Celestial Mech Dyn Astr (2007) 99:85–104
DOI 10.1007/s10569-007-9088-3

ORIGINAL ARTICLE

The manifolds of families of 3D periodic orbits associated to Sitnikov motions in the restricted three-body problem

E. A. Perdios

Received: 5 February 2007 / Revised: 11 May 2007 / Accepted: 25 May 2007 /
Published online: 30 August 2007

Abstract This paper deals with the Sitnikov family of straight-line motions of the circular restricted three-body problem, viewed as generator of families of three-dimensional periodic orbits. We study the linear stability of the family, determine several new critical orbits at which families of three dimensional periodic orbits of the same or double period bifurcate and present an extensive numerical exploration of the bifurcating families. In the case of the same period bifurcations, 44 families are determined. All these families are computed for equal as well as for nearly equal primaries ($\mu = 0.5,\ \mu = 0.4995$). Some of the bifurcating families are determined for all values of the mass parameter μ for which they exist. Examples of families of three dimensional periodic orbits bifurcating from the Sitnikov family at double period bifurcations are also given. These are the only families of three-dimensional periodic orbits presented in the paper which do not terminate with coplanar orbits and some of them contain stable parts. By contrast, all families bifurcating at single-period bifurcations consist entirely of unstable orbits and terminate with coplanar orbits.

Keywords Sitnikov motions · Restricted three-body problem · Three dimensional periodic orbits · Stability

1 Introduction

The Sitnikov problem is an interesting special case of the restricted three-body problem when the two primaries are of equal mass and the third body of negligible mass performs straight-line oscillations along the z-axis (perpendicular to the plane of the primaries) (Sitnikov 1960). The case of elliptic motion of the primaries (the elliptic Sitnikov problem) has attracted the interest of many researchers (see Liu and Sun 1990; Hagel 1992; Alfaro and Chiralt 1993; Dvorak 1993; Faruque 2003; Hagel and Lhotka 2005, among others).

E. A. Perdios (✉)
Department of Engineering Sciences, University of Patras, Patras 26500, Greece
e-mail: e.perdios@des.upatras.gr

In the case of circular motion of the primaries (the circular Sitnikov problem) Perdios and Markellos (1988) studied numerically the stability and bifurcations of Sitnikov motions and presented examples of the continuation of these bifurcations into the case of non-equal masses of the primaries. They showed that the Sitnikov problem is of importance in the sense that it can be used as a generator of families of three-dimensional periodic orbits of the restricted problem. Belbruno et al. (1994) extended these results and also studied the problem analytically using elliptic functions.

In this paper we study the classical (circular) Sitnikov problem. Without the loss of generality we take the origin of time to be at the instant when the particle leaves the plane of the primaries upwards, and consider only the motions reaching a maximum height on the z-axis: $z(T/4) \leqslant 10$, where T is the period of the orbit. Up to this value 44 critical orbits of the Sitnikov family are determined, from which families of three-dimensional periodic orbits of the same period bifurcate. We compute the families of three-dimensional periodic orbits bifurcating from the computed critical Sitnikov orbits. We also compute these families for nearly equal masses of the primaries, i.e. $\mu = 0.4995$. We plot the bifurcating family characteristics in the appropriate plane of parameters (initial conditions), and thus illustrate graphically the resulting manifold of the families. For some of the families computed for $\mu = 0.4995$ we also give their critical orbits at which other families of three-dimensional periodic orbits, of the same or double period, bifurcate. Some of the families are continued numerically for all values of the mass parameter for which they exist.

Finally, we explore the existence of period-doubling bifurcations from the Sitnikov motions occurring when a stability index takes the value 2. Several such orbits for $z(T/4) \leqslant 10$ are found and we present some bifurcating families of three-dimensional periodic orbits bifurcating at these points. These are the only families of three-dimensional periodic orbits presented in the paper which do not terminate with coplanar orbits and some of them consist of stable parts. By contrast, the families bifurcating at single-period bifurcations contain only unstable orbits and terminate with coplanar orbits.

2 Equations of motion

We consider the usual barycentric, rotating and dimensionless coordinate system $Oxyz$, where the Ox axis always contains the two main bodies having masses $m_1 = 1 - \mu$ and $m_2 = \mu \leqslant 1/2$. The equations of motion of a third body of negligible mass moving under the gravitational attraction of the two primaries are (Szebehely 1967):

$$\ddot{x} - 2\dot{y} = x - \frac{1-\mu}{r_1^3}(x+\mu) - \frac{\mu}{r_2^3}(x+\mu-1) = \frac{\partial \Omega}{\partial x},$$

$$\ddot{y} + 2\dot{x} = y\left(1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right) = \frac{\partial \Omega}{\partial y}, \tag{1}$$

$$\ddot{z} = z\left(-\frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right) = \frac{\partial \Omega}{\partial z},$$

where Ω is the potential function, $\mu = m_2/(m_1 + m_2)$ and:

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}, \qquad r_2 = \sqrt{(x+\mu-1)^2 + y^2 + z^2}, \tag{2}$$

are the distances of the moving body from the two primaries. System (1) admits the following integral:

$$C = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}(x^2 + y^2) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \tag{3}$$

where C is the Jacobi constant. The Sitnikov motion $z(t)$ of the restricted three-body problem is described by:

$$\ddot{z} = -\frac{z}{(z^2 + \frac{1}{4})^{3/2}}, \tag{4}$$

and can be easily obtained from System (1) for $\mu = 0.5$ and $x(t) = y(t) = 0$, while the equation of the Jacobi integral becomes:

$$C = \frac{1}{2}\left[\dot{z}^2 - \frac{2}{(z^2 + \frac{1}{4})^{1/2}}\right]. \tag{5}$$

3 Stability of the Sitnikov family—critical orbits

We now consider small perturbations $x = \xi$ and $y = \eta$ of the zero horizontal components of the rectilinear motion. The linearized equations of the perturbed motion are:

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= [F_1(z) + F_2(z)]\xi + F_3(z), \\ \ddot{\eta} + 2\dot{\xi} &= F_1(z)\eta, \\ \ddot{z} &= [F_1(z) - 1]z + F_4(z)\xi z, \end{aligned} \tag{6}$$

as determined from (1), where we have abbreviated:

$$\begin{aligned} F_1(z) &= 1 - \left(\frac{1-\mu}{g_1^{3/2}} + \frac{\mu}{g_2^{3/2}}\right), & F_2(z) &= 3\mu(1-\mu)\left(\frac{\mu}{g_1^{5/2}} + \frac{1-\mu}{g_2^{5/2}}\right), \\ F_3(z) &= -\mu(1-\mu)\left(\frac{1}{g_1^{3/2}} - \frac{1}{g_2^{3/2}}\right), & F_4(z) &= 3\mu(1-\mu)\left(\frac{1}{g_1^{5/2}} - \frac{1}{g_2^{5/2}}\right), \end{aligned} \tag{7}$$

where $g_1 = \mu^2 + z^2$ and $g_2 = (\mu - 1)^2 + z^2$. For $\mu = 1/2$ we obtain $F_3(z) = F_4(z) = 0$ and:

$$F_1(z) = 1 - \frac{1}{(z^2 + 1/4)^{3/2}} = F_{10}(z), \qquad F_2(z) = \frac{3}{4(z^2 + 1/4)^{5/2}} = F_{20}(z). \tag{8}$$

Now, the first two equations of System (6) can be written in the form:

$$\dot{\Xi} = A(z(t))\Xi, \tag{9}$$

where $\Xi = (\xi, \eta, \dot{\xi}, \dot{\eta})^{\mathrm{T}}$ and:

$$A(z(t)) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ F_{10}(z) + F_{20}(z) & 0 & 0 & 2 \\ 0 & F_{10}(z) & -2 & 0 \end{bmatrix}. \tag{10}$$

