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ORIGINAL ARTICLE

Approximate analysis for relative motion of satellite formation flying in elliptical orbits

Fanghua Jiang · Junfeng Li · Hexi Baoyin

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Abstract This paper studies the relative motion of satellite formation flying in arbitrary elliptical orbits with no perturbation. The trajectories of the leader and follower satellites are projected onto the celestial sphere. These two projections and celestial equator intersect each other to form a spherical triangle, in which the vertex angles and arc-distances are used to describe the relative motion equations. This method is entitled the reference orbital element approach. Here the dimensionless distance is defined as the ratio of the maximal distance between the leader and follower satellites to the semi-major axis of the leader satellite. In close formations, this dimensionless distance, as well as some vertex angles and arc-distances of this spherical triangle, and the orbital element differences are small quantities. A series of order-of-magnitude analyses about these quantities are conducted. Consequently, the relative motion equations are approximated by expansions truncated to the second order, i.e. square of the dimensionless distance. In order to study the problem of periodicity of relative motion, the semi-major axis of the follower is expanded as Taylor series around that of the leader, by regarding relative position and velocity as small quantities. Using this expansion, it is proved that the periodicity condition derived from Lawden's equations is equivalent to the condition that the Taylor series of order one is zero. The first-order relative motion equations, simplified from the second-order ones, possess the same forms as the periodic solutions of Lawden's equations. It is presented that the latter are further first-order approximations to the former; and moreover, compared with the latter more suitable to research spacecraft rendezvous and docking, the former are more suitable to research relative orbit configurations. The first-order relative

F. Jiang (✉)· J. Li · H. Baoyin
School of Aerospace, Tsinghua University,
Beijing 100084, China
e-mail: jiangfh04@mails.thu.edu.cn

J. Li
e-mail: lijunf@tsinghua.edu.cn

H. Baoyin
e-mail: baoyin@tsinghua.edu.cn

motion equations are expanded as trigonometric series with eccentric anomaly as the angle variable. Except the terms of order one, the trigonometric series' amplitudes are geometric series, and corresponding phases are constant both in the radial and in-track directions. When the trajectory of the in-plane relative motion is similar to an ellipse, a method to seek this ellipse is presented. The advantage of this method is shown by an example.

Keywords Formation flying · Taylor expansion · Periodicity condition · Trigonometric series · Elliptical orbit

Nomenclature

(A, B)	semi-major and semi-minor axes of the ellipse approximate to the in-plane relative motion
a	semi-major axis of the leader satellite
d_j	the j-th integration constants in Lawden's equations
E	eccentric anomaly of the leader satellite
e	eccentricity of the leader satellite
(err_x, err_y)	maximum errors between truncated elliptical motion and accurate relative motion in the x- and y-axes
(erx, ery)	indexes to evaluate (err_x, err_y)
F	coefficient with respect to amplitudes of trigonometric series
f	true anomaly of the leader satellite
(G_j, H_j)	the j-th coefficients of the trigonometric series of x^2/a^2
i	orbit inclination of the leader satellite
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	unit vector in the X-, Y- and Z-axes of Earth-centered-inertial frame
$\boldsymbol{i_c}, \boldsymbol{j_c}, \boldsymbol{k_c}$	unit vector in the x-, y- and z-axes of the leader's LVLH frame
(J_j, K_j)	the j-th coefficients of the trigonometric series of y^2/a^2
l^r	reference mean ascension
M	mean anomaly of the leader satellite
$O(10^{-k})$	a value whose order of magnitude is not larger than 10^{-k}
(P_j, Q_j)	the j-th coefficients of the trigonometric series of xy/a^2
(R_j, S_j)	the j-th coefficients of the trigonometric series of $(x^2 + y^2)/a^2$
$\boldsymbol{r}$	position vector from the Earth center to the leader satellite
r	magnitude of $\boldsymbol{r}$
$\boldsymbol{v}$	velocity vector of the leader satellite with respect to the Earth
v	magnitude of $\boldsymbol{v}$
(X_j, U_j)	the j-th coefficients of the trigonometric series of x/a
(x, y, z)	radial, in-track and cross-track position distances from the origin of the LVLH frame
$(\hat{x}, \hat{y})$	approximate elliptical motion equations in the $x-y$ plane
(x_0, y_0)	center coordinates of the approximate ellipse
(Y_j, V_j)	the j-th coefficients of the trigonometric series of y/a
α	angle parameter
$\Delta\beta$	difference angle parameter
ε	index function to weigh the similarity between approximate elliptical motion and the first-order relative motion
$(\varepsilon_j, \upsilon_j)$	the j-th coefficients of the trigonometric series of ε

η spherical angle with respect to the leader satellite
θ argument of latitude of the leader satellite
ϑ included angle between the semi-major axis of the approximate ellipse and the x-axis
λ common ratio, a function of e
μ gravitational parameter of the Earth, 3.986005×10^{14} m^3/s^2
$\chi(E)$ phase function of the approximate elliptical motion in the x–y plane
Ω right ascension of ascending node of the leader satellite
ω argument of perigee of the leader satellite

Operators

$d(\cdot)$ differential of $(\cdot)$
$\Delta(\cdot)$ difference of $(\cdot)$ between the leader and follower satellites
$\nabla(\cdot)$ gradient of $(\cdot)$
$\delta(\cdot)$ variation of $(\cdot)$
$\dot{(\cdot)}$ derivative of $(\cdot)$ with respect to time in inertial frame
$\dot{\tilde{(\cdot)}}$ derivative of $(\cdot)$ with respect to time in LVLH frame
$(\cdot)'$ derivative of $(\cdot)$ with respect to f

Subscript

f value of the follower satellite

Superscript

r reference value of the follower with respect to the leader

1 Introduction

Spacecraft formation flying has been paid much attention lately because of its advantages such as lower cost and risk dispersion, as well as stronger flexibility and redundancy, compared with a single, complex spacecraft. Satellite formation flying has been considered an important development trend of future satellite technology. One of the most important and fundamental problems in formation flying is to work out the trajectory of relative motion of the follower with respect to the leader in natural conditions. Nonlinearity of the differential gravitational acceleration (DGA), eccentricity of the reference orbit, and non-central gravitational perturbation such as Earth's oblateness are the three most complicated problems. In this paper, the non-central gravitational perturbations are not considered.

Obtaining the accurate differential equations describing relative motion is not difficult, but getting the analytic solutions with no approximation seems impossible, even when the reference orbit is circular. In close formation, the distance between the follower and the leader is a small quantity, compared to their orbit radii; hence the relative motion equations can be solved analytically by approximating the DGA. The DGA expanded to n-order series leads to the n-order solutions. Previous research on relative motion can be sorted into two classes. One, named the dynamical method, involves solving the differential equations after approximating the DGA mainly. The accurate kinematical equations, which describe relative motion by orbital elements, are also not difficult to obtain. Another, named the kinematical method, is mainly approximating the kinematical equations based on the precondition of knowing very well the order of magnitudes of all the small quantities used in these equations.

The dynamical method on relative motion may be first seen in Hill's paper (Hill 1878), where the author studied the motion of the Moon with respect to the Sun–Earth system. The solutions are first-order and the reference orbit is assumed circular. The Clohessy–Wiltshire equations (Clohessy and Wiltshire 1960) also linearized the DGA with a circular reference orbit. Gómez and Marcote (2006) applied the Lindstedt–Poincaré Procedure to Hill's equations, i.e. Clohessy–Wiltshire equations, to obtain high-order analytical solutions. Lawden (1963) presented a solution of the linearized homogenous equations, called Lawden's equations, of relative motion for an elliptical reference orbit. One deficiency of Lawden's equations, which was improved by Carter (1990), is that the solutions become singular when the true anomaly is a multiple of 180 degree. London (1963) obtained the second-order solutions for a circular reference orbit by using perturbation method to solve differential equations. Anthony and Sasaki's work (Anthony and Sasaki 1965) is similar to London's, while the main difference is that the former studied the problem of a reference orbit with slight eccentricity. The solutions are in the form of a series expansion in eccentricity. Melton (2000) derived first-order relative motion equations which were explicit in time and discarded the terms higher than squared eccentricity. Inalhan et al. (2002) obtained the periodicity conditions for bounded relative motion solutions to the linearized elliptical problem. Vaddi et al. (2003) obtained time-explicit bounded solutions by combining the second-order solutions for a circular reference orbit and the linearized solutions for an elliptical reference orbit. Kasdin et al. (2005) presented a Hamiltonian approach to modeling relative motion to a circular reference orbit, incorporating the influence of nonlinearity and perturbations such as Earth's oblateness.

