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ORIGINAL ARTICLE

New central configurations for the planar 5-body problem

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Abstract In this paper we show the existence of three new families of planar central configurations for the 5-body problem with the following properties: three bodies are on the vertices of an equilateral triangle and the other two bodies are on a perpendicular bisector.

Keywords Planar central configurations · 5-Body problem

1 Introduction

The classical *n-body problem* in celestial mechanics consists in studying the motion of n pointlike masses, interacting amount themselves through no other forces than their mutual gravitational attraction according to Newton's gravitational law (Newton 1687). The equations of motion are given by

$$\ddot{r}_i = -\sum_{\substack{j=1 \\ j \neq i}}^{n} m_j \frac{r_i - r_j}{r_{ij}^3}, \tag{1}$$

for $i = 1, 2, \ldots, n$. Here the gravitational constant is taken equal to one, $r_j \in \mathbb{R}^d$ for $d = 2, 3$ is the position vector of the punctual mass m_j in an inertial system and $r_{ij} = |r_i - r_j|$ is the Euclidean distance between m_i and m_j.

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The center of mass of the system, given by $\sum_{j=1}^{n} m_j r_j / M$, where $M = m_1 + \cdots + m_n$ is the total mass, is considered at the origin of the inertial system. Usually this inertial system is called the *inertial barycentric* system.

Since the general solution of the n-body problem cannot be given, great importance has been attached from the very beginning to the search of particular solutions where the n mass points fulfill certain initial conditions. A *homographic* solution of the n-body problem is a solution such that the configuration of the n-bodies at the instant t (with respect to the inertial barycentric system) remains similar to itself as t varies. Here *similar* means that we can pass from one configuration to the other doing a dilation and a rotation of $\mathbb{R}^d$.

The first three homographic solutions were found in 1767 by Euler (1767) in the 3-body problem, for which three bodies are *collinear* at any time. In 1772 Lagrange (1873) found two additional homographic solutions in the 3-body problem, where the three bodies are at any time in the vertices of an *equilateral triangle*.

At a given instant $t = t_0$ the configuration of the n bodies is *central* if the gravitational acceleration $\ddot{r}_j$ acting on every mass point m_j is proportional to its position r_j (referred to the inertial barycentric system), that is $\ddot{r}_j = \lambda r_j$ with $\lambda \neq 0$ for all $j = 1, \ldots, n$. So Eq. 1 can be written as

$$\lambda r_i = - \sum_{\substack{j=1 \\ j \neq i}}^{n} m_j \frac{r_i - r_j}{r_{ij}^3}, \tag{2}$$

for $i = 1, 2, \ldots, n$. If we have a central configuration, any dilation and any rotation (centered at the center of mass) of it provides another central configuration. Two central configurations are *related* if we can pass from one to another through a dilation and a rotation. So we can study the classes of central configurations defined by the above equivalence relation. Thus the 3-body problem has exactly 5 classes of central configurations for any value of the positive masses.

Central configurations and homographic solutions are linked by the Laplace theorem (see for instance Boccaletti and Pucacco 1996; Wintner 1941): the configuration of the n bodies in a homographic solution is central at any instant of time.

Central configurations of the n-body problem are important because: they allow to compute homographic solutions; if the n bodies are going to a simultaneous collision then the bodies tend to a central configuration (see Saari 1980); there is a relation between central configurations and the bifurcations of the hypersurfaces of constant energy and angular momentum (see Smale 1970). See also the Refs. Llibre 1991; Moeckel 1990; Moulton 1910.

In this paper we are only interested in planar central configurations, that is $d = 2$. The main general open problem for these central configurations is due to Wintner 1941 and Smale (1998): *Is the number of classes of planar central configurations finite for any choice of the (positive) masses* $m_1, \ldots, m_n$?

Recently Hampton and Moeckel in (2006) proved this conjecture for the 4-body problem. The conjecture remains open for $n > 4$. But if one mass can be negative Roberts (1999) proved that there exists a one-parameter not equivalent family of planar central configurations for the 5-body problem.