Equation (9) is the variational equations of the rectilinear motion and describe the evolution of the planar perturbations ξ and η of the Sitnikov motion in the restricted three-body problem. The characteristic roots for the equations of variation will determine the "transversal" stability of the Sitnikov motions and can be computed by a numerical technique based on the classical Floquet theory. The characteristic roots s_k, $k = 1, 2, 3, 4$, are the solutions of the characteristic polynomial $\det(B - Is) = 0$, where I is the four–dimensional identity matrix and:

$$B = X^{-1}(t)X(t+T), \tag{11}$$

where $X(t)$ is a fundamental solution of Equation (9) and T is the period of a particular solution of System (4). Without the loss of generality we can set $X(0) = I$, so $B = X(T)$. In case where the roots of the characteristic equation are distinct there are four independent solutions x_k satisfying the property:

$$x_k(t+T) = s_k x_k(t), \qquad k = 1, 2, 3, 4. \tag{12}$$

Thus, a solution is periodic if s_k is unity, while in case of $|s_k| < 1$ ($|s_k| > 1$) the motion is bounded (unbounded). The characteristic equation is quartic and can be written as the product of two quadratic factors:

$$(s^2 + a_1 s + 1)\,(s^2 + a_2 s + 1) = 0, \tag{13}$$

with

$$a_1 = \frac{1}{2}(p_1 + \sqrt{\Delta}), \qquad a_2 = \frac{1}{2}(p_1 - \sqrt{\Delta}), \qquad \Delta = p_1^2 - 4(p_2 - 2), \tag{14}$$

where we have abbreviated:

$$p_1 = -\mathrm{Tr}\, B, \qquad p_2 = \sum_{j=i+1}^{4} \sum_{i=1}^{4} (b_{ii}b_{jj} - b_{ij}b_{ji}), \tag{15}$$

and b_{ij}, $i, j = 1, 2, 3, 4$ are the elements of the matrix B. The stability conditions are:

$$\Delta > 0, \qquad |a_1| \leqslant 2, \qquad |a_2| \leqslant 2. \tag{16}$$

For economy in computing time the matrix B can be determined by integrating numerically the fundamental solution matrix from $t = 0$ to $t = T/4$, and applying the transformation:

$$X(T) = [MX^{-1}(T/4)MX(T/4)]^2, \tag{17}$$

where M is the constant symmetry matrix $M = \mathrm{diag}\{1, -1, -1, 1\}$. The above described method for the determination of the stability has been proposed by Perdios and Markellos (1988) and successfully applied by them and by Belbruno et al. (1994) in the case of the classical Sitnikov problem.

In Table 1 we give the stability intervals of the Sitnikov family for $z(T/4) \leqslant 10$. The stability diagrams of the Sitnikov family can be found in Perdios and Markellos (1988) and Belbruno et al. (1994).

Of importance here are the critical orbits of the Sitnikov family which mark the bifurcations from the rectilinear motion of other families of three-dimensional periodic orbits of the same or double period. We call the critical orbits of the rectilinear motion *one-to-one critical points/orbits* when at these points families of 3D periodic orbits of the same period bifurcate and *one-to-two critical points/orbits* in case where families of 3D periodic orbits of double

Table 1 Intervals of $\dot{z}_0$ for $z(T/4) \leqslant 10$ at which the Sitnikov motion is stable

[1.89880,1.89895],	[1.90655,1.90680],	[1.91295,1.91325],	[1.91835,1.91870],	[1.92300,1.92335],
[1.92705,1.92740],	[1.93060,1.93095],	[1.93375,1.93410],	[1.93660,1.93690],	[1.93915,1.93940],
[1.94145,1.94170],	[1.94355,1.94380],	[1.94545,1.94570],	[1.94720,1.94745],	[1.94885,1.94905]

period bifurcate. A point of the Sitnikov family is considered to be one-to-one or one-to-two critical if:

$$a_i(\dot{z}_0) = \pm 2, \qquad i = 1, 2. \tag{18}$$

Linearizing the above criticality condition we easily obtain the following corrector scheme:

$$\frac{\partial a_i}{\partial \dot{z}_0}\delta \dot{z}_0 = \pm 2 - a_i, \qquad i = 1, 2, \tag{19}$$

where the partial derivatives involved in this equation can be computed by additional integrations.

4 The manifold of families of 3D periodic orbits

4.1 One-to-one bifurcations

In this study we compute critical orbits of the Sitnikov family for which $z(T/4) \leqslant 10$, where T is the period of the orbit. Our main results refer to one-to-one critical points occurring when $a_1 = -2$. Up to this value of $z(T/4)$, 44 one-to-one critical points of the Sitnikov family exist. The first 34 critical points have been determined by Belbruno et al. (1994). The remaining critical orbits are presented in Table 2 where the initial velocity $\dot{z}_0$, the value of z at $T/4$, the stability parameters a_i, $i = 1, 2$, the quarter period $T/4$ and the Jacobi constant C are given.

Table 2 One-to-one critical points of the Sitnikov family

	$\dot{z}_0$	$z(t/4)$	a_1	a_2	$T/4$	C
B35	1.94141103	8.64644301	1.70385	−2.00000	28.52406634	−0.11546160
B36	1.94172797	8.69292647	0.37073	−2.00000	28.75193500	−0.11484624
B37	1.94350828	8.96371059	1.76351	−2.00000	30.09146734	−0.11138779
B38	1.94380999	9.01129889	0.45120	−2.00000	30.32900096	−0.11080136
B39	1.94542819	9.27550139	1.81195	−2.00000	31.65915515	−0.10765459
B40	1.94571532	9.32402432	0.52333	−2.00000	31.90553597	−0.10709595
B41	1.94719353	9.58217285	1.85137	−2.00000	33.22709292	−0.10421868
B42	1.94746682	9.63148840	0.58835	−2.00000	33.48160853	−0.10368649
B43	1.94882325	9.88404371	1.88349	−2.00000	34.79525003	−0.10104396
B44	1.94908348	9.93403249	0.64725	−2.00000	35.05727555	−0.10053679

4.2 Determination of families of 3D periodic orbits and their stability

Three-dimensional periodic orbits of the families bifurcating at the critical orbits of the Sitnikov family are of two types of symmetries:

(S1) double symmetry with respect to the Ox-axis and the Oxz plane and
(S2) double symmetry with respect to the Ox-axis and the Oyz plane.

Note that when $\mu \neq 0.5$ the symmetry w.r.t. the Oyz plane does not exist and the orbits can be computed using the simple symmetry w.r.t. the Ox–axis. In this paper we compute the bifurcating families of three-dimensional periodic orbits using the following periodicity conditions:

$$\begin{aligned} &x_2(x_{01}, x_{05}, x_{06}; T/4) = 0, \quad x_4(x_{01}, x_{05}, x_{06}; T/4) = 0, \\ &x_6(x_{01}, x_{05}, x_{06}; T/4) = 0 \quad \text{(S1)}, \end{aligned} \tag{20}$$

$$\begin{aligned} &x_1(x_{02}, x_{03}, x_{04}; T/2) = 0, \quad x_5(x_{02}, x_{03}, x_{04}; T/2) = 0, \\ &x_6(x_{02}, x_{03}, x_{04}; T/2) = 0 \quad \text{(S2)}, \end{aligned} \tag{21}$$

where $(x_1, x_2, x_3, x_4, x_5, x_6) = (x, y, z, \dot{x}, \dot{y}, \dot{z})$, $x_{0j} = x_j(t = 0)$, $j = 1, \ldots, 6$ and T is the period of the orbit. Conditions (20) are used for the computation of orbits of symmetry (S1) for both $\mu = 1/2$ and $\mu \neq 1/2$, while conditions (21) are used for the computation of orbits of symmetry (S2) for $\mu = 1/2$. When $\mu \neq 1/2$ for the determination of orbits of symmetry (S2) we use the following periodicity conditions:

$$\begin{aligned} &x_2(x_{01}, x_{05}, x_{06}; T/2) = 0, \quad x_3(x_{01}, x_{05}, x_{06}; T/2) = 0, \\ &x_4(x_{01}, x_{05}, x_{06}; T/2) = 0 \ (\mathrm{O}x - \mathrm{O}x). \end{aligned} \tag{22}$$

From these periodicity conditions appropriate linear predictor–corrector schemes can be constructed. For example, from Conditions (20) we obtain:

$$\begin{aligned} x_2 + v_{21}\delta x_{01} + v_{25}\delta x_{05} + v_{26}\delta x_{06} + f_2\delta\tau = 0, \\ x_4 + v_{41}\delta x_{01} + v_{45}\delta x_{05} + v_{46}\delta x_{06} + f_4\delta\tau = 0, \\ x_6 + v_{61}\delta x_{01} + v_{65}\delta x_{05} + v_{66}\delta x_{06} + f_6\delta\tau = 0, \end{aligned} \tag{23}$$

where $v_{j1} = \partial x_j/\partial x_{01}$, $v_{j5} = \partial x_j/\partial x_{05}$, $v_{j6} = \partial x_j/\partial x_{06}$, $f_j = \partial x_j/\partial t$ and $\tau = T/4$. The partial derivatives involved in this system can be computed by integrating the equations of motion simultaneously with the variational equations. A corrector can be obtained from System (23) for fixed τ, i.e. $\delta\tau = 0$, in which case the corresponding predictor is obtained by considering an arbitrary step of τ, i.e. $\tau = \tau + \Delta\tau$.