The kinematical method is the series expansion in fact. In close formation, some parameters in the relative motion equations are small quantities. Karlgaard and Lutze (2001) described the equations of relative motion in spherical coordinates and obtained the second-order solutions for a circular reference orbit. Schaub and Alfriend (2002) used orbital element differences to describe relative orbit geometry and obtained an invertible matrix mapping the relative motion coordinates to the corresponding orbital element difference. Schaub (2002) developed the direct relationships between the orbital element differences and the resulting relative orbit geometry for reference orbits with arbitrary eccentricity. Gim and Alfriend (2003) obtained a state transition matrix relating relative position and velocity with the orbital element differences. The matrix's singularity existing at zero inclination was removed by Gim and Alfriend (2005). Broucke (2003) and Lane and Axelrad (2006) used a simple geometrical method to obtain the first-order solutions of relative motion equations for elliptical orbits. The solutions are expressed as functions of a constant set of orbital element differences and formulated with time as the independent variable. Baoyin et al. (2002) presented a method, which is suitable to elliptical reference orbits, to describe relative motion, based on relative orbital elements. Li et al. (2005) and Meng et al. (2005) also used relative orbital elements to analyze relative orbital configurations with and without the perturbation of the Earth's oblateness for slightly elliptical reference orbits. Wang et al. (2005) presented a method, using the so-called reference orbital elements (ROE) to describe relative motion. The introduced parameters are more geometrical and concise than classical orbital elements (COE) in the spherical triangle.

This paper will also use the reference orbital elements to study the relative motion of satellite formation flying in arbitrary elliptical orbits. In Sect. 2, the notion of reference orbital elements will be introduced. Then the relative motion equations will be presented, which are expressed by a set of reference orbital elements. In close formations, a series of order-of-magnitude analyses about these orbital element differences will be conducted. In Sect. 3, the second-order relative motion equations will be derived by Taylor expansion. The first-order equations are easy to reach by simplifying from the second-order ones. These equations are formulated using time-independent constants and an alternative time-dependent variable between true anomaly and eccentric anomaly. In Sect. 4, by expanding the semi-major axis of the follower as a Taylor series around that of the leader, it will be proved that the periodicity condition derived from Lawden's equations is equivalent to the condition that this Taylor series of order one is zero. The periodic solutions of Lawden's equations will be proved to be further approximations to the first-order relative motion equations. Consequently the latter are more suitable to research relative orbit configurations. In Sect. 5, the first-order relative motion equations are expanded as trigonometric series with eccentric anomaly as the angle variable. By this expansion, a method to seek an ellipse to approximate the trajectory of in-plane relative motion is presented.

2 Order-of-magnitude analysis of the small quantities in relative motion

2.1 Reference orbital element approach

This section introduces the ROE approach (Wang et al. 2005) briefly. The essence of the approach is describing the trajectories of the leader and the follower and corresponding relative motion on the celestial sphere. As shown in Fig. 1, *OXYZ* denotes the Earth-centered-inertial (ECI) reference frame. *Lxyz* denotes the leader's LVLH frame, a non-inertial frame. *L* denotes the instantaneous position of the leader. All the frames are right-handed. The inclined ellipses denote the projections of the spacecrafts' trajectories on the celestial sphere, with the solid line denoting the leader, broken line denoting the follower and dotted lines denoting the shaded regions of corresponding projections. *F* denotes the instantaneous position of the follower. *P* denotes the spacecraft's perigee.

It is well known that in Keplerian elliptical motion, the argument of latitude θ and the distance between the spacecraft and the Earth r are given by following expressions:

$$\theta = \omega + f \tag{1}$$

$$r = \frac{a\left(1-e^2\right)}{1+e\cos f} = a\left(1-e\cos E\right) \tag{2}$$

Furthermore, the relationships between the mean anomaly, eccentric anomaly and true anomaly are

$$M = E - e\sin E \tag{3}$$

$$\cos f = \frac{\cos E - e}{1-e\cos E}, \quad \sin f = \frac{\sqrt{1-e^2}\sin E}{1-e\cos E} \tag{4}$$

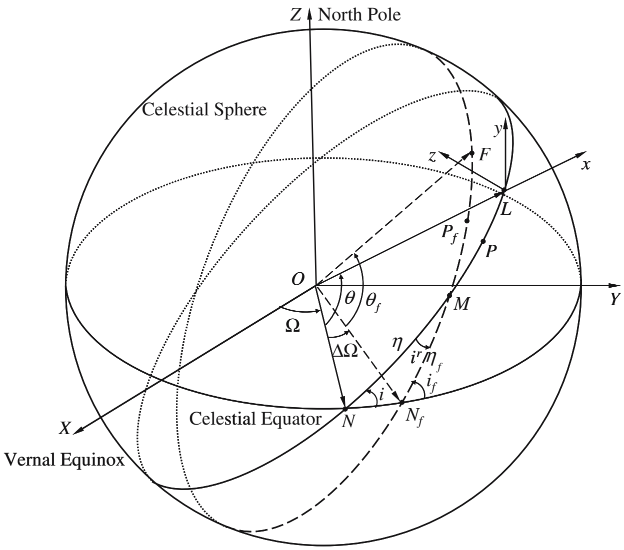

Fig. 1 Projections of the leader and follower trajectories on the celestial sphere

$$\cos E = \frac{\cos f + e}{1 + e\cos f}, \quad \sin E = \frac{\sqrt{1-e^2}\sin f}{1 + e\cos f} \tag{5}$$

The projections of the leader trajectory, the follower trajectory and the Earth equator onto the celestial sphere intersect each other and form the spherical triangle NN_fM, of which the three vertex angles are i, $\pi^- i_f$ and i^r and the three arc-distances are $\Delta\Omega$, η_f and η. The point M is the intersection of the projections of the leader and follower trajectories. The arrows point to positive directions. Symbols η and η_f denote the arc-distance from N to M and from N_f to M, respectively. The vertex angle i^r, named reference inclination, is the inclination angle between the orbital planes of the leader and follower. Define the reference ascension of ascending node Ω^r, the reference orbit angle θ^r, the reference mean ascension l^r and the reference argument of perigee ω^r as follows:

$$\begin{aligned} &\Omega^r = \theta - \eta, \quad \theta^r = \theta_f - \eta_f \\ &l^r = \theta^r - \Omega^r = \Delta\theta - \Delta\eta, \quad \omega^r = \omega_f - \eta_f \end{aligned} \tag{6}$$

Note that the definition of Ω^r here is contrary to that done by Wang et al. (2005). By the knowledge of spherical trigonometry (Todhunter and Leathem 1929), the six spherical parameters relate with each other as

$$\cos i^r = \cos i \cos i_f + \sin i \sin i_f \cos\Delta\Omega \tag{7}$$

$$\sin\eta = \frac{\sin i_f \sin\Delta\Omega}{\sin i^r} \tag{8}$$

$$\sin\eta_f = \frac{\sin i \sin\Delta\Omega}{\sin i^r} \tag{9}$$

$$\cos\eta = \cos\Delta\Omega\cos\eta_f - \sin\Delta\Omega\sin\eta_f\cos i_f \tag{10}$$

$$\cos\eta_f = \cos\Delta\Omega\cos\eta + \sin\Delta\Omega\sin\eta\cos i \tag{11}$$

The governing equations of relative motion expressed by these parameters defined above were presented by Wang et al. (2005), who originally presented them in a paper in Chinese. Here we give a brief presentation of how to obtain the governing equations. The rotation matrix, transforming any vector in the LVLH frame of the follower into that of the leader, can be characterized by an Euler 3-1-3 rotation. When consider counterclockwise as positive, the three Eulerian angles are $-\theta^r$, $-i^r$ and Ω^r in turn, and the three matrices (Taff 1985) are

$$T_3\left(-\theta^r\right) = \begin{bmatrix} c\theta^r & -s\theta^r & 0 \\ s\theta^r & c\theta^r & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_1\left(-i^r\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & ci^r & -si^r \\ 0 & si^r & ci^r \end{bmatrix}, \quad T_3\left(\Omega^r\right) = \begin{bmatrix} c\Omega^r & s\Omega^r & 0 \\ -s\Omega^r & c\Omega^r & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

respectively, where we have used the compact notation $sx = \sin(x), cx = \cos(x)$. So the composite rotation matrix is

$$T_3\left(\Omega^r\right)T_1\left(-i^r\right)T_3\left(-\theta^r\right) = \begin{bmatrix} c\Omega^r c\theta^r + s\Omega^r s\theta^r ci^r & -c\Omega^r s\theta^r + s\Omega^r c\theta^r ci^r & -s\Omega^r si^r \\ -s\Omega^r c\theta^r + c\Omega^r s\theta^r ci^r & s\Omega^r s\theta^r + c\Omega^r c\theta^r ci^r & -c\Omega^r si^r \\ s\theta^r si^r & c\theta^r si^r & ci^r \end{bmatrix} \tag{13}$$