In 2005 Hampton (2005) provides a new family of planar central configurations for the 5-body problem with an interesting property: the central configuration has a subset of three bodies forming a central configuration of the 3-body problem, in fact these three bodies are in an equilateral triangle. Other central configurations in the spatial 5-body and 7-body problems have been studied by Santos (2004) and Hampton and Santoprete (2007), respectively.

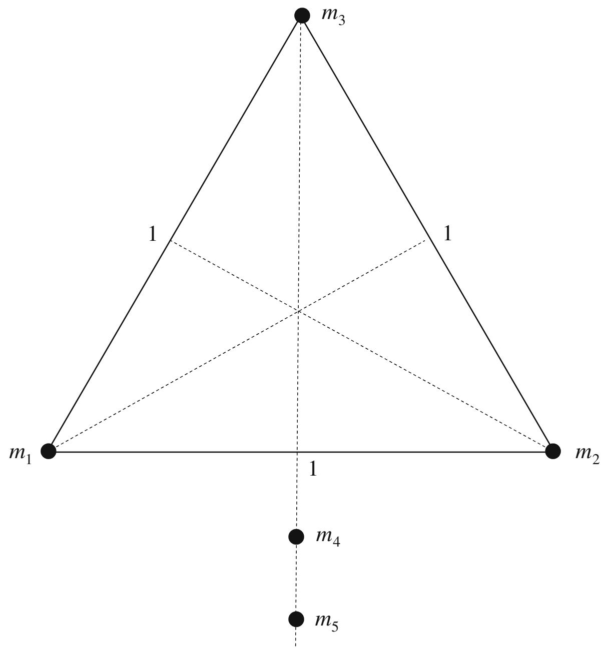

Fig. 1 Five bodies in the plane

In this paper we find a new class of central configurations of the 5-bodies which as the ones studied by Hampton (2005) have three bodies in the vertices of an equilateral triangle, but the other two instead of be located symmetrically with respect to a perpendicular bisector (the ones found by Hampton), are on the perpendicular bisector, see Fig. 1.

Consider an equilateral triangle whose sides have length 1 and let A, B, C, D, E and F be the following points on the perpendicular bisector (see Fig. 2): A is the barycenter of the triangle, B is the vertex at m_3, C is the point symmetric of the point B relative to the side that contains m_1 and m_2, D and F are the points where the circle with center at B and radius 1 meet the perpendicular bisector and E is the middle point of the side that contains m_1 and m_2. Let AB be the segment on the perpendicular bisector whose endpoints are A and B, and in a similar way are defined the segments BF, CD and EA.

Theorem 1 *Assume that we have the masses m_1, m_2 and m_3 at the vertices of an equilateral triangle whose sides have length* 1 *and masses m_4 and m_5 on the perpendicular bisector according to Fig.* 1*. In order that the five masses can be in a central configuration the following statements hold.*

(a) *The two masses m_1 and m_2 are equal.*
(b) *Only one of the masses m_4 or m_5 must be in the segment AB. See Fig.* 2.

Without loss of generality we can assume that $m_4 \in AB$. Then there exist a position $G \in AB$, non-empty segments $I^1(G) \subset CD$, $I^2(G) \subset EA$, $I^3(G) \subset BF$ and positive masses $m_1 = m_2$, m_3, m_4 and $m_5(i)$, $i = 1, 2, 3$, such that m_1, m_2 and m_3 are at the vertices of the triangle, m_4 is at G and $m_5(1) \in I^1(G)$, or $m_5(2) \in I^2(G)$, or $m_5(3) \in I^3(G)$, form three central configurations. See Fig. 2.

Numerical results presented at the end of the paper provides numerical evidence of the following improvement of Theorem 1.