In order to compute the stability parameters P and Q of a three-dimensional periodic orbit the matrix of first order variations $V(T)$ in the whole period is used (Bray and Goudas 1967). A three-dimensional periodic orbit is stable when $|P| < 2$ and $|Q| < 2$. When $|P| = 2$ or $|Q| = 2$ the orbit is considered to be critical (for details on the stability and criticality of 3D orbits see also Markellos 1977). For economy in the computations, according to the type of symmetry of the orbit the variational matrix can be determined by using the following formulae (Robin and Markellos 1980) applicable for the symmetry type (S1) orbits:

$$\begin{aligned} &\text{Case I}: V(T) = LV^{-1}(T/2)LV(T/2), && (\mathrm{O}xz \text{ plane symmetry}), \\ &\text{Case II}: V(T) = MV^{-1}(T/2)MV(T/2), && (\mathrm{O}x \text{ axis symmetry}), \\ &\text{Case III}: V(T) = [MV^{-1}(T/4)LV(T/4)]^2, && (\mathrm{O}x \text{ axis}-\mathrm{O}xz \text{ plane symmetry}), \\ &\text{Case IV}: V(T) = [LV^{-1}(T/4)MV(T/4)]^2, && (\mathrm{O}x \text{ axis}-\mathrm{O}xz \text{ plane symmetry}), \end{aligned} \tag{24}$$

where L and M are the 6×6 diagonal matrices defined by $L = \mathrm{diag}\{1, -1, 1, -1, 1, -1\}$ and $M = \mathrm{diag}\{1, -1, -1, -1, 1, 1\}$. The third relation of the above formulae is used in the case where we start the integration from the axis while the fourth is used when the integration is started from the plane. In the case of symmetry type (S2) orbits the matrix $V(T)$ is computed using:

$$\text{Case V}: V(T) = NV^{-1}(T/2)NV(T/2), \quad (\text{O}yz \text{ plane symmetry}), \tag{25}$$

where $N = \mathrm{diag}\{-1, 1, 1, 1, -1, -1\}$. When $\mu \neq 1/2$ these orbits are computed using Case II above.

4.2.1 Case I: equal masses

We have determined all the bifurcating families of the Sitnikov family at the one-to-one critical points in the classical restricted problem for $z(T/4) \leqslant 10$. In particular, 44 families of 3D periodic orbits have been computed. At the critical points of the Sitnikov family B_2, B_3, B_6, B_7, B_{10}, B_{11}, B_{13}, B_{16}, B_{17}, B_{20}, B_{21}, B_{24}, B_{25}, B_{28}, B_{29}, B_{32}, B_{33}, B_{36}, B_{37}, B_{40}, B_{41} and B_{44}, families of 3D periodic orbits of symmetry (S1) bifurcate, while at the critical orbits B_1, B_4, B_5, B_8, B_9, B_{12}, B_{14}, B_{15}, B_{18}, B_{19}, B_{22}, B_{23}, B_{26}, B_{27}, B_{30}, B_{31}, B_{34}, B_{35}, B_{38}, B_{39}, B_{42} and B_{43}, families of 3D periodic orbits of symmetry (S2) bifurcate.

The computed families are denoted by f_i^j where the superscript denotes the running number of the critical orbit of the Sitnikov family, and the subscript denotes the number of cuts of the orbits of the family with the Oxz plane for symmetry type (S1) orbits, or the Oyz plane for symmetry type (S2) orbits. In Fig. 1 we show all these families of 3D periodic orbits. As we can see from this figure all these families terminate at planar periodic orbits in the (x, y) plane. In Tables 3 and 4 we give one member of each family.

We have also determined the stability and all critical orbits of some of the families of three-dimensional periodic orbits. At these critical orbits other families of three dimensional periodic orbits of the same or double period bifurcate (for $\mu = 0.5$). In the appendix, in

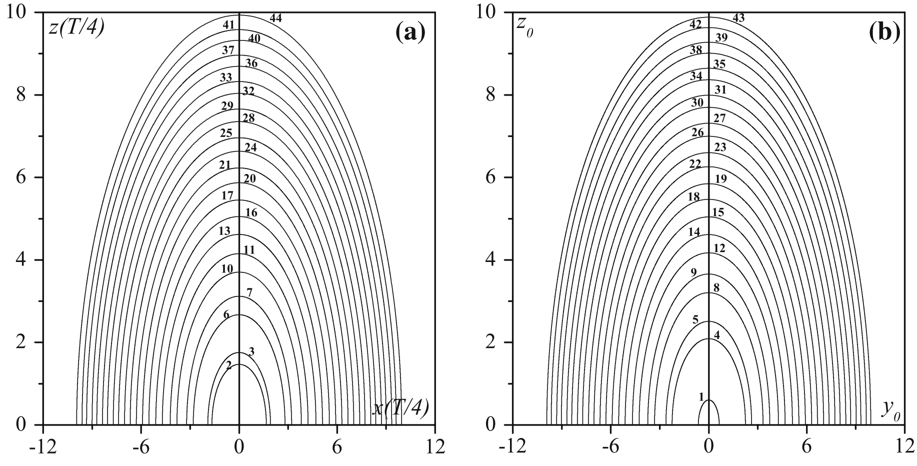


Fig. 1 Families of three-dimensional periodic orbits for $\mu = 0.5$. The orbits of these families are of **(a)** symmetry (S1) and **(b)** symmetry (S2)

Table 3 Three-dimensional periodic orbits of symmetry (S1) for $\mu = 0.5$ and $\dot{z}_0 = 0.5$

	$T/4$	x_0	$\dot{y}_0$	$x(T/4)$	$z(T/4)$	C
f_1^2	2.34914798	−0.28502234	2.16803752	1.61277141	0.37710849	1.05634632
f_1^3	2.85576350	−0.36465663	2.66230112	1.89878032	0.25897920	1.34027010
f_2^6	5.22105585	−0.28524703	2.33287623	−2.72737306	0.63819485	0.31905190
f_2^7	6.25949204	−0.42751505	3.70372186	−3.20461735	0.23330786	1.08933024
f_3^{10}	8.27755584	−0.28558704	2.38745481	3.70353600	0.86151553	0.06845009
f_3^{11}	9.49347077	−0.44525483	4.27527758	4.21323178	0.23296388	0.99462233
f_4^{13}	11.37531164	−0.28566082	2.41492395	−4.57695587	1.06076220	−0.06193912
f_4^{16}	12.68663776	−0.45400801	4.66971770	−5.09910480	0.23779603	0.94098648
f_5^{17}	14.48961391	−0.28566068	2.43175050	5.37873072	1.24344029	−0.14349480
f_5^{20}	15.86205897	−0.45930152	4.96677055	5.90868982	0.24451529	0.90551388
f_6^{21}	17.61234222	−0.28563697	2.44324840	−6.12719014	1.41385539	−0.20003820
f_6^{24}	19.02789855	−0.46287902	5.20209661	−6.66380009	0.25204447	0.87994326
f_7^{25}	20.73999543	−0.28560595	2.45167110	6.83393559	1.57470465	−0.24190906
f_7^{28}	22.18789082	−0.46547399	5.39496445	7.37704317	0.25993272	0.86045624
f_8^{29}	23.87079991	−0.28557358	2.45814541	−7.50682522	1.72780477	−0.27436701
f_8^{32}	25.34401625	−0.46745105	5.55698661	−8.05658734	0.26796441	0.84501293
f_9^{33}	27.00375452	−0.28554214	2.46330081	8.15147075	1.87444856	−0.30038970
f_9^{36}	28.49743396	−0.46901279	5.69569274	8.70812686	0.27602776	0.83241273
f_{10}^{37}	30.13824826	−0.28551249	2.46751823	−8.77204763	2.01559561	−0.32179841
f_{10}^{40}	31.64887232	−0.47028113	5.81623180	−9.33584022	0.28406196	0.82189784
f_{11}^{41}	33.27388558	−0.28548489	2.47104256	9.37176512	2.15198221	−0.33977442
f_{11}^{44}	34.79881434	−0.47133405	5.92227118	9.94291279	0.29203319	0.81296380

Tables 7 and 8, we give these critical orbits for those families of three-dimensional periodic orbits for which they were determined.

