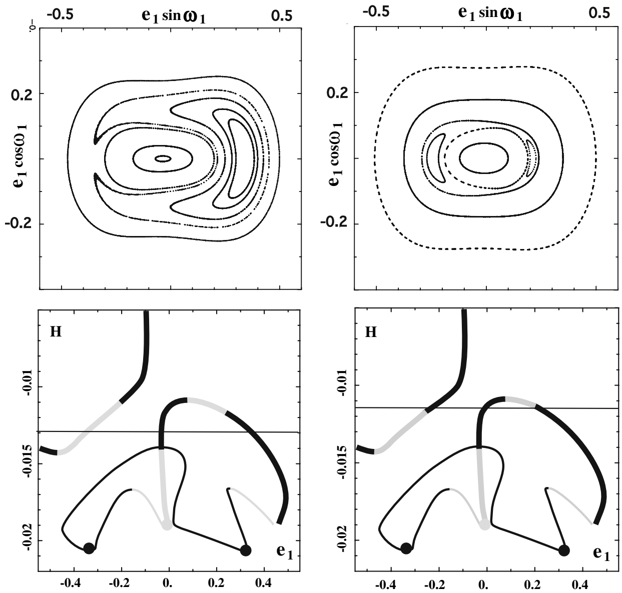

Fig. 7 Levels of energy $\mathcal{K} = -0.013$ (to the left) and $\mathcal{K} = -0.0115$ (to the right). See text for comments

is the member of $\mathcal{C}$ which is unstable and the member of $\mathcal{C}'$ which is stable. For values of $\mathcal{K}$ larger than -0.011, only the family $\mathcal{C}'$ is present, and surrounded by regular quasi-periodic motions (see the right panel of Fig. 8).

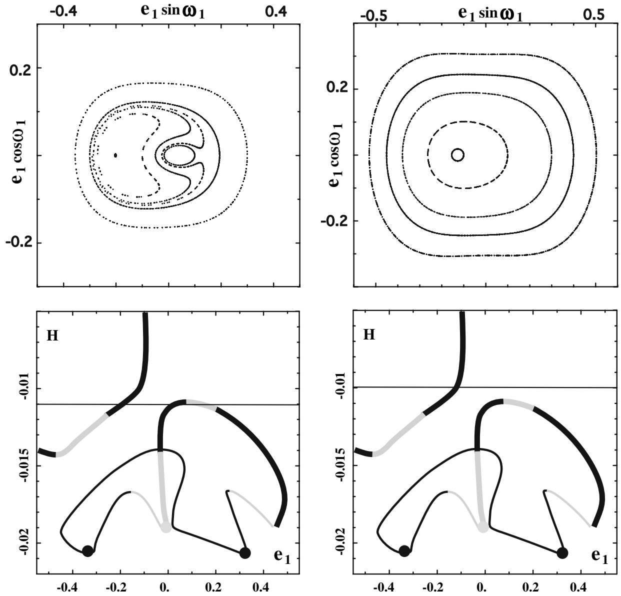

Fig. 8 Levels of energy $\mathcal{K} = -0.011$ (to the left) and $\mathcal{K} = -0.01$ (to the right). See text for comments

6 Conclusion

We have shown that the dynamics of a typical case of the 3D secular planetary problem is governed by the evolution and the stability of the families of periodic orbits emanating from three equilibria: the central equilibrium which, for the value of the parameters we have chosen, is unstable and generates an unstable family of periodic orbits and the two *Kozai equilibria* which are stable, each of them generating two Lyapunov families of stable periodic orbits.

The tangles of stable and unstable manifolds associated with the unstable periodic orbits emanating from the central equilibrium, are responsible for a large chaotic sea, punctured by the islands of regular motions surrounding the Lyapunov families emanating from the Kozai equilibria.

For larger values of the energy $\mathcal{K} > -0.0139$, the families emanating from the Kozai equilibria have disappeared, one after the other and at $\mathcal{K} \approx -0.0139$ the family emanating from the central equilibrium becomes stable. The chaotic domain has disappeared and the surfaces of section are filled with what looks like regular orbits: the problem has become quasi integrable. Up to $\mathcal{K} \approx -0.011$, the dynamics organizes itself around three periodic orbits which change their stability several times. The unstable periodic orbits do not seem to generate macroscopic domain of chaoticity.

For yet larger values of the energy $\mathcal{K} > -0.011$, only one periodic orbit remains and governs the dynamics.

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