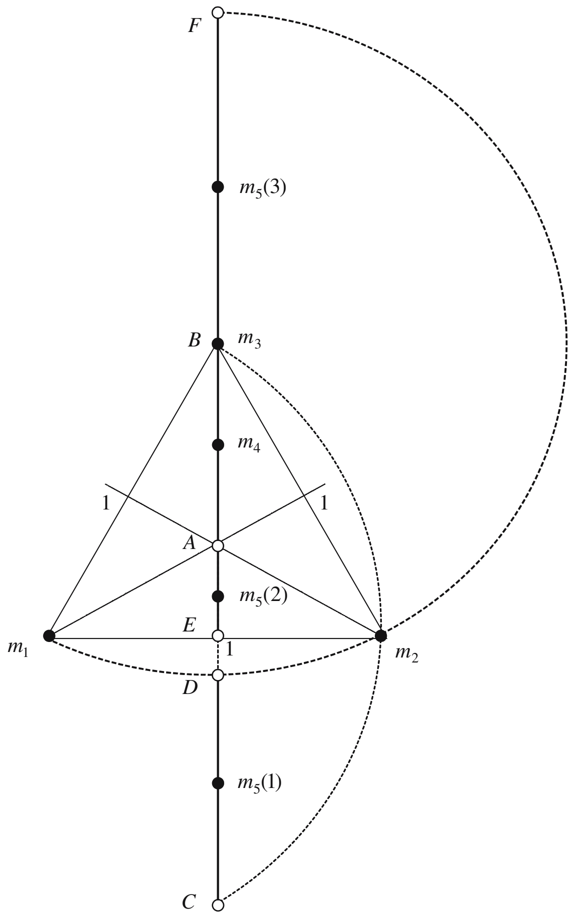

Fig. 2 Five bodies in a central configuration

Without loss of generality we can assume that $m_4 \in AB$. Then for all position $G \in AB$, there exist non–empty segments $I^1(G) \subset CD$, $I^2(G) \subset EA$, $I^3(G) \subset BF$ and positive masses $m_1 = m_2$, m_3, m_4 and $m_5(i)$, $i = 1, 2, 3$, such that m_1, m_2 and m_3 are at the vertices of the triangle, m_4 is at G and $m_5(1) \in I^1(G)$, or $m_5(2) \in I^2(G)$, or $m_5(3) \in I^3(G)$, form three different central configurations.

The proof of Theorem 1, given in the next section, and the proof of Hampton's theorem (2005) use the Laura-Andoyer or Dziobek equations, but they are very different.

2 Proof of Theorem 1

For the planar central configurations instead of working with Eq. 2 we shall do with the Dziobek equations (see Hagihara 2005, p. 241)

$$f_{ij} = \sum_{\substack{k=1 \\ k \neq i,j}}^{n} m_k \,(R_{ik} - R_{jk})\, \Delta_{ijk} = 0, \tag{3}$$

for $1 \leq i < j \leq n$, where $R_{ij} = 1/r_{ij}^3$ and $\Delta_{ijk} = (r_i - r_j) \wedge (r_i - r_k)$.

For the 5-body problem equations (3) is a set of 10 equations. Our class of configurations with five bodies as in Fig. 1 without collisions must satisfy

$$r_{12} = r_{23} = r_{13} = 1, \quad r_{14} = r_{24}, \quad r_{15} = r_{25}, \quad r_{34} \neq 0, \quad r_{35} \neq 0, \quad r_{45} \neq 0,$$
$$\Delta_{345} = 0, \quad \Delta_{153} = \Delta_{235}, \quad \Delta_{143} = \Delta_{234}, \quad \Delta_{154} = \Delta_{245}.$$

Without loss of generality we can take a coordinate system such that the masses m_1, m_2 and m_3 are at $(-1/2, 0)$, $(1/2, 0)$ and $(0, \sqrt{3}/2)$ respectively. Thus $A = (0, \sqrt{3}/6)$, $B = (0, \sqrt{3}/2)$, $C = (0, -\sqrt{3}/2)$, $D = (0, -1+\sqrt{3}/2)$, $E = (0, 0)$ and $F = (0, 1+\sqrt{3}/2)$.