4.2.2 Case II: non-equal masses

The bifurcations of the Sitnikov family exist also for non-equal primaries, i.e. $\mu \neq 0.5$. Consider now the linearized Equations (6) for $\mu = 1/2 + \epsilon$. Linearizing this system with respect to ϵ we obtain:

$$\begin{aligned} F_1(z) &= 1 - \Phi_0^{-3/2}, \qquad & F_2(z) &= \tfrac{3}{4}\Phi_0^{-5/2}, \\ F_3(z) &= \tfrac{3}{4}\Phi_0^{-5/2}\epsilon, \qquad & F_4(z) &= \tfrac{-15}{4}\Phi_0^{-7/2}\epsilon, \end{aligned} \tag{26}$$

where $\Phi_0 = z^2 + 1/4$ and System (6) becomes:

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \left(1 - \frac{1}{\Phi_0^{3/2}} + \frac{3}{4\Phi_0^{5/2}}\right)\xi + \frac{3\epsilon}{4\Phi_0^{5/2}}, \\ \ddot{\eta} + 2\dot{\xi} &= \left(1 - \frac{1}{\Phi_0^{3/2}}\right)\eta, \\ \ddot{z} &= -\frac{1}{\Phi_0^{3/2}}\, z. \end{aligned} \tag{27}$$

Table 4 Three-dimensional periodic orbits of symmetry (S2) for $\mu = 0.5$ and $z_0 = 0.5$

	$T/2$	y_0	$\dot{x}_0$	C
f_1^1	1.88664719	0.36114524	0.24232617	−2.59061581
f_3^4	7.47092498	2.18150460	2.25099297	−0.56412044
f_3^5	9.21442015	2.57122935	2.48937459	−1.16422873
f_5^8	13.48440605	3.28118958	3.34733838	−0.15738661
f_5^9	15.77400564	3.70025350	3.63253518	−1.02746095
f_7^{12}	19.65019583	4.23442384	4.29168332	0.02233207
f_7^{14}	22.18982374	4.64563833	4.58774131	−0.96019457
f_9^{15}	25.86570646	5.09562486	5.14590284	0.12615504
f_9^{18}	28.55445146	5.49370578	5.44274903	−0.91836051
f_{11}^{19}	32.10415154	5.89194638	5.93690910	0.19482962
f_{11}^{22}	34.89383072	6.27676292	6.23098219	−0.88924650
f_{13}^{23}	38.35522692	6.63915917	6.67997815	0.24412475
f_{13}^{26}	41.21860544	7.01168158	6.96992937	−0.86756190
f_{15}^{27}	44.61406726	7.34739683	7.38489635	0.28150052
f_{15}^{30}	47.53406215	7.70877064	7.67025765	−0.85065244
f_{17}^{31}	50.87805537	8.02364094	8.05841721	0.31097301
f_{17}^{34}	53.84316015	8.37494546	8.33910377	−0.83702073
f_{19}^{35}	57.14564924	8.67295240	8.70544956	0.33490929
f_{19}^{38}	60.14769966	9.01514406	8.98155009	−0.82575009
f_{21}^{39}	63.41587707	9.29914454	9.32970230	0.35480139
f_{21}^{42}	66.44884537	9.63305465	9.60138317	−0.81624438
f_{23}^{43}	69.68809368	9.90517939	9.93406365	0.37163993

Equations (27) are the equations of motion for the basic family for μ slightly different from $1/2$. We observe that for $\mu = 1/2 + \epsilon$ the equation for z does not change, so the period of a member of the Sitnikov family remains the same. Using (27) we can compute the initial conditions for the determination of three-dimensional periodic orbits for $\mu \neq 1/2$, following the corresponding procedure described in Perdios and Markellos (1988).

In Fig. 2 we show the family characteristics of the families of three-dimensional orbits of the restricted three-body problem for the value of the mass parameter $\mu = 0.4995$. In this figure we see that family f_1^2 consists now of two separate branches named $2a$ and $2b$. Branch $2a$ terminates with a coplanar orbit, as in the case $\mu = 0.5$, while branch $2b$ is joined with the branch $3b$ of family f_1^3. The branch $3a$ of f_1^3 is joined now with branch $6b$ of family f_2^6 and so on. As we can see from Fig. 2 all these families terminate at planar periodic orbits in the (x, y) plane. The coplanar termination orbits of the first 8 families of 3D periodic orbits f_1^1, f_1^2, f_1^3, f_3^4, f_3^5, f_2^6, f_2^7, f_5^8 are shown in Fig. 3. With respect to Fig. 2 note that this is the exact network of the bifurcating families showing clearly the interconnections of the branches. A similar picture of these interconnections has been given by Belbruno et al. (1994) (Fig. 19, p. 127) in schematic form and in a different plane of initial conditions. Comparison of that diagram to our present Fig. 2 shows that the given schematic representation is confirmed.

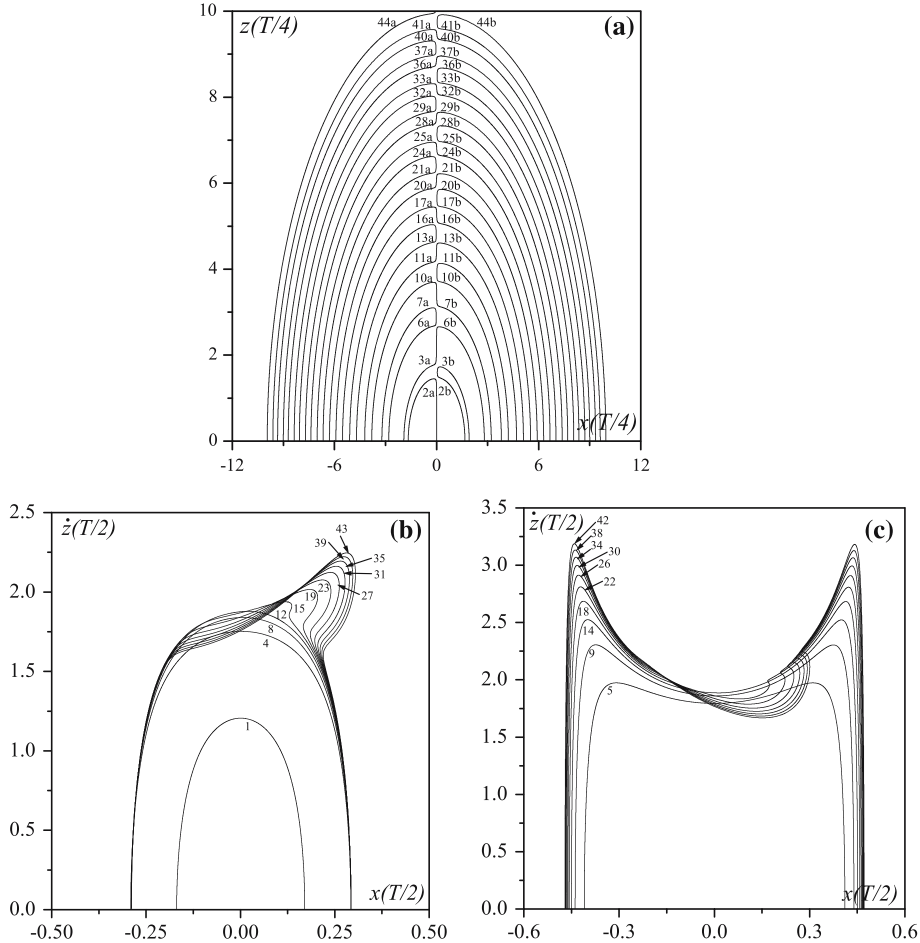


Fig. 2 Families of three-dimensional periodic orbits for $\mu = 0.4995$. The orbits of these families are of (**a**) symmetry (S1) and (**b**), (**c**) symmetry with respect to the Ox-axis

We consider now the families of 3D periodic orbits of symmetry (S1) f_1^2, f_1^3, f_2^6 and f_2^7 and follow their evolution when the mass parameter μ is varied. The continuation of these families with respect to the mass parameter is shown in Fig. 4. From this figure we can see that family f_1^2 (Fig. 4a), which is the three-dimensional Lyapunov family emanating from the collinear equilibrium point L_1, exists for all values of the mass parameter in the range of $(0, 0.5]$ while family f_1^3 (Fig. 4b) exists until the value of the mass parameter $\mu \cong 0.4840$. This happens because the vertical critical orbit of the family of plane symmetric periodic orbits at which family f_1^3 bifurcates into three-dimensions does not exist for lower values of the mass parameter (for details on vertical stability of planar periodic orbits see Hénon 1973; Markellos 1978). Family f_2^6 (Fig. 4c) at $\mu \cong 0.001$ goes to collision. Finally, family f_2^7 (Fig. 4d) exists until the value $\mu \cong 0.4400$ for the same reason for which family f_1^3 does not exist for all values of the mass parameter. The above observations can be corroborated from Fig. 5 where we have plotted the series of vertical critical symmetric plane periodic orbits