The leader's and follower's position vectors in their own LVLH frames are $[r, 0, 0]^T$ and $[r_f, 0, 0]^T$, respectively; hence the relative position vector in the leader's LVLH frame can be written as

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= T_3\left(\Omega^r\right)T_1\left(-i^r\right)T_3\left(-\theta^r\right)\begin{bmatrix} r_f \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \\ &= r_f\begin{bmatrix} c\Omega^r c\theta^r + s\Omega^r s\theta^r ci^r \\ -s\Omega^r c\theta^r + c\Omega^r s\theta^r ci^r \\ s\theta^r si^r \end{bmatrix} - r\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{14}$$

For convenience, Eq. 14 can be rebuilt in the following forms:

$$x = r_f\left[\sin^2\frac{i^r}{2}\cos\left(\theta^r + \Omega^r\right) + \cos^2\frac{i^r}{2}\cos l^r - 1\right] + \Delta r \tag{15}$$

$$y = r_f\left[-\sin^2\frac{i^r}{2}\sin\left(\theta^r + \Omega^r\right) + \cos^2\frac{i^r}{2}\sin l^r\right] \tag{16}$$

$$z = r_f \sin i^r \sin\theta^r \tag{17}$$

where the denotation $l^r = \theta^r - \Omega^r$ is used.

2.2 Order-of-magnitude analysis

In close formation, the leader and follower satellites are close to each other. Here the dimensionless distance is defined as the ratio of the maximal distance between the leader and follower satellites to the semi-major axis of the leader satellite. This dimensionless distance, some vertex angles and arc-distances of the spherical triangle, and the orbital element differences are small quantities. For the purpose of approximating the relative motion equations, this section presents comprehensive and reasonable order-of-magnitude analysis of these small quantities. When it comes to the problem of spacecraft flying in close formation, many previous papers directly assumed that the orbital element differences were small. This paper will give the explanations.

Now that the follower is close to the leader, it can be assumed that the dimensionless distance is of the order of magnitude of 10^{-k}, generally $k \geqslant 3$. Therefore, the corresponding dimensionless distances in the x-, y-, and z-axes are all not more than 10^{-k}, denoted by $O(10^{-k})$. So there are two cases. One is that the dimensionless distances in the x-, y-, and z-axes are of the same order of magnitude. Another is that they are not of the same order of magnitude. In either case, they can be denoted as $O(10^{-k})$.

Use d_x, d_y and d_z to denote the dimensionless distances in the x-, y-, and z-axes, respectively. Analyzing Eq. 17, the conditions $d_z = O(10^{-k})$ and unbounded range of θ^r lead to $i^r = O(10^{-k})$. Then analyzing Eq. 16, the conditions $d_y = O(10^{-k})$ and $i^r = O(10^{-k})$ lead to $l^r = O(10^{-k})$. Finally, analyzing Eq. 15, the conditions $d_x = O(10^{-k})$, $i^r = O(10^{-k})$ and $l^r = O(10^{-k})$ lead to $\Delta r/a = O(10^{-k})$. Summarizing the analyses above, the necessary and sufficient conditions for $d_x = O(10^{-k}), d_y = O(10^{-k})$ and $d_z = O(10^{-k})$ are

$$i^r = O\left(10^{-k}\right), \quad l^r = O\left(10^{-k}\right), \quad \Delta r/a = O\left(10^{-k}\right) \tag{18}$$

Note that the conditions $d_z = O(10^{-k})$ and unbounded range of θ^r directly imply $\sin i^r = O(10^{-k})$, which means i^r is approximate to 0 or $\pm\pi$. On the condition of $i^r \approx \pm\pi$, deriving $|y| \approx r_f |\sin(\theta^r + \Omega^r)| = r_f |x \sin(\theta + \theta_f - \eta - \eta_f)|$, it is impossible to make $d_y = O(10^{-k})$ because of the unbounded range of $\theta + \theta_f$. So $i^r \approx 0$ is the only reasonable solution.

On the basic conditions Eq. 18, the orders of magnitude of the orbital element differences and some other small quantities in the spherical triangle can be analyzed. Employing some basic transformations of trigonometric functions, Eq. 7 yields

$$\begin{aligned}
2\sin^2\frac{i^r}{2} &= 1 - \cos i^r = 1 - \frac{1}{2}(1+\cos\Delta\Omega)\cos\left(i_f - i\right) - \frac{1}{2}(1-\cos\Delta\Omega)\cos\left(i_f + i\right) \\
&= \cos^2\frac{\Delta\Omega}{2} + \sin^2\frac{\Delta\Omega}{2} - \cos^2\frac{\Delta\Omega}{2}\cos\Delta i - \sin^2\frac{\Delta\Omega}{2}\cos(2i+\Delta i) \\
&= 2\sin^2\frac{\Delta i}{2}\cos^2\frac{\Delta\Omega}{2} + 2\sin^2\left(i + \frac{\Delta i}{2}\right)\sin^2\frac{\Delta\Omega}{2}
\end{aligned} \tag{19}$$

When $i^r = O(10^{-k})$, namely $\sin^2(i^r/2) = O(10^{-2k})$, and furthermore, the problem considered is general so the leader's inclination is arbitrary, Eq. 19 yields

$$\Delta i = O\left(10^{-k}\right), \quad \Delta\Omega = O\left(10^{-k}\right) \tag{20}$$

Because $\Delta\eta$ is constant and $\Delta\theta$ varies with time, the necessary condition for $l^r = \Delta\theta - \Delta\eta = O(10^{-k})$ is that the period of the follower should equal to that of the leader.

$$a_f = a \tag{21}$$

Substituting Eqs. 8 and 9 into Eqs. 11 and 10, respectively, and solving them by regarding $\cos\eta$ and $\cos\eta_f$ as given, we can obtain

$$\cos\eta = \frac{\cos i \sin i_f \cos\Delta\Omega - \sin i \cos i_f}{\sin i^r} \tag{22}$$

$$\cos\eta_f = \frac{\cos i \sin i_f - \sin i \cos i_f \cos\Delta\Omega}{\sin i^r} \tag{23}$$

It is well known that

$$\sin\left(\eta_f - \eta\right) = \sin\eta_f \cos\eta - \cos\eta_f \sin\eta \tag{24}$$

Substitution of Eqs. 8, 9, 22 and 23 into Eq. 24 yields

$$\sin\left(\eta_f - \eta\right) = -\frac{\sin\Delta\Omega\left(\cos i + \cos i_f\right)\left(1 - \cos i \cos i_f - \sin i \sin i_f \cos\Delta\Omega\right)}{\sin^2 i^r} \tag{25}$$

By adopting Eq. 7, the equation above can be simplified into

$$\sin\Delta\eta = -\frac{\sin\Delta\Omega\left(\cos i + \cos i_f\right)(1 - \cos i^r)}{\sin^2 i^r} = -\frac{\sin\Delta\Omega \cos\frac{\Delta i}{2}\cos\left(i + \frac{\Delta i}{2}\right)}{\cos^2\frac{i^r}{2}} \tag{26}$$

When the order of magnitudes of i^r, Δi, $\Delta\Omega$ and $\Delta\theta - \Delta\eta$ are all $O(10^{-k})$, Eq. 26 leads to

$$\Delta\eta = -\Delta\Omega\cos i + O\left(10^{-2k}\right) = O\left(10^{-k}\right), \quad \Delta\theta = O\left(10^{-k}\right) \tag{27}$$