Suppose $r_{14} = r_{12} = 1$. Then m_4 must be at the point $(0, -\sqrt{3}/2)$. The equations $f_{13} = 0$ and $f_{14} = 0$ of (3) become

$$m_4(R_{12} - R_{34})\Delta_{134} + m_5(R_{15} - R_{35})\Delta_{135} = 0,$$
$$m_3(R_{12} - R_{34})\Delta_{143} + m_5(R_{15} - R_{45})\Delta_{145} = 0,$$

respectively. As $(R_{12} - R_{34})\Delta_{134} < 0$ and $(R_{12} - R_{34})\Delta_{143} > 0$ in order to have positive solutions of the above equations one must have $(R_{15} - R_{35})\Delta_{135} > 0$ in the first equation and $(R_{15} - R_{45})\Delta_{145} < 0$ in the second one. But there is no position for m_5 such that these inequalities hold. The same arguments are valid if $r_{15} = r_{12} = 1$.

From the previous paragraph we can assume that $r_{14} \neq r_{12}$ and $r_{15} \neq r_{12}$. Then equations $f_{34} = 0$ and $f_{35} = 0$ of (3) become

$$(R_{12} - R_{14})\Delta_{134}(m_1 - m_2) = 0, \quad (R_{12} - R_{15})\Delta_{135}(m_1 - m_2) = 0,$$

respectively. Thus $m_1 = m_2$. So statement (a) of Theorem 1 is proved.

Taking into account that $m_1 = m_2$ in the other 8 equations of (3), they reduce to the following 3 equations:

$$(R_{14} - R_{34})\Delta_{134}m_4 + (R_{15} - R_{35})\Delta_{135}m_5 = 0, \tag{4}$$
$$(R_{12} - R_{14})\Delta_{124}m_1 + (R_{12} - R_{34})\Delta_{134}m_3 + (R_{15} - R_{45})\Delta_{154}m_5 = 0, \tag{5}$$
$$(R_{12} - R_{15})\Delta_{125}m_1 + (R_{12} - R_{35})\Delta_{135}m_3 + (R_{14} - R_{45})\Delta_{145}m_4 = 0. \tag{6}$$

If $r_{14} = r_{34}$, that is m_4 is at the barycenter of the equilateral triangle, then from (4) one has $r_{15} = r_{35}$, for positive mass m_5. Therefore $r_{45} = 0$, and we do not allow that two masses are colliding in a central configuration. Hence $r_{14} \neq r_{34}$. Analogous conclusion occurs when $r_{15} = r_{35}$. In short in what follows we consider only the cases where $r_{14} \neq r_{34}$ and $r_{15} \neq r_{35}$.

From (4) one has that $(R_{14} - R_{34})\Delta_{134}$ and $(R_{35} - R_{15})\Delta_{135}$ must have the same sign. We distinguish two cases.

Case 1. Consider $(R_{14} - R_{34})\Delta_{134} > 0$ and $(R_{35} - R_{15})\Delta_{135} > 0$. If $\Delta_{134} > 0$ then $r_{14} > r_{34}$ and this implies that $R_{14} < R_{34}$. Thus the product $(R_{14} - R_{34})\Delta_{134}$ is negative. So the only case of interest is when $\Delta_{134} < 0$. Thus $r_{14} > r_{34}$ in order to have $(R_{14} - R_{34})\Delta_{134} > 0$. But this implies that m_4 must be in the segment AB. From $(R_{35} - R_{15})\Delta_{135} > 0$ one has two possibilities. If $\Delta_{135} > 0$ then $r_{35} < r_{15}$, and if $\Delta_{135} < 0$ then $r_{35} > r_{15}$.

Case 2. Consider $(R_{14} - R_{34})\Delta_{134} < 0$ and $(R_{35} - R_{15})\Delta_{135} < 0$. If $\Delta_{135} > 0$ then $r_{15} > r_{35}$ and this implies that $R_{15} < R_{35}$. Thus the product $(R_{35} - R_{15})\Delta_{135}$ is positive. So the only case of interest is when $\Delta_{135} < 0$. Thus $r_{15} > r_{35}$ in order to have $(R_{35} - R_{15})\Delta_{135} < 0$. But this implies that m_5 must be in the segment AB. From $(R_{14} - R_{34})\Delta_{134} < 0$ one has two possibilities. If $\Delta_{134} > 0$ then $r_{34} < r_{14}$ and if $\Delta_{135} < 0$ then $r_{34} > r_{14}$.