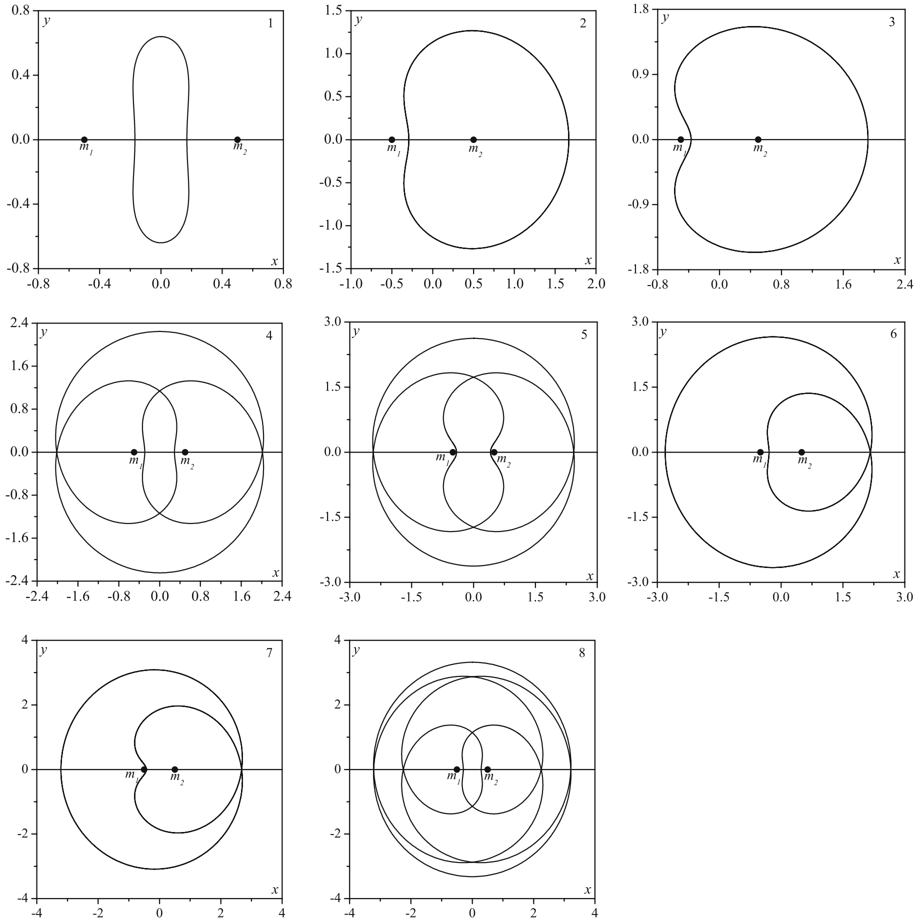


Fig. 3 Terminations in the plane (x, y) of the first 8 families of three-dimensional periodic orbits f_1^1, f_1^2, f_1^3, f_3^4, f_3^5, f_2^6, f_2^7, f_5^8 of the restricted three-body problem

at which these families of three-dimensional periodic orbits end on the plane (for details on vertical stability parameters see Markellos 1978).

The same work has been done for the families of 3D periodic orbits of symmetry (S2) f_1^1, f_3^4 and f_3^5. The continuation of these families with respect to the mass parameter is shown in Fig. 6. We can see that family f_1^1 exists for all values of μ in the range of (0, 0.5] (Fig. 6a) while families f_3^4 and f_3^5 meet each other at $\mu \cong 0.2935$ (see Fig. 6b) forming thus a closed curve which is reduced to a point at the value of the mass parameter $\mu \cong 0.2836$ and does not exist for lower values. In Fig. 7 we have plotted the series of vertical critical symmetric plane periodic orbits at which these families of three-dimensional periodic orbits end on the plane.

In Figs. 8 and 9 we show the orbits of the families f_1^3 and f_1^1 for sample values of the mass parameter. As we can see from these two figures family f_1^3 ends on the plane while the orbits of family f_1^1 approach the smaller primary for lower values of the mass parameter and the family terminates at a coplanar orbit.

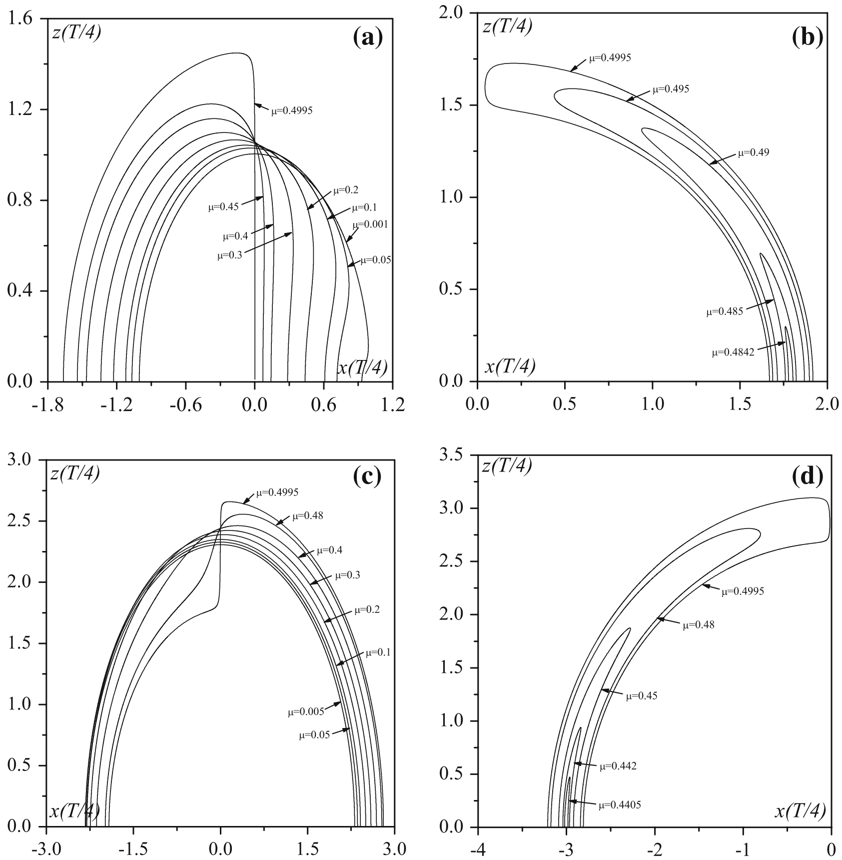
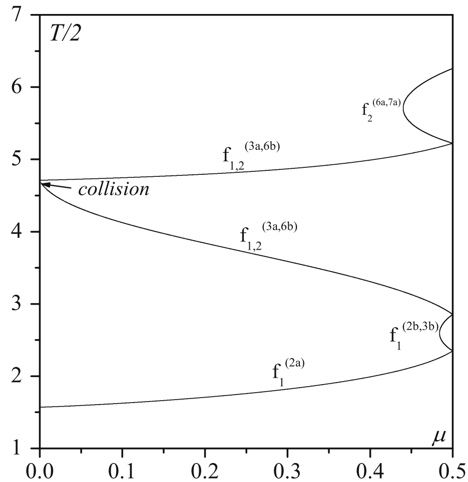


Fig. 4 Continuation of the families (a) f_1^2, $(b)\, f_1^3$, $(c)\, f_2^6$ and (d) f_2^7 of three-dimensional periodic orbits of symmetry (S1) with respect to the mass parameter μ

Fig. 5 Series of vertical critical periodic orbits belonging to the families of plane symmetric periodic orbits at which families f_1^2, f_1^3, f_2^6 and f_2^7 end on the plane. The vertical stability parameters of all these orbits are: $a_v = -1, b_v = 0$ and $c_v \neq 0$



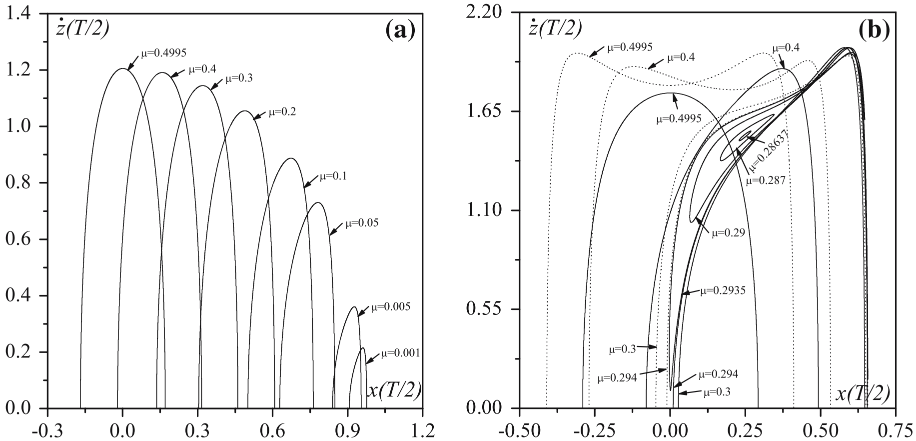


Fig. 6 Continuation of the families (**a**) f_1^1 and (**b**) f_3^4 (continuous lines), f_3^5 (dotted lines) of three-dimensional periodic orbits of symmetry (S2) with respect to the mass parameter μ

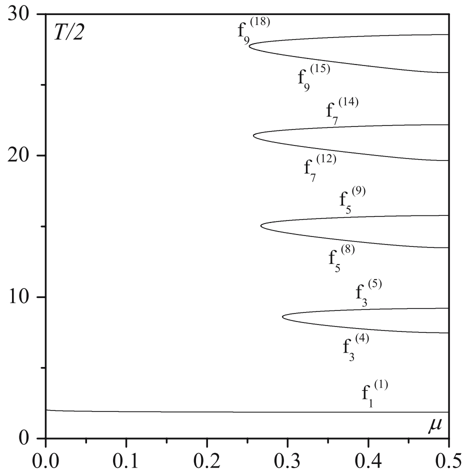


Fig. 7 Series of vertical critical periodic orbits belonging to the families of plane symmetric periodic orbits at which families f_1^1, f_3^4, f_3^5, f_5^8, f_5^9, f_7^{12}, f_7^{14}, f_9^{15} and f_9^{18} end on the plane. The vertical stability parameters of all these orbits are: $a_v = -1,\ b_v = 0$ and $c_v \neq 0$

5 One-to-two critical orbits and the bifurcating families

In this section we study the one-to-two critical orbits of the Sitnikov problem for which $a_1 = 2$. These critical orbits are of importance because at these points families of three-dimensional periodic orbits of double period bifurcate. In Table 5 we give these critical orbits for which $z(T/4) \leqslant 10$.