When $a_f = a$ and $\Delta r/a = O(10^{-k})$, derived from Eq. 2, the following expression can be obtained

$$\begin{aligned}\Delta r/a &= e\cos E - e_f\cos E_f = \left(e\cos\Delta E - e_f\right)\cos E_f + e\sin\Delta E\sin E_f \\ &= \sqrt{\left(e\cos\Delta E - e_f\right)^2 + (e\sin\Delta E)^2}\cos\left(E_f - \psi\right) \\ &= \sqrt{4e\,(e + \Delta e)\sin^2\frac{\Delta E}{2} + (\Delta e)^2}\cos\left(E_f - \psi\right) = O\left(10^{-k}\right)\end{aligned} \tag{28}$$

where $\tan\psi = e\sin\Delta E/(e\cos\Delta E - e_f)$ is a function with respect to e, e_f and ΔE. Analyzing the equation above, when the interval of e is $[0, 1)$, and because E_f varies arbitrarily, it yields that the amplitude of $\cos(E_f - \psi)$ should be small, namely

$$\Delta e = O\left(10^{-k}\right), \quad \Delta E = O\left(10^{-k}\right) \tag{29}$$

On the conditions of Eqs. 29, derived from Eq. 3, it can be shown that

$$\begin{aligned}\Delta M &= \Delta E - e_f\sin E_f + e\sin E = \Delta E + \left(e\cos\Delta E - e_f\right)\sin E_f - e\sin\Delta E\cos E_f \\ &= \Delta E + \sqrt{\left(e\cos\Delta E - e_f\right)^2 + (e\sin\Delta E)^2}\sin\left(E_f - \psi\right) \\ &= \Delta E + \sqrt{4e\,(e + \Delta e)\sin^2\frac{\Delta E}{2} + (\Delta e)^2}\sin\left(E_f - \psi\right)\end{aligned} \tag{30}$$

Note that when semi-major axes are equal, the difference of M is constant. Since $\Delta e = O(10^{-k})$ and $\Delta E = O(10^{-k})$, it is evident that

$$\Delta M = O\left(10^{-k}\right) \tag{31}$$

Analyzing the order of magnitude of Δf is not as easy as ΔE or ΔM. Derived from Eqs. 4, the variation of f with respect to E and e is

$$\delta f=\frac{\left(1-e^2\right)\delta E+\sin E\delta e}{\sqrt{1-e^2}\left(1-e\cos E\right)}=\frac{1}{1-e^2}\left[\sqrt{1-e^2}\left(1+e\cos f\right)\delta E+\sin f\delta e\right] \quad (32)$$

In mathematics, the variation of a function is the linear component of its difference. Choosing E and e as the basic variables, then $\delta E=\Delta E$ and $\delta e=\Delta e$, and applying the inequality $|ac+bd|\leqslant\sqrt{a^2+b^2}\sqrt{c^2+d^2}$, the expression above can be derived as

$$\begin{aligned}|\delta f| &\leqslant \frac{1}{1-e^2}\sqrt{\left(1-e^2\right)\left(1+e\cos f\right)^2+\sin^2 f}\sqrt{(\Delta E)^2+(\Delta e)^2}\\ &=\frac{\sqrt{(\Delta E)^2+(\Delta e)^2}}{1-e^2}\sqrt{1+\frac{1-e^2}{1-e^2+e^4}-\left(1-e^2+e^4\right)\left[\cos f-\frac{e\left(1-e^2\right)}{1-e^2+e^4}\right]^2}\\ &\leqslant\frac{1}{1-e^2}\sqrt{1+\frac{1-e^2}{1-e^2+e^4}}\sqrt{(\Delta E)^2+(\Delta e)^2} \end{aligned} \quad (33)$$

It is not difficult to prove that both $\sqrt{1/\left(1-e^2\right)+1/\left(1-e^2+e^4\right)}$ and $1/\sqrt{1-e^2}$ are monotone increasing positive functions with respect to e on the interval [0, 1); hence their product function $\sqrt{1+\left(1-e^2\right)/\left(1-e^2+e^4\right)}/\left(1-e^2\right)$ possesses the same properties. Though this product function tends to infinity when e tends to 1, it increases very slowly when e is not too close to 1, such as $e\leqslant 0.8$ resulting in the value of the product function not larger than 3.4. Because $\Delta e=O(10^{-k})$ and $\Delta E=O(10^{-k})$, based on the analyses above, we can obtain $\delta f=O(10^{-k})$. Generally speaking, the difference of f is unlikely too different from its variation, so it is reasonable that

$$\Delta f=O\left(10^{-k}\right),\quad \Delta\omega=\Delta\theta-\Delta f=O\left(10^{-k}\right) \quad (34)$$

Equations 18, 20, 21, 27, 29, 31 and 34 are the order-of-magnitude relations of the small quantities in close formation. Figure 2 sums up these processes in a flow chart.

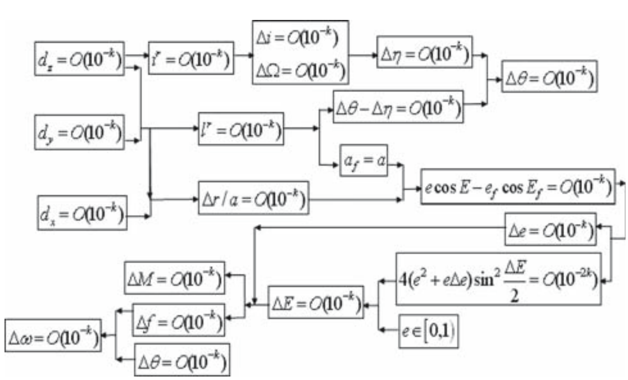

Fig. 2 The flow chart of order-of-magnitude analysis of small quantities

3 Second-order approximation to relative motion equations

In the above section, the order-of-magnitude relations between the small quantities of relative motion equations and the dimensionless distance are obtained. This section will approximate the accurate relative motion Eqs. 15, 16 and 17 to the second order, i.e. truncating to the terms of order of magnitudes of $a \times O(10^{-2k})$. In the approximate equations, the leader's true anomaly and eccentric anomaly are chosen alternatively as the only variable varying with time, and all other quantities are constant when perturbation is not considered.

Derived from Eqs. 15–17 under the order-of-magnitude relations, the second order relative motion equations can be expressed as follows (see the derivation in Appendix A):

$$\begin{aligned}\frac{x}{a} = {} & \left(1-e^2\right)\sin^2\frac{i^r}{2}\frac{\cos(2f+\varphi)-1}{1+e\cos f} + \cos^2\frac{i^r}{2}\left\{-\Delta e\cos f + \frac{e\Delta M}{\sqrt{1-e^2}}\sin f\right. \\ & - \frac{\Delta e\Delta M\sin f(1+e\cos f)}{\left(1-e^2\right)^{\frac{3}{2}}} - \Delta e\Delta\beta\sin f\left(1+\frac{1}{1+e\cos f}\right) \\ & - \frac{\Delta\beta\Delta M}{\sqrt{1-e^2}}(1+e\cos f) - \frac{\left(1-e^2\right)(\Delta\beta)^2}{2(1+e\cos f)} - \frac{(\Delta M)^2(1+e\cos f)^2}{2\left(1-e^2\right)^2} \\ & \left. - \frac{(\Delta e)^2\sin^2 f}{2\left(1-e^2\right)}\left(1+\frac{1}{1+e\cos f}\right)\right\} + O\left(10^{-3k}\right)\end{aligned} \quad (35)$$

$$\begin{aligned}\frac{y}{a} = {} & -\left(1-e^2\right)\sin^2\frac{i^r}{2}\frac{\sin(2f+\varphi)}{1+e\cos f} + \cos^2\frac{i^r}{2}\left\{\left(1+\frac{1}{1+e\cos f}\right)\Delta e\sin f\right. \\ & + \frac{\Delta M}{\sqrt{1-e^2}}(1+e\cos f) + \frac{\left(1-e^2\right)\Delta\beta}{1+e\cos f} + \frac{\Delta e\Delta M}{\left(1-e^2\right)^{\frac{3}{2}}}(\cos f + e) \\ & \left. -\Delta e\Delta\beta\cos f + \frac{e\Delta M\Delta\beta\sin f}{\sqrt{1-e^2}} + \frac{(\Delta e)^2\sin f(\cos f+2e)}{2\left(1-e^2\right)(1+e\cos f)}\right\} + O\left(10^{-3k}\right)\end{aligned} \quad (36)$$

$$\begin{aligned}\frac{z}{a} = {} & \sin i^r\left\{\frac{\left(1-e^2\right)\sin(\alpha+f)}{1+e\cos f} + \Delta e\left[-\sin\alpha + \frac{\cos(\alpha+f)\sin f}{1+e\cos f}\right]\right. \\ & \left. + \frac{\Delta M}{\sqrt{1-e^2}}\left[\cos(\alpha+f) + e\cos\alpha\right]\right\} + O\left(10^{-3k}\right)\end{aligned} \quad (37)$$

where

$$\varphi = 2\omega + \Delta\omega - \eta - \eta_f, \quad \Delta\beta = \Delta\omega - \Delta\eta, \quad \alpha = \omega + \Delta\omega - \eta_f \quad (38)$$