From these two cases we see that only one of the masses m_4 or m_5 must be in the segment AB. Hence statement (b) of Theorem 1 is proved.

Without loss of generality we can assume that m_4 is located at the point $(0, x) \in AB$ and that m_5 at the point $(0, y)$ satisfies either $y < \sqrt{3}/6$ or $y > \sqrt{3}/2$. Thus from (4) the masses m_4 and m_5 are related by

$$m_4 = \frac{(R_{35} - R_{15})\Delta_{135}}{(R_{14} - R_{34})\Delta_{134}} m_5. \tag{7}$$

Substituting (7) into equation (6) results

$$\begin{aligned}(R_{12} - R_{15})\Delta_{125}m_1 + (R_{12} - R_{35})\Delta_{135}m_3 \\ + \frac{(R_{14} - R_{45})(R_{35} - R_{15})\Delta_{145}\Delta_{135}}{(R_{14} - R_{34})\Delta_{134}} m_5 = 0.\end{aligned} \tag{8}$$

Case 1. Consider $y < \sqrt{3}/6$. One has three possibilities: either $\Delta_{125} > 0$, or $\Delta_{125} = 0$, or $\Delta_{125} < 0$. If $\Delta_{125} = 0$, that is $y = 0$, then the first coefficient of the left–hand side of Eq. (8) vanishes while the second one is positive since $R_{12} - R_{35} < 0$ and $\Delta_{135} < 0$. As $R_{35} - R_{15} < 0$, $R_{14} - R_{45} < 0$, $R_{14} - R_{34} < 0$, $\Delta_{145} < 0$ and $\Delta_{134} < 0$ then the third coefficient of the left–hand side of (8) is positive too. Therefore there are no positive masses m_3 and m_5 satisfying this equation.

Consider $\Delta_{125} < 0$, that is $y < 0$. By the same arguments as above one see that if either $-1 + \sqrt{3}/2 \le y < 0$ or $y \le -\sqrt{3}/2$ then there are no positive masses m_1, m_3 and m_5 satisfying equation (8). Thus the only possibility of solution of Eq. (8) in this case is $-\sqrt{3}/2 < y < -1 + \sqrt{3}/2$, that is $(0, y) \in CD$.

If $\Delta_{125} > 0$ then $0 < y < \sqrt{3}/6$, that is $(0, y) \in EA$.

Case 2. Consider $y > \sqrt{3}/2$. In this case $R_{12} - R_{15} > 0$, $\Delta_{125} > 0$, $\Delta_{135} > 0$, $R_{35} - R_{15} > 0$, $\Delta_{145} > 0$, $R_{14} - R_{34} < 0$ and $\Delta_{134} < 0$. If $y > 1 + \sqrt{3}/2$ then $R_{14} - R_{45} > 0$ and thus the first and third coefficients of the left–hand side of (8) are positive. Therefore the second one must be negative, that is $R_{12} - R_{35} < 0$. But this implies that $r_{35} < 1$ and one has a contradiction. If $y = 1 + \sqrt{3}/2$ then $R_{12} = R_{35}$ and $R_{14} - R_{45} \ge 0$. Thus the second coefficient of the left–hand side of (8) vanishes while the third one is larger than or equal to zero. Therefore there are no positive masses m_1 and m_5 satisfying this equation. In short $(0, y) \in BF$.

From the above cases 1 and 2 the only possibilities for positive masses m_1, m_3 and m_5 to be solutions of the system defined by Eq. 5, 7 and 8 are $(0, x) \in AB$ and either $(0, y) \in CD$, or $(0, y) \in EA$, or $(0, y) \in BF$.