In Fig. 10 we show the first six families emanating from the rectilinear motion of the Sitnikov problem. These families are symmetric with respect the Ox-axis and also with respect to the Oy-axis. As we see the families emanate from the rectilinear motion in pairs, namely the family emanating from the first critical point C_1 of the Sitnikov motion (for $a_1 = 2$) returns to the second critical point C_2, the family emanating from the third critical point C_3 returns to the fourth critical point C_4, and so on. In Table 6 we give one member of each family.

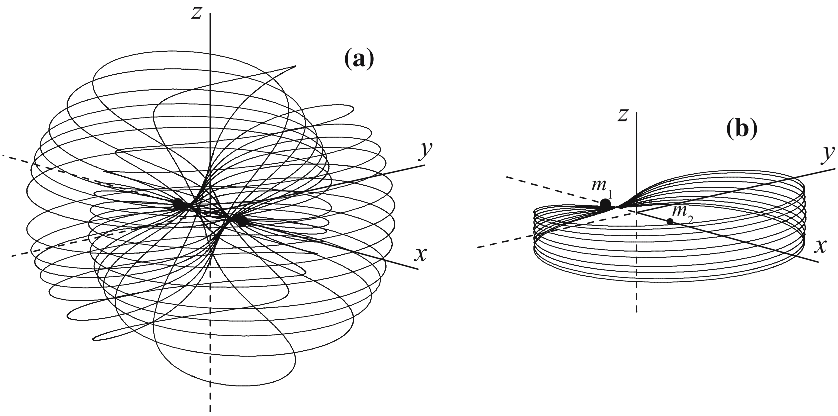


Fig. 8 Three-dimensional orbits of family f_1^3 for (**a**) $\mu = 0.5$ and (**b**) $\mu = 0.4842$

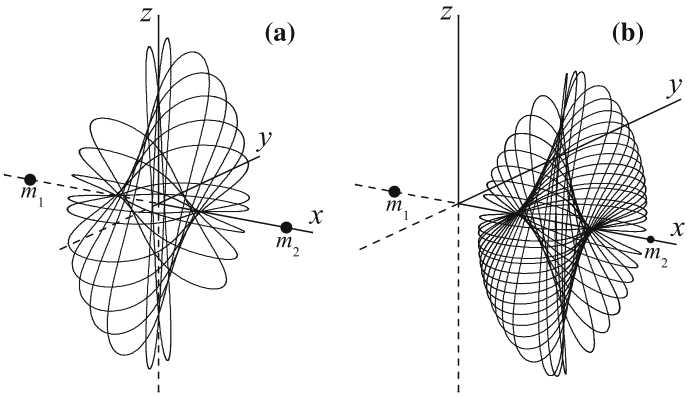


Fig. 9 Three-dimensional orbits of family f_1^1 for (**a**) $\mu = 0.5$ and (**b**) $\mu = 0.25$

6 Summary and conclusions

We have studied the classical Sitnikov problem when the primaries perform circular motion. The stability parameters of the Sitnikov motions have always values less than or equal to 2. For $z(T/4) \leqslant 10$, 44 one-to-one critical points of the rectilinear motion of the Sitnikov problem have been determined. We have computed the families of three-dimensional periodic orbits bifurcating from the one-to-one critical points of the Sitnikov motions ($\mu = 0.5$). All the families have also been computed for nearly equal masses ($\mu = 0.4995$). The manifolds of the families have been illustrated graphically, by plotting all the computed families in the space of the appropriate initial conditions. In both cases these families terminate at vertical self-resonant coplanar periodic orbits. This indicates the importance of the Sitnikov family as a generator of families of three-dimensional periodic orbits, which would alternatively have to be generated from the coplanar vertical self-resonant orbits, a more complicated procedure. Some of the families have been computed for all values of the mass parameter for which they exist.

Finally, we examined the existence of period-doubling bifurcations from the Sitnikov motions. These bifurcations exist when a stability index takes the value 2. We found many

Table 5 Critical orbits of the Sitnikov family from which families of three-dimensional periodic orbits of double period bifurcate (case of $a_1 = 2$)

	$\dot{z}_0$	$z(t/4)$	a_2	$T/4$	C
C1	1.91095530	5.72119610	−17.66662	15.49778795	0.34824986
C2	1.91118207	5.73557979	−15.40339	15.55505319	0.34738311
C3	1.91638024	6.08661442	−20.56122	16.97480584	0.32748678
C4	1.91720951	6.14668042	−11.38497	17.22196198	0.32430768
C5	1.92120049	6.45338872	−21.32563	18.50283492	0.30898866
C6	1.92216644	6.53233727	−9.56691	18.83757987	0.30527617
C7	1.92541581	6.81288591	−21.59012	20.04352678	0.29277395
C8	1.92641286	6.90392621	−8.32539	20.44030815	0.28893349
C9	1.92912602	7.16460395	−21.62362	21.59094170	0.27847281
C10	1.93011051	7.26418713	−7.39016	22.03612339	0.27467342
C11	1.93241673	7.50880011	−21.53425	23.14265929	0.26576559
C12	1.93336887	7.61471969	−6.64850	23.62747585	0.26208481
C13	1.93535678	7.84590038	−21.37602	24.69740100	0.25439415
C14	1.93626782	7.95666011	−6.04054	25.21566761	0.25086691
C15	1.93800113	8.17635690	−21.17893	26.25439414	0.24415163
C16	1.93886791	8.29088667	−5.53020	26.80149186	0.24079125
C17	1.94039391	8.50060385	−20.96067	27.81313066	0.23487146
C18	1.94121614	8.61810905	−5.09409	28.38546998	0.23167988
C19	1.94257089	8.81904395	−20.73208	29.37325789	0.22641835
C20	1.94334981	8.93891863	−4.71602	29.96797002	0.22339152
C21	1.94456127	9.13204493	−20.49993	30.93451812	0.21868146
C22	1.94529889	9.25381695	−4.38446	31.54925837	0.21581224
C23	1.94638918	9.43994106	−20.26854	32.49671827	0.21156917
C24	1.94708783	9.56323644	−4.09083	33.12953696	0.20884896
C25	1.94807468	9.74303619	−20.04062	34.05971204	0.20500505
C26	1.94873683	9.86755417	−3.82865	34.70896237	0.20242477

such orbits and presented some examples of the bifurcating families in this case. A remarkable result is that the bifurcating families of three-dimensional periodic orbits at these critical points (one-to-two) of the Sitnikov motions do not end on families of planar periodic orbits as in the case of the one-to-one bifurcating families. Again this shows the importance of the Sitnikov family as a generator of families of three-dimensional periodic orbits, since in this case these families could not have been generated by the above mentioned alternative procedure (from vertical self-resonant orbits) applicable to the one-to-one bifurcating families.

Another interesting property of the computed manifolds is that while all the families emanating from the one-to-one critical points consist of unstable orbits, some families emanating from the one-to-two critical points have stable parts. In particular, the families emanating from the critical points $C3$, $C4$, $C5$ and $C6$ have stable parts.