The variable f can be replaced by E through Eqs. 4, if necessary. When i^r, Δe, ΔM and $\Delta\beta$ (namely $\Delta\omega - \Delta\eta$) are all of first order, neglecting all the quadratic terms of these small quantities in Eqs. 35–37, yields the first-order approximate equations of relative motion

$$\begin{cases} x = -a\Delta e\cos f + \dfrac{a\Delta M}{\sqrt{1-e^2}}e\sin f \\ y = \dfrac{a\Delta M}{\sqrt{1-e^2}} + \dfrac{a\Delta M}{\sqrt{1-e^2}}e\cos f + a\Delta e\sin f + \dfrac{a\Delta e\sin f}{1+e\cos f} + \dfrac{a\left(1-e^2\right)\Delta\beta}{1+e\cos f} \\ z = a\left(1-e^2\right)\sin i^r\left(\dfrac{\sin\alpha\cos f}{1+e\cos f} + \dfrac{\cos\alpha\sin f}{1+e\cos f}\right) \end{cases} \quad (39)$$

Note that when i^r is of first order, then $\sin^2(i^r/2)$ is of second order and $\cos^2(i^r/2)$ is approximately 1.

The relationships between Eqs. 39 and the periodic solutions of famous Lawden's equations will be presented in next section. When the dimensionless distances in the x-, y-, and z-axes are not of the same orders of magnitude, for example, if the distance in the z-axis is larger than that in the x-and y-axes, the first-order equations (39) are not sufficient to describe relative motion because it becomes zero in the x- and y-axes. The case above results when $\Delta e = 0$, $\Delta M = 0$ and $\Delta\beta = 0$, and i^r not equal to zero. Applying them to the second-order equations (35–37) yields

$$\frac{x}{a} = \left(1 - e^2\right) \sin^2 \frac{i^r}{2} \frac{\cos(2f + \varphi) - 1}{1 + e\cos f} + O\left(10^{-3k}\right) \tag{40}$$

$$\frac{y}{a} = -\left(1 - e^2\right) \sin^2 \frac{i^r}{2} \frac{\sin(2f + \varphi)}{1 + e\cos f} + O\left(10^{-3k}\right) \tag{41}$$

$$\frac{z}{a} = \sin i^r \frac{\left(1 - e^2\right) \sin(\alpha + f)}{1 + e\cos f} \tag{42}$$

It is obvious that when $i^r = O(10^{-k})$, $d_z = O(10^{-k})$, but d_x, $d_y = O(10^{-2k})$ as well. So the derivation for the second-order equations of relative motion is necessary for this special case.

4 Relationship between the first-order relative motion equations and the periodic solutions of Lawden's equations

It is known that in order to keep the relative motion periodic with no secular changes, the semi-major axis of the follower should be equal to that of the leader. It is different from the periodicity condition derived from Lawden's equations and expressed by LVLH coordinates. In the above section, the first-order relative motion equations (39) are derived, of which the variable is true anomaly and constants are orbit elements, while the constants of the famous Lawden's equations are expressed by initial relative position and velocity. It seems that the two kinds of equations are different, in form at least. Schaub (2002) obtained that at the perigee of the leader orbit, the difference of semi-major axis being zero is actually equal to the periodicity condition of Lawden's equations. In fact what the author meant is not the 'difference' but the 'variation', because the conclusion was based upon the invertible matrix mapping the LVLH coordinates to the orbital element differences, while the mapping was linearized. As emphasized before, variation is the linear component of difference. This section will reveal the relationship between the periodicity condition and the equivalence of the semi-major axes by Taylor expansion of a multivariable function, which is easily understandable, though a little complicated. The relationship between the first-order relative motion equations and the periodic solutions of Lawden's equations will also be presented.

The leader's position vector and velocity vector with respect to the Earth center are expressed in the LVLH frame as

$$\boldsymbol{r} = r\boldsymbol{i}_c, \quad \boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\boldsymbol{i}_c + r\dot{\theta}\boldsymbol{j}_c \tag{43}$$

where $\boldsymbol{i}_c = \boldsymbol{r}/r$, $\boldsymbol{k}_c = (\boldsymbol{r} \times \dot{\boldsymbol{r}}) / |\boldsymbol{r} \times \dot{\boldsymbol{r}}|$ and $\boldsymbol{j}_c = \boldsymbol{k}_c \times \boldsymbol{i}_c$ are the unit vectors of the x-, y- and z-axes of the leader's LVLH frame, and for Keplerian motion of an ellipse

$$\dot{f} = \dot{\theta} = \frac{\sqrt{\mu a \left(1 - e^2\right)}}{r^2} \tag{44}$$

Then deriving from Eq. 2, it yields

$$\dot{r} = \frac{a\left(1 - e^2\right) e \sin f}{\left(1 + e \cos f\right)^2} \dot{f} = \sqrt{\frac{\mu}{a\left(1 - e^2\right)}} e \sin f \tag{45}$$

Substitution of Eqs. 44 and 45 into Eqs. 43 yields

$$\boldsymbol{r} = r\boldsymbol{i}_c, \quad \boldsymbol{v} = \dot{\boldsymbol{r}} = \sqrt{\frac{\mu}{a\left(1 - e^2\right)}} \left[e \sin f \boldsymbol{i}_c + (1 + e\cos f)\boldsymbol{j}_c\right] \tag{46}$$

Use $(x, y, z)^T$ and $(\dot{x}, \dot{y}, \dot{z})^T$ to denote the position vector and velocity vector of the follower with respect to the leader in the LVLH frame, respectively (where the velocity is observed in an inertial reference frame).

$$\Delta\boldsymbol{r} = x\boldsymbol{i}_c + y\boldsymbol{j}_c + z\boldsymbol{k}_c, \quad \Delta\dot{\boldsymbol{r}} = \Delta\boldsymbol{v} = \dot{x}\boldsymbol{i}_c + \dot{y}\boldsymbol{j}_c + \dot{z}\boldsymbol{k}_c \tag{47}$$

Using $\left(\dot{\tilde{x}}, \dot{\tilde{y}}, \dot{\tilde{z}}\right)^T$ to denote the relative velocity vector observed in the LVLH frame, the relationship between them is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \\ \dot{\tilde{z}} \end{pmatrix} + \begin{pmatrix} 0 & -\dot{f} & 0 \\ \dot{f} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{48}$$

When f is used as the free variable instead of time t, denoting $(\cdot)' = d(\cdot)/df$ to get the relationship $\dot{(\cdot)} = (\cdot)'\dot{f}$, Eq. 48 yields

$$\begin{cases} \dot{x} = \dfrac{\sqrt{\mu a \left(1 - e^2\right)}}{r^2} \left(\tilde{x}' - y\right) \\ \dot{y} = \dfrac{\sqrt{\mu a \left(1 - e^2\right)}}{r^2} \left(\tilde{y}' + x\right) \\ \dot{z} = \dfrac{\sqrt{\mu a \left(1 - e^2\right)}}{r^2} \tilde{z}' \end{cases} \tag{49}$$

Then substituting Eqs. 49 into Eqs. 47, yields

$$\Delta\boldsymbol{r} = x\boldsymbol{i}_c + y\boldsymbol{j}_c + z\boldsymbol{k}_c, \quad \Delta\dot{\boldsymbol{r}} = \Delta\boldsymbol{v} = \frac{\sqrt{\mu a \left(1 - e^2\right)}}{r^2} \left[\left(\tilde{x}' - y\right)\boldsymbol{i}_c + \left(\tilde{y}' + x\right)\boldsymbol{j}_c + \tilde{z}'\boldsymbol{k}_c\right] \tag{50}$$