Equations 5 and 8 define two planes through the origin in the space (m_1, m_3, m_5). The normal vectors of these planes are, respectively, $n_1 = (n_{11}, n_{12}, n_{13})$ and $n_2 = (n_{21}, n_{22}, n_{23})$, where

$$\begin{aligned}
n_{11} &= x\left(1 - \frac{8}{(1 + 4x^2)^{3/2}}\right), \\
n_{12} &= \frac{2x - \sqrt{3}}{4}\left(1 - \frac{8}{(\sqrt{3} - 2x)^3}\right), \\
n_{13} &= \frac{x - y}{2}\left(\frac{8}{(1 + 4y^2)^{3/2}} - \frac{1}{((x - y)^2)^{3/2}}\right),
\end{aligned}$$

$$n_{21} = y\left(1 - \frac{8}{(1+4y^2)^{3/2}}\right),$$
$$n_{22} = \frac{2y-\sqrt{3}}{4}\left(1 - \frac{8}{((2y-\sqrt{3})^2)^{3/2}}\right),$$
$$n_{23} = \frac{(y-x)(2y-\sqrt{3})}{2(2x-\sqrt{3})}\left(\frac{8}{(1+4x^2)^{3/2}} - \frac{1}{((x-y)^2)^{3/2}}\right)\frac{\left(\frac{1}{((2y-\sqrt{3})^2)^{3/2}} - \frac{1}{(1+4y^2)^{3/2}}\right)}{\left(\frac{1}{(1+4x^2)^{3/2}} - \frac{1}{(\sqrt{3}-2x)^3}\right)}.$$

Let $T = (T_1, T_2, T_3) = n_1 \wedge n_2 = (n_{12}n_{23} - n_{13}n_{22}, n_{13}n_{21} - n_{11}n_{23}, n_{11}n_{22} - n_{12}n_{21})$ be the vector parallel to the straight line defined by the intersection of the two planes. Thus there will be positive masses m_1, m_3 and m_5 solutions of Eqs. 5 and 8 if and only if the components of the vector T have the same sign.

Let $G = (0, 2/5) \in AB$, $H_1 = (0, -2/5) \in CD$, $H_2 = (0, 1/5) \in EA$ and $H_3 = (0, 6/5) \in BF$. From G and H_1 define the point $(x_0, y_0) = (2/5, -2/5)$, from G and H_2 define the point $(x_0, y_1) = (2/5, 1/5)$ and from G and H_3 define the point $(x_0, y_2) = (2/5, 6/5)$. By a straightforward calculation we have $T_1(x_0, y_0) < 0$, $T_2(x_0, y_0) < 0$, $T_3(x_0, y_0) < 0$, $T_1(x_0, y_1) > 0$, $T_2(x_0, y_1) > 0$, $T_3(x_0, y_1) > 0$, $T_1(x_0, y_2) > 0$, $T_2(x_0, y_2) > 0$ and $T_3(x_0, y_2) > 0$. Therefore there are neighborhoods of (x_0, y_0), (x_0, y_1) and (x_0, y_2) such that the signs of the functions T_1, T_2 and T_3 are as above.

The existence of the segments $I^1(G) \subset CD$, $I^2(G) \subset EA$ and $I^3(G) \subset BF$ follows from the intersections of the above neighborhoods with the straight line $x = x_0$.

In short we have proved Theorem 1.

The curves $T_1(x, y) = 0$ (dot), $T_2(x, y) = 0$ (solid) and $T_3(x, y) = 0$ (dash) are shown in Fig. 3 for $\sqrt{3}/6 < x < \sqrt{3}/2$ and $-\sqrt{3}/2 < y < -\sqrt{3}/6$ (region 1). So if we write $y_i(x)$ for the explicit form of the implicit curve $T_i(x, y_i) = 0$ in the region 1, then we have $y_3(x) < y_2(x) < y_1(x) < 0$. Then the three components T_1, T_2 and T_3 of the vector T are negative in the subset of the region 1 defined by $\{(x, y) : \sqrt{3}/6 < x < \sqrt{3}/2$ and $y_3(x) < y < y_2(x)\}$. This subset contains the point (x_0, y_0).