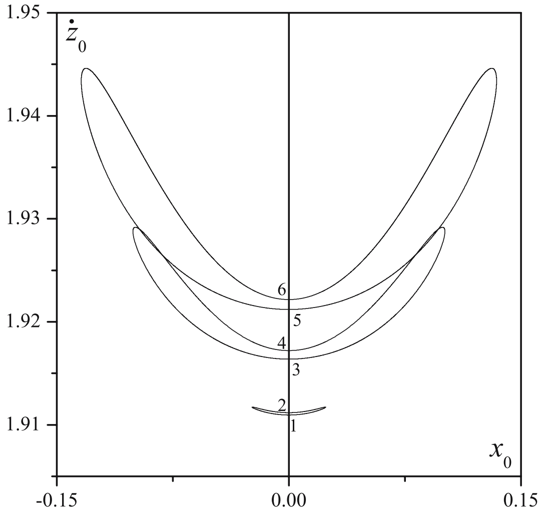


Fig. 10 Families of three-dimensional periodic orbits of double period emanating from critical orbits of the Sitnikov family at which $a_1 = 2$

Table 6 Three-dimensional orbits of double period of families bifurcating from the Sitnikov motion when $a_1 = 2$

	$T/2$	x_0	$\dot{y}_0$	$\dot{z}_0$	c
$g_{21}^{(1,2)}$	62.04381802	0.02002436	−0.07187450	1.91144695	0.34803148
$g_{23}^{(3,4)}$	68.87831788	0.02007905	−0.06584173	1.91782555	0.32447430
$g_{25}^{(5,6)}$	75.34320000	0.02008056	−0.06374727	1.92284342	0.30547479

Appendix A

Table 7 Critical orbits of families of 3D periodic orbits of symmetry (S1) for $\mu = 0.5$

	$T/4$	x_0	$\dot{y}_0$	$\dot{z}_0$	$x(T/4)$	$z(T/4)$	C	P
f_1^2	2.34949854	−0.29211084	2.26167662	0.00000000	1.66672765	0.00000000	1.04285278	−2.0
	2.34673489	−0.22705976	1.54000061	1.25328546	1.21261982	1.01882805	−0.57421913	2.0
	2.34568089	−0.19646596	1.27563431	1.39854463	1.02445342	1.16827475	−0.59288636	−2.0
f_1^3	2.85424973	−0.36785478	2.74046312	0.00000000	1.91937674	0.00000000	1.34487765	−2.0
	2.91114876	−0.21331404	0.89585079	1.71974874	1.03094996	1.48627685	−0.58773148	2.0
	2.92529952	−0.14998576	0.56776436	1.72083624	0.71131973	1.63398754	−0.56719192	−2.0
f_2^6	5.22305555	−0.29158821	2.41837859	0.00000000	−2.80807464	0.00000000	0.29794478	−2.0
	5.20674978	−0.23172200	1.72345946	1.30623157	−2.10501238	1.77237853	−0.23562877	2.0
	5.19046546	−0.13775846	0.92933379	1.68452561	−1.18121663	2.42594112	−0.32313514	−2.0
f_2^7	6.25860382	−0.42852292	3.76290681	0.00000000	−3.21359889	0.00000000	1.09164181	−2.0
	6.33313013	−0.33302799	1.33441020	2.14884347	−2.40769503	2.06745408	−0.45109820	2.0
	6.43375250	−0.10203675	0.22659385	1.86429237	−0.70322421	3.04439427	−0.32865185	−2.0

Table 7 continued

	$T/4$	x_0	$\dot{y}_0$	$\dot{z}_0$	$x(T/4)$	$z(T/4)$	C	P
f_3^{10}	8.27972892	−0.29160929	2.47006260	0.00000000	3.80775300	0.00000000	0.04575461	−2.0
	8.26441635	−0.24388267	1.88887795	1.23850800	3.03360660	2.24016634	−0.10323765	2.0
	8.23910476	−0.10761828	0.72942305	1.78844532	1.23532345	3.50409511	−0.23764792	−2.0
f_3^{11}	9.49288130	−0.44585268	4.32759239	0.00000000	4.21987979	0.00000000	0.99611071	−2.0
	9.55966818	−0.37177467	1.57199412	2.40465566	3.42727519	2.42227132	−0.41526928	2.0
	9.69923175	−0.09035238	0.16227883	1.90229525	0.78892126	4.07881373	−0.24906396	−2.0
f_4^{13}	11.37750600	−0.29151294	2.49604979	0.00000000	−4.70262986	0.00000000	−0.08542070	−2.0
	11.36370648	−0.25086242	1.98658887	1.18293494	−3.87672223	2.61450271	−0.03135494	2.0
	11.33370150	−0.08963375	0.60865254	1.83929948	−1.26048532	4.44928289	−0.19368466	−2.0
f_4^{16}	12.68619063	−0.45443947	4.71815359	0.00000000	−5.10476603	0.00000000	0.94210066	−2.0
	12.74858465	−0.38942834	1.69594922	2.57965078	−4.27855727	2.75497670	−0.39451958	2.0
	12.90806251	−0.08464348	0.13373147	1.92238000	−0.87843530	4.97670000	−0.20587496	−2.0
f_5^{17}	14.49180042	−0.29140485	2.51195747	0.00000000	5.52430115	0.00000000	−0.16746257	−2.0
	14.47913809	−0.25507683	2.04814617	1.14273871	4.64823490	2.94644477	0.01420435	2.0
	14.44856651	−0.10124750	0.69533068	1.84100333	1.67452878	5.19632580	−0.15425145	−2.0
f_5^{20}	15.86169334	−0.45964436	5.01264303	0.00000000	5.91382415	0.00000000	0.90641820	−2.0
	15.92127149	−0.39973840	1.77301985	2.70932187	5.04169188	3.06798659	−0.38055418	2.0
	16.07281172	−0.15875968	0.25165834	2.00180467	1.90169879	5.55939881	−0.20157055	−2.0
f_6^{21}	17.61451469	−0.29130550	2.52282144	0.00000000	−6.29144515	0.00000000	−0.22434158	−2.0
	17.60268826	−0.25786693	2.09032366	1.11284235	−5.36579628	3.25195579	0.04596120	2.0
	17.57499468	−0.13366790	0.93933584	1.79588964	−2.54505093	5.69688230	−0.10908645	−2.0
	17.56822291	−0.06871808	0.46743342	1.88933640	−1.27987416	6.09937996	−0.14682276	−2.0
f_6^{24}	19.02758533	−0.46316724	5.24612169	0.00000000	−6.66861899	0.00000000	0.88071426	−2.0
	19.08520003	−0.40656650	1.82552970	2.81031910	−5.74649790	3.36478118	−0.37035340	2.0
	19.23198636	−0.19127253	0.30221263	2.05833052	−2.57570836	6.11870944	−0.19711921	−2.0
	19.26509442	−0.07972608	0.10787730	1.94482267	−1.06132370	6.54941714	−0.15836833	−2.0
f_7^{25}	20.74215338	−0.29121801	2.53077609	0.00000000	7.01590284	0.00000000	−0.26646031	−2.0
	20.73095934	−0.25984941	2.12111846	1.08975337	6.04148328	3.53836363	0.06954002	2.0
	20.70526501	−0.15169713	1.08351186	1.76124380	3.24501215	6.17158342	−0.07627695	−2.0
	20.69529196	−0.06197398	0.42177283	1.90354552	1.28275400	6.84419265	−0.13243707	−2.0
f_7^{28}	22.18761401	−0.46572530	5.43758180	0.00000000	7.38166105	0.00000000	0.86113531	−2.0
	22.24373732	−0.41149413	1.86451729	2.89198781	6.40904964	3.64684197	−0.36254749	2.0
	22.24373784	−0.41149360	1.86450381	2.89198498	6.40904030	3.64685825	−0.36254686	2.0
	22.38700146	−0.21174063	0.33334129	2.10255728	3.14885271	6.64981395	−0.19353739	−2.0
	22.42958245	−0.07903938	0.10157063	1.95221542	1.15832101	7.26192712	−0.14365148	−2.0
f_8^{29}	23.87294437	−0.29114157	2.53688825	0.00000000	−7.70570242	0.00000000	−0.29911001	−2.0
	23.86224422	−0.26133352	2.14468239	1.07133429	−6.68350890	3.80987374	0.08784613	2.0
	23.83802629	−0.16351940	1.18227538	1.73368185	−3.86299057	6.62478393	−0.05118264	−2.0
	23.82570023	−0.05661277	0.38543928	1.91420821	−1.28338603	7.55037707	−0.12119726	−2.0

Table 7 continued

	$T/4$	x_0	$\dot{y}_0$	$\dot{z}_0$	$x(T/4)$	$z(T/4)$	C	P
f_8^{32}	25.34376619	−0.46767580	5.59848838	0.00000000	−8.06107288	0.00000000	0.84562488	−2.0
	25.39874088	−0.41520957	1.89393576	2.95959835	−7.03780081	3.91763891	−0.35630702	2.0
	25.53924587	−0.22611256	0.35467327	2.13833871	−3.66530485	7.15761808	−0.19058661	−2.0
	25.58902601	−0.07917352	0.09761071	1.95836440	−1.26174012	7.93822809	−0.13221203	−2.0
f_9^{33}	27.00588684	−0.29107462	2.54175366	0.00000000	8.36658079	0.00000000	−0.32528615	−2.0
	26.99558385	−0.26249093	2.16338573	1.05622067	7.29774757	4.06911391	0.10254092	2.0
	26.97250435	−0.17196898	1.25526223	1.71114604	4.42979486	7.05975416	−0.03126271	−2.0
	26.95837267	−0.05222994	0.35571460	1.92252048	1.28256750	8.22478522	−0.11211954	−2.0
f_9^{36}	28.49720437	−0.46921751	5.73628432	0.00000000	8.71252434	0.00000000	0.83297353	−2.0
	28.55127043	−0.41811586	1.91669647	3.01667267	7.63886344	4.17893251	−0.35117268	2.0
	28.68953867	−0.23687266	0.37025077	2.16802567	4.14382277	7.64571268	−0.18810776	−2.0
	28.74494032	−0.08001829	0.09535210	1.96374156	1.37364020	8.58406480	−0.12308481	−2.0