It is known that for Keplerian motion of an ellipse, the semi-major axis of the leader satellite can be expressed as

$$a = \frac{\mu r}{2\mu - r v^2} \tag{51}$$

The follower's semi-major axis can be expanded around the leader's position and velocity, i.e. the leader's semi-major axis, by Taylor series of a multivariable function, while Δr and Δv are treated as small quantities.

$$\Delta a = \left[\frac{\partial a}{\partial r}, \frac{\partial a}{\partial v}\right]\begin{bmatrix}\Delta r \\ \Delta v\end{bmatrix} + \frac{1}{2}\left[\Delta r, \Delta v\right]\begin{bmatrix}\dfrac{\partial^2 a}{\partial r^2} & \dfrac{\partial^2 a}{\partial r \partial v} \\ \dfrac{\partial^2 a}{\partial r \partial v} & \dfrac{\partial^2 a}{\partial v^2}\end{bmatrix}\begin{bmatrix}\Delta r \\ \Delta v\end{bmatrix} + \cdots \tag{52}$$

Differentiating Eq. 51 with respect to r and v, respectively, yields

$$\frac{\partial a}{\partial r} = \frac{2a^2}{r^2}, \quad \frac{\partial a}{\partial v} = \frac{2a^2 v}{\mu} \tag{53}$$

$$\frac{\partial^2 a}{\partial r^2} = \frac{4a^3 v^2}{\mu r^3}, \quad \frac{\partial^2 a}{\partial r \partial v} = \frac{8a^3 v}{\mu r^2}, \quad \frac{\partial^2 a}{\partial v^2} = \frac{2a^2\left(8a - 3r\right)}{\mu r} \tag{54}$$

The increments of r and v with respect to small increments of the leader's position and velocity, namely $\Delta \boldsymbol{r}$ and $\Delta \boldsymbol{v}$, can be expressed by Taylor expansion as

$$\begin{aligned}\Delta r &= \nabla r \cdot \Delta \boldsymbol{r} + \frac{1}{2}\Delta \boldsymbol{r} \cdot \nabla\left(\nabla r\right) \cdot \Delta \boldsymbol{r} + \cdots, \\ \Delta v &= \nabla v \cdot \Delta \boldsymbol{v} + \frac{1}{2}\Delta \boldsymbol{v} \cdot \nabla\left(\nabla v\right) \cdot \Delta \boldsymbol{v} + \cdots\end{aligned} \tag{55}$$

Note that ∇ denotes the gradient operator. It is well known that the gradient of a scalar function is a vector, and that of a vector function is a tensor. It is also well known that

$$\nabla r = \boldsymbol{r}/r, \quad \nabla\left(\nabla r\right) = \nabla\left(\frac{\boldsymbol{r}}{r}\right) = \frac{\nabla \boldsymbol{r}}{r} - \frac{\boldsymbol{r}\boldsymbol{r}}{r^3} \tag{56}$$

The function v possesses the same property as r when we replace r by v in Eqs. 56. Substituting Eqs. 46, 50 and 56 into Eqs. 55, we obtain

$$\begin{aligned}\Delta r &= \nabla r \cdot \Delta \boldsymbol{r} + \frac{1}{2}\Delta \boldsymbol{r} \cdot \nabla\left(\nabla r\right) \cdot \Delta \boldsymbol{r} + \cdots = \frac{\boldsymbol{r} \cdot \Delta \boldsymbol{r}}{r} + \frac{1}{2r}\Delta \boldsymbol{r} \cdot \left(\nabla \boldsymbol{r} - \frac{\boldsymbol{r}\boldsymbol{r}}{r^2}\right) \cdot \Delta \boldsymbol{r} + \cdots \\ &= \left[\boldsymbol{i}_c + \frac{1}{2}\left(x\boldsymbol{i}_c + y\boldsymbol{j}_c + z\boldsymbol{k}_c\right) \cdot \frac{1}{r}\left(\boldsymbol{j}_c\boldsymbol{j}_c + \boldsymbol{k}_c\boldsymbol{k}_c\right)\right] \cdot \left(x\boldsymbol{i}_c + y\boldsymbol{j}_c + z\boldsymbol{k}_c\right) + \cdots \\ &= x + \frac{y^2 + z^2}{2r} + \cdots\end{aligned} \tag{57}$$

In order to work out Δv, first we can derive that

$$\begin{aligned}\Delta \boldsymbol{v} \cdot \left(\nabla \boldsymbol{v} - \frac{\boldsymbol{v}\boldsymbol{v}}{v^2}\right) \cdot \Delta \boldsymbol{v} &= \Delta \boldsymbol{v} \cdot \nabla \boldsymbol{v} \cdot \Delta \boldsymbol{v} - \left(\frac{\Delta \boldsymbol{v} \cdot \boldsymbol{v}}{v}\right)^2 \\ &= \frac{\mu a\left(1 - e^2\right)}{r^4}\left[\left(\tilde{x'} - y\right)^2 + \left(\tilde{y'} + x\right)^2 + \left(\tilde{z'}\right)^2\right] \\ &\quad - \frac{\mu^2}{r^4 v^2}\left[e \sin f\left(\tilde{x'} - y\right) + \left(1 + e\cos f\right)\left(\tilde{y'} + x\right)\right]^2\end{aligned} \tag{58}$$

then derive that

$$\begin{aligned}\Delta v &= \frac{\boldsymbol{v}\cdot\Delta\boldsymbol{v}}{v} + \frac{1}{2v}\Delta\boldsymbol{v}\cdot\left(\nabla\boldsymbol{v} - \frac{\boldsymbol{v}\boldsymbol{v}}{v^2}\right)\cdot\Delta\boldsymbol{v} + \cdots \\ &= \frac{\mu}{r^2 v}\left[e\sin f\left(\tilde{x}' - y\right) + (1+e\cos f)\left(\tilde{y}' + x\right)\right] \\ &\quad + \frac{\mu a\left(1-e^2\right)}{2r^4 v}\left[\left(\tilde{x}' - y\right)^2 + \left(\tilde{y}' + x\right)^2 + \left(\tilde{z}'\right)^2\right] \\ &\quad - \frac{\mu^2}{2r^4 v^3}\left[e\sin f\left(\tilde{x}' - y\right) + (1+e\cos f)\left(\tilde{y}' + x\right)\right]^2 + \cdots \end{aligned} \tag{59}$$

Substitution of Eqs. 53, 54, 57 and 59 into Eq. 52 yields

$$\Delta a = a^{(1)} + a^{(2)} + \cdots \tag{60}$$

where $a^{(1)}$ and $a^{(2)}$ are the first-order and second-order terms of Δa, respectively.

$$a^{(1)} = \frac{2a^2}{r^2}\left[(2+e\cos f)\,x - (e\sin f)\,y + (e\sin f)\,\tilde{x}' + (1+e\cos f)\,\tilde{y}'\right] \tag{61}$$

$$\begin{aligned} a^{(2)} = \frac{\left(a^{(1)}\right)^2}{a} + \frac{a^3}{r^4}\Big\{\left(1-e^2\right)\left[\left(x+\tilde{y}'\right)^2 + \left(\tilde{x}' - y\right)^2 + \left(\tilde{z}'\right)^2\right] \\ -\frac{r}{a}\left[2x^2 - y^2 - z^2\right]\Big\} \end{aligned} \tag{62}$$

Setting $a^{(1)} = 0$ leads to

$$(2+e\cos f)x - (e\sin f)\,y + (e\sin f)\,\tilde{x}' + (1+e\cos f)\tilde{y}' = 0 \tag{63}$$

In fact, the equation above is the periodicity condition of Lawden's equations with arbitrary initial conditions. When $f = 0$, it leads to $\tilde{y}'(0)/x(0) = -(2+e)/(1+e)$, which is just the periodicity condition obtained by Inalhan et al. (2002). When $e = 0$, Eq. 63 leads to $\tilde{y}'(0) = -2x(0)$, which is the periodicity condition of the C-W equations (Clohessy and Wiltshire 1960). When $a^{(1)} = 0$, it does not mean $\Delta a = 0$, so the periodicity condition in Lawden's equations is just equivalent to the follower semi-major axis's Taylor series of order one being zero. That is to say, this condition can be formulated as $\delta a = 0$, which is a first-order approximation to the precise periodicity condition $\Delta a = 0$.