The curves $T_2(x, y) = 0$ (solid) and $T_3(x, y) = 0$ (dash) are given in Fig. 4 for $\sqrt{3}/6 < x < \sqrt{3}/2$ and $0 < y < \sqrt{3}/6$ (region 2). We note that the curve $T_1(x, y) = 0$ has no points in region 2. Using the notation of the previous paragraph the components T_1, T_2 and T_3 of

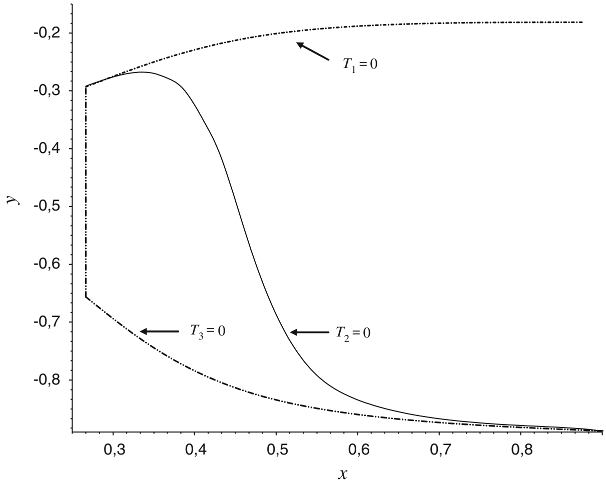

Fig. 3 Curves $T_1 = 0$, $T_2 = 0$ and $T_3 = 0$ for region 1

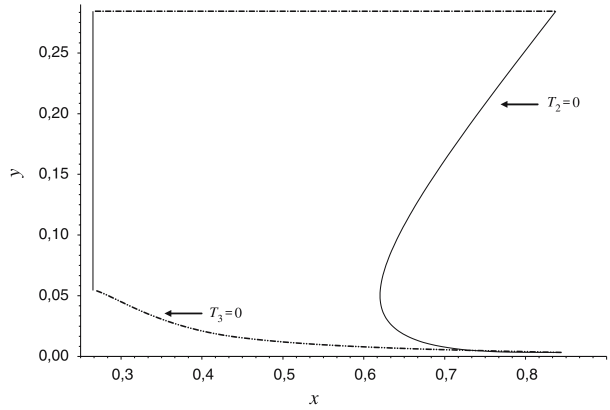

Fig. 4 Curves $T_2 = 0$ and $T_3 = 0$ for region 2

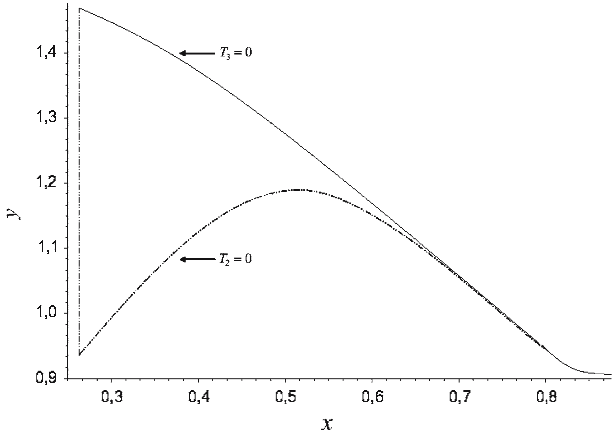

Fig. 5 Curves $T_2 = 0$ and $T_3 = 0$ for region 3

the vector T are positive in the subset of region 2 limited by the four curves $\{x = \sqrt{3}/6\}$, $\{y = \sqrt{3}/6\}$, $\{(x, y_2(x))\}$ and $\{(x, y_3(x))\}$. Note that in fact the curve $T_2 = 0$ is not a graph with respect to the x-axis. Really it must be decomposed as two graphs of the form $\{(x, y_2(x))\}$. The above mentioned subset contains the point (x_0, y_1).

The curves $T_2(x, y) = 0$ (dash) and $T_3(x, y) = 0$ (solid) are illustrated in Fig. 5 for $\sqrt{3}/6 < x < \sqrt{3}/2$ and $\sqrt{3}/2 < y < 1 + \sqrt{3}/2$ (region 3). We note that the curve $T_1(x, y) = 0$ has no points in region 3. The components T_1, T_2 and T_3 of the vector T are positive in the subset of the region 3 defined by $\{(x, y) : \sqrt{3}/6 < x < \sqrt{3}/2$ and $y_2(x) < y < y_3(x)\}$. This subset contains the point (x_0, y_2).

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