Table 8 Critical orbits of families of 3D periodic orbits of symmetry (S2) for $\mu = 0.5$

	$T/2$	y_0	z_0	$\dot{x}_0$	C	P
f_1^1	1.85554740	0.63965340	0.00000000	0.43167507	−2.68621763	−2.0
f_3^4	7.47408245	2.24577926	0.00000000	2.31672209	−0.54559891	−2.0
	7.44610586	1.63189565	1.44263739	1.68737325	−0.71079362	2.0
	7.42774884	1.12875859	1.81206364	1.16900233	−0.81967191	−2.0
f_3^5	9.20395875	2.62348844	0.00000000	2.54031921	−1.17833447	−2.0
	9.36282809	1.75447358	1.86980153	1.69531601	−0.96967747	2.0
	9.48969275	0.67407375	2.42836534	0.65025764	−0.80996073	−2.0
f_5^8	13.48627370	3.32168314	0.00000000	3.38840416	−0.14769298	−2.0
	13.45410954	2.57860909	2.02341239	2.63375865	−0.31574324	2.0
	13.41098234	1.21429974	2.98513418	1.24253003	−0.54391058	−2.0
f_5^9	15.76741414	3.73529438	0.00000000	3.66704956	−1.03587120	−2.0
	15.90755861	2.94784106	2.24690316	2.89205511	−0.86052937	2.0
	16.15528700	0.74495594	3.58453048	0.72998285	−0.56334066	−2.0
f_7^{12}	19.65137049	4.26511966	0.00000000	4.32267396	0.02853375	−2.0
	19.62234464	3.46516381	2.43379497	3.51440036	−0.12540608	2.0
	19.56632857	1.25019598	3.98977690	1.26982429	−0.42549907	−2.0
f_7^{14}	22.18522464	4.67316560	0.00000000	4.61498320	−0.96595341	−2.0
	22.31386071	3.86668070	2.59089216	3.81719489	−0.80748068	2.0
	22.61564611	0.83370640	4.54026083	0.82232424	−0.44958933	−2.0
f_9^{15}	25.86652555	5.12084546	0.00000000	5.17130165	0.13059062	−2.0
	25.84015366	4.27002651	2.78418730	4.31404366	−0.01262657	2.0
	25.77709517	1.26800613	4.88682566	1.28258558	−0.35703043	−2.0
	25.77504784	1.06067433	4.93453594	1.07291303	−0.36821333	−2.0

Table 8 continued

	$T/2$	y_0	z_0	$\dot{x}_0$	C	P
f_9^{18}	28.55099229	5.51681774	0.00000000	5.46568159	−0.92265079	−2.0
	28.67269340	4.66858616	2.91358291	4.62425908	−0.77386201	2.0
	28.98186836	1.48848433	5.26704998	1.47343769	−0.40845962	−2.0
	29.00513204	0.92348064	5.39283762	0.91410102	−0.38126041	−2.0
f_{11}^{19}	32.10476440	5.91361162	0.00000000	5.95869334	0.19822355	−2.0
	32.08033738	5.01280015	3.10178237	5.05261934	0.06273143	2.0
	32.02235915	2.14255077	5.45011627	2.16134523	−0.25939488	−2.0
	32.01297368	1.27716093	5.70954905	1.28854956	−0.31138098	−2.0
f_{11}^{22}	34.89109515	6.29690509	0.00000000	6.25100102	−0.89261969	−2.0
	35.00812167	5.40019366	3.21802243	5.35994840	−0.75019442	2.0
	35.30608197	2.25628904	5.84291997	2.23846096	−0.39843404	−2.0
	35.36041414	1.01454976	6.17709386	1.00644075	−0.33487006	−2.0
f_{13}^{23}	38.35570836	6.65830143	0.00000000	6.69920537	0.24684112	−2.0
	38.33272868	5.70819910	3.39719012	5.74463861	0.11709922	2.0
	38.27950513	2.90844839	5.93712114	2.92877675	−0.18299249	−2.0
	38.26249486	1.28164291	6.47712705	1.29086977	−0.27830493	−2.0
f_{13}^{26}	41.21636431	7.02966205	0.00000000	6.98781981	−0.87031461	−2.0
	41.33001874	6.08224968	3.50756752	6.04528619	−0.73240937	2.0
	41.61981792	2.87117713	6.38741335	2.85272662	−0.39047367	−2.0
	41.69619527	1.10905989	6.91067564	1.10181755	−0.30103644	−2.0
f_{15}^{27}	44.61445903	7.36464073	0.00000000	7.40220391	0.28374542	−2.0
	44.59259283	6.36621974	3.67577714	6.39989415	0.15845390	2.0
	44.54278366	3.56176611	6.40061344	3.58227200	−0.12591282	−2.0
	44.52031821	1.28324915	7.20161853	1.29093227	−0.25299452	−2.0
f_{15}^{30}	47.53217747	7.72509323	0.00000000	7.68651160	−0.85296080	−2.0
	47.64325610	6.72669507	3.78432467	6.69242445	−0.71846953	2.0
	47.92681920	3.41270462	6.90635651	3.39435414	−0.38398900	−2.0
	48.01964460	1.20901873	7.60421947	1.20239015	−0.27518756	−2.0
f_{17}^{31}	50.87838272	8.03939596	0.00000000	8.07422169	0.31287336	−2.0
	50.85740101	6.99382770	3.94075242	7.02519967	0.19114751	2.0
	50.81017115	4.15165371	6.84435009	4.17183520	−0.08137495	−2.0
	50.78360235	1.28318648	7.89121272	1.28971939	−0.23286434	−2.0
f_{17}^{34}	53.84154308	8.38994833	0.00000000	8.35405256	−0.83899710	−2.0
	53.95056729	7.34111711	4.04996154	7.30909828	−0.70720460	2.0
	54.22919743	3.90856265	7.40392388	3.89059494	−0.37859190	−2.0
	54.33471188	1.31670389	8.26461133	1.31051887	−0.25480576	−2.0

References

Alfaro, J.M., Chiralt, C.: Invariant rotational curves in Sitnikov's problem. Celest. Mech. Dyn. Astron. **55**, 351–367 (1993)

Belbruno, E., Llibre, J., Ollé, M.: On the families of periodic orbits which bifurcate from the circular Sitnikov motions. Celest. Mech. Dyn. Astron. **60**, 99–129 (1994)

Bray, T.A., Goudas, C.L.: Three dimensional oscillations about L_1, L_2, L_3. Adv. Astron. Astrophys. **5**, 71–130 (1967)

Dvorak, R.: Numerical results to the Sitnikov problem. Celest. Mech. Dyn. Astron. **56**, 71–80 (1993)

Faruque, S.B.: Solution of the Sitnikov problem. Celest. Mech. Dyn. Astron. **87**, 353–369 (2003)

Hagel, J.: A new analytic approach to the Sitnikov problem. Celest. Mech. Dyn. Astron. **53**, 267–292 (1992)

Hagel, J., Lhotka, Ch.: A high order perturbation analysis of the Sitnikov problem. Celest. Mech. Dyn. Astron. **93**, 201–228 (2005)

Hénon, M.: Vertical stability of periodic orbits in the restricted problem. I. Equal Masses. Asrton. Astrophys. **28**, 415–426 (1973)

Liu, J., Sun, Y.S.: On the Sitnikov problem. Celest. Mech. Dyn. Astron. **49**, 285–302 (1990)

Markellos, V.V.: Bifurcations of plane with three-dimensional asymmetric periodic orbits in the restricted three-body problem. Mon. Not. R. Asrt. Soc. **180**, 103–116 (1977)

Markellos, V.V.: Bifurcations of straight-line oscillations. Asrton. Astrophys. **67**, 229–240 (1978)

Perdios, E.A., Markellos, V.V.: Stability and bifurcations of Sitnikov motion. Celest. Mech. **42**, 187–200 (1988)

Robin, I.A., Markellos, V.V.: Numerical determination of three-dimensional periodic orbits generated from vertical self-resonant satellite orbits. Celest. Mech. **21**, 395–434 (1980)

Sitnikov, K.: The existence of oscillatory motions in the three-body problem. Dokl. Akad. Nauk. **133**, 303–306 (1960)

Szebehely, V.: Theory of Orbits. Academic Press, Orlando, Florida (1967)