The periodic solutions of Lawden's equations (Inalhan et al. 2002) are written here as

$$\begin{cases} x = -d_3\cos f + d_1 e\sin f + d_2\left[2e^2 H(f)\sin f - \dfrac{e\cos f}{(1+e\cos f)^2}\right] \\ y = d_1(1+e\cos f) + d_3\sin f\left(1 + \dfrac{1}{1+e\cos f}\right) + \dfrac{d_4}{1+e\cos f} + 2d_2 eH(f)(1+e\cos f) \\ z = \dfrac{d_5\sin f}{1+e\cos f} + \dfrac{d_6\cos f}{1+e\cos f} \end{cases} \tag{64}$$

where the integration constants are expressed by initial ($f = 0$) position and velocity as

$$\begin{aligned} & d_1 = \tilde{x}'(0)\big/e, \quad d_2 = (1+e)^2\left[(2+e)\,x(0) + (1+e)\,\tilde{y}'(0)\right]\big/e^2 = 0 \\ & d_3 = -(1+e)\left[2x(0) + \tilde{y}'(0)\right]\big/e, \quad d_4 = (1+e)\left[-(1+e)\,\tilde{x}'(0) + ey(0)\right]\big/e \\ & d_5 = (1+e)\,\tilde{z}'(0)\,, \quad d_6 = (1+e)\,z(0) \end{aligned} \tag{65}$$

Based on Eq. 27, the constant $\Delta\beta$ defined in Eqs. 38 can be expressed by classical orbit elements as

$$\Delta\beta = \Delta\omega - \Delta\eta \approx \Delta\omega + \Delta\Omega\cos i \tag{66}$$

Deriving from Eqs. 9 and 23, the relationships for α defined in Eqs. 38 are derived as

$$\begin{aligned} \sin i^r \sin\alpha &= \sin i^r \sin\left(\omega + \Delta\omega - \eta_f\right) \approx \sin\omega\sin i^r\cos\eta_f - \cos\omega\sin i^r\sin\eta_f \\ &= \sin\omega\left(\cos i\sin i_f - \sin i\cos i_f\cos\Delta\Omega\right) - \cos\omega\sin i\sin\Delta\Omega \\ &\approx \Delta i\sin\omega - \Delta\Omega\cos\omega\sin i \end{aligned} \tag{67}$$

$$\sin i^r\cos\alpha \approx \cos\omega\sin i^r\cos\eta_f + \sin\omega\sin i^r\sin\eta_f \approx \Delta i\cos\omega + \Delta\Omega\sin\omega\sin i \tag{68}$$

Note that the approximations $\sin x \approx x$, $\cos x \approx 1$ for x is small are applied.

As analyzed at the beginning of this section, $d_2 = 0$ is equivalent to $\delta a = a^{(1)} = 0$, namely $\tilde{y}'(0)\big/x(0) = -(2+e)/(1+e)$. Setting relative position and velocity as basic variables, the variations of unit vectors in the LVLH frame are developed as

$$\delta \boldsymbol{i}_c = \delta\left(\frac{\boldsymbol{r}}{r}\right) = \frac{\delta\boldsymbol{r}}{r} - \frac{\boldsymbol{r}}{r^2}\delta r = \frac{x\boldsymbol{i}_c + y\boldsymbol{j}_c + z\boldsymbol{k}_c}{r} - \frac{r\boldsymbol{i}_c}{r^2}x = \frac{(1+e\cos f)\left(y\boldsymbol{j}_c + z\boldsymbol{k}_c\right)}{a\left(1-e^2\right)} \tag{69}$$

$$\begin{aligned} \delta\boldsymbol{k}_c &= \delta\left(\frac{\boldsymbol{r}\times\dot{\boldsymbol{r}}}{|\boldsymbol{r}\times\dot{\boldsymbol{r}}|}\right) = \frac{\delta(\boldsymbol{r}\times\dot{\boldsymbol{r}}) - \left[\boldsymbol{k}_c\cdot\delta(\boldsymbol{r}\times\dot{\boldsymbol{r}})\right]\boldsymbol{k}_c}{|\boldsymbol{r}\times\dot{\boldsymbol{r}}|} \\ &= \frac{-(1+e\cos f)\,z\boldsymbol{i}_c + \left[(e\sin f)\,z - (1+e\cos f)\,\tilde{z}'\right]\boldsymbol{j}_c}{a\left(1-e^2\right)} \end{aligned} \tag{70}$$

$$\begin{aligned} \delta\boldsymbol{j}_c &= \delta\left(\boldsymbol{k}_c\times\boldsymbol{i}_c\right) = (\delta\boldsymbol{k}_c)\times\boldsymbol{i}_c + \boldsymbol{k}_c\times(\delta\boldsymbol{i}_c) \\ &= \frac{-(1+e\cos f)\,y\boldsymbol{i}_c - \left[(e\sin f)\,z - (1+e\cos f)\,\tilde{z}'\right]\boldsymbol{k}_c}{a\left(1-e^2\right)} \end{aligned} \tag{71}$$

where since the leader's position and velocity are set as basic variables, $\delta\boldsymbol{r}$ and $\delta\dot{\boldsymbol{r}}$ are equivalent to $\Delta\boldsymbol{r}$ and $\Delta\dot{\boldsymbol{r}}$ given by Eqs. 50, respectively.

For Keplerian motion of an ellipse,

$$\begin{cases} e\cos E = 1 - r/a \\ e\sin E = r\cdot\dot{r}\big/\sqrt{\mu a} \end{cases} \tag{72}$$

It can be derived from Eqs. 72 that

$$\delta\left(e^2\right) = \delta\left[(1-r/a)^2\right] + \delta\left[\left(\boldsymbol{r}\cdot\dot{\boldsymbol{r}}\big/\sqrt{\mu a}\right)^2\right] \tag{73}$$

$$\cos E\delta e - e\sin E\delta E = -\delta\,(r/a) \tag{74}$$

When $\delta a = 0$, they become

$$\delta e = \frac{1}{a}\left(\tilde{x}' \sin f - x \cos f\right) \tag{75}$$

$$\delta E = \frac{x \sin f + \tilde{x}' (\cos f + e)}{ae\sqrt{1-e^2}} \tag{76}$$

By deriving the variation of M from Eq. 3, then replacing its variable E by f through Eqs. 5, we can obtain

$$\delta M = \frac{\sqrt{1-e^2}\left[x \sin f + \tilde{x}' \cos f\right]}{ae} \tag{77}$$

Substituting Eqs. 75 and 76 into Eq. 32 yields

$$\delta f = \frac{x \sin f + \tilde{x}'\left[2e + \left(1+e^2\right)\cos f\right]}{ae\left(1-e^2\right)} \tag{78}$$

The rotation matrix, transforming any vector in the LVLH frame of the leader to the ECI frame, can be characterized by an Euler 3-1-3 rotation. When consider counterclockwise as positive, the three Eulerian angles are $-\theta$, $-i$ and $-\Omega$ in turn. As in Sect. 2, we can write the composite rotation matrix, i.e. the direction cosine matrix, as

$$\begin{bmatrix} \boldsymbol{i}\cdot\boldsymbol{i_c} & \boldsymbol{i}\cdot\boldsymbol{j_c} & \boldsymbol{i}\cdot\boldsymbol{k_c} \\ \boldsymbol{j}\cdot\boldsymbol{i_c} & \boldsymbol{j}\cdot\boldsymbol{j_c} & \boldsymbol{j}\cdot\boldsymbol{k_c} \\ \boldsymbol{k}\cdot\boldsymbol{i_c} & \boldsymbol{k}\cdot\boldsymbol{j_c} & \boldsymbol{k}\cdot\boldsymbol{k_c} \end{bmatrix} = \begin{bmatrix} c\Omega c\theta - s\Omega s\theta ci & -c\Omega s\theta - s\Omega c\theta ci & s\Omega si \\ s\Omega c\theta + c\Omega s\theta ci & -s\Omega s\theta + c\Omega c\theta ci & -c\Omega si \\ s\theta si & c\theta si & ci \end{bmatrix} \tag{79}$$

where $\boldsymbol{i}$,$\boldsymbol{j}$ and $\boldsymbol{k}$ are the unit vectors of the X-, Y- and Z-axes of the ECI frame. It can be determined from Eq. 79 that

$$\delta\left(\boldsymbol{k}\cdot\boldsymbol{k}_c\right) = \delta\left(\cos i\right), \quad \delta\left(\boldsymbol{k}\cdot\boldsymbol{i}_c\right) = \delta\left(\sin\theta \sin i\right), \quad \delta\left(\boldsymbol{i}\cdot\boldsymbol{k}_c\right) = \delta\left(\sin\Omega \sin i\right) \tag{80}$$

Because $\boldsymbol{i}$,$\boldsymbol{j}$ and $\boldsymbol{k}$ are the unit vectors of the Earth-centered-inertial frame, $\delta\boldsymbol{i} = \delta\boldsymbol{j} = \delta\boldsymbol{k} = 0$. Applying Eqs. 69 and 70 to Eqs. 80 and solving them, results in

$$\delta i = \frac{z\left(1+e\cos f\right)\sin\theta + \left[\tilde{z}'\left(1+e\cos f\right) - ze\sin f\right]\cos\theta}{a\left(1-e^2\right)} \tag{81}$$

$$\delta\Omega = \frac{-z\left(1+e\cos f\right)\cos\theta + \left[\tilde{z}'\left(1+e\cos f\right) - ze\sin f\right]\sin\theta}{a\left(1-e^2\right)\sin i} \tag{82}$$

$$\delta\theta = \frac{y\left(1+e\cos f\right)\tan i + z\left(\cos\theta + e\cos\omega\right) - \tilde{z}'\left(1+e\cos f\right)\sin\theta}{a\left(1-e^2\right)\tan i} \tag{83}$$

When $f = 0$, Eqs. 75, 77, 78, 81, 82 and 83 are reduced to

$$\delta M = \frac{\sqrt{1-e^2}\tilde{x}'(0)}{ae}, \quad \delta\Omega = \frac{-z(0)\cos\omega + \tilde{z}'(0)\sin\omega}{a(1-e)\sin i}, \quad \delta i = \frac{z(0)\sin\omega + \tilde{z}'(0)\cos\omega}{a(1-e)}$$
$$\delta e = -\frac{x(0)}{a}, \quad \delta\omega = \delta\theta - \delta f = \frac{z(0)\cos\omega - \tilde{z}'(0)\sin\omega}{a(1-e)\tan i} + \frac{ey(0) - (1+e)\tilde{x}'(0)}{ae(1-e)} \tag{84}$$

From Eqs. 84 and the periodicity condition $\tilde{y}'(0)\big/x(0) = -(2+e)/(1+e)$, it is not difficult to substantiate that Eqs. 65 are equivalent to

$$\begin{aligned} &d_1 = a\delta M\Big/\sqrt{1-e^2}, \quad d_2 = 0, \quad d_3 = a\delta e, \quad d_4 = a\left(1-e^2\right)(\delta\omega + \delta\Omega\cos i) \\ &d_5 = a\left(1-e^2\right)(\delta i\cos\omega + \delta\Omega\sin\omega\sin i), \\ &d_6 = a\left(1-e^2\right)(\delta i\sin\omega - \delta\Omega\cos\omega\sin i) \end{aligned} \tag{85}$$

Then the periodic solutions of Lawden's Eqs. 64 can be written as

$$\begin{aligned} &\text{periodicity condition: } \delta a = 0 \\ &\begin{cases} x = -a\delta e\cos f + \dfrac{a\delta M}{\sqrt{1-e^2}} e\sin f \\ y = \dfrac{a\delta M}{\sqrt{1-e^2}}(1+e\cos f) + a\delta e\sin f\left(1 + \dfrac{1}{1+e\cos f}\right) \\ \quad + \dfrac{a\left(1-e^2\right)(\delta\omega + \delta\Omega\cos i)}{1+e\cos f} \\ z = \dfrac{a\left(1-e^2\right)}{1+e\cos f}\left[(\delta i\cos\omega + \delta\Omega\sin\omega\sin i)\sin f + (\delta i\sin\omega - \delta\Omega\cos\omega\sin i)\cos f\right] \end{cases} \end{aligned} \tag{86}$$

For comparison, after substituting Eqs. 66, 67 and 68 into them, Eqs. 39 can be written as

$$\begin{aligned} &\text{periodicity condition: } \Delta a = 0 \\ &\begin{cases} x = -a\Delta e\cos f + \dfrac{a\Delta M}{\sqrt{1-e^2}} e\sin f \\ y = \dfrac{a\Delta M}{\sqrt{1-e^2}}(1+e\cos f) + a\Delta e\sin f\left(1 + \dfrac{1}{1+e\cos f}\right) \\ \quad + \dfrac{a\left(1-e^2\right)(\Delta\omega + \Delta\Omega\cos i)}{1+e\cos f} \\ z = \dfrac{a\left(1-e^2\right)}{1+e\cos f}\left[(\Delta i\cos\omega + \Delta\Omega\sin\omega\sin i)\sin f + (\Delta i\sin\omega - \Delta\Omega\cos\omega\sin i)\cos f\right] \end{cases} \end{aligned} \tag{87}$$

The first-order relative motion equations (87) possess the same forms as Eqs. 86, which are a set of equivalent transformations of the periodic solutions of Lawden's equations. They differ in two ways, the periodicity condition and the constants. As discussed at the beginning of this section, the periodicity condition of Lawden's equations is just a first-order approximation to the accurate periodicity condition of Eqs. 87. As mentioned above, the variation of a function is the linear component of its increment, i.e. difference, with respect to its variables. So including the periodicity conditions, the periodic solutions of Lawden's equations can be considered as further first-order approximations to Eqs. 87, even though they are derived two different ways. The former are the linearized solutions of a set of nonlinear differential equations, of which the integration constants are expressed by initial relative position and velocity. The latter is developed by Taylor expansion after regarding orbit elements and their small differences as basic quantities. It should be pointed out that though Eqs. 86 can be considered as further first-order approximations to Eqs. 87, it does not mean that the latter are more accurate than the former, compared with the precise relative motion equations (14). A simple example can substantiate this assertion. Let $g(x) = x + x^2$, where $x = \varepsilon - 2\varepsilon^2$ and ε is small. Both $g_1 = g\prime(0)\Delta x = \varepsilon - 2\varepsilon^2$ and $g_2 = g\prime(0)\delta x = \varepsilon$ are first-order approximations to g, and g_2 is also a first-order approximation to g_1.

A simple derivation yields that $|g - g_1| > |g - g_2|$ when $-1 \ll \varepsilon < 0$, which means g_2 is sometimes closer to g than g_1.

As first-order approximations to relative motion, the periodic solutions of Lawden's equations (86) and the first-order relative motion equations (87) are both of the same order of precision, but they are in a measure different. Equations (86) are more suitable for researching the problem over a small-scale time, such as spacecraft rendezvous and docking. Though the corresponding periodicity condition $\delta a = 0$ is not precise, the follower can remain close to the leader for a short time. The first-order equations (87) are more suitable for researching the problem over a large-scale time such as that required for the configuration of a relative orbit, because the periodicity condition $\Delta a = 0$ is so precise that the practical relative orbit is closed under no perturbation. But, Eqs. 86 are not suitable for researching relative orbit configurations, because though the periodicity. condition is satisfied, the practical distance between the follower and leader will increase over large enough time. Similarly, Eqs. 87 are not suitable for researching spacecraft rendezvous and docking, because the periodicity condition is too precise to allow researching scenarios where the follower only remains close to the leader momentarily, and/or where the semi-major axes may not be equal to each other.

5 Trigonometric series expansion and ellipse approximation to in-plane relative motion

In Sect. 3, the second-order equations of relative motion, described by orbital elements and true anomaly, are obtained. In this section, they will be expanded by trigonometric series, where eccentric anomaly is used as the only variable. By this transformation, some interesting characteristics of relative motion will appear. The method to approximate the trajectory by an ellipse is presented, and simulations are given to substantiate the feasibility of this method.

5.1 Trigonometric series expansion for in-plane relative motion

Substitution of Eqs. 4 into Eqs. 39 results in the relative motion equations using eccentric anomaly as the only variable.

$$\begin{cases} \dfrac{x}{a} = \dfrac{\Delta e}{e} - \dfrac{1-e^2}{e}\dfrac{\Delta e}{1-e\cos E} + \dfrac{e\Delta M \sin E}{1-e\cos E} \\ \dfrac{y}{a} = \Delta\beta - e\Delta\beta\cos E + \dfrac{\Delta e}{\sqrt{1-e^2}}\sin E + \dfrac{\sqrt{1-e^2}\Delta M}{1-e\cos E} + \dfrac{\sqrt{1-e^2}\Delta e \sin E}{1-e\cos E} \\ \dfrac{z}{a} = \sin i^r \left(-e\sin\alpha + \sin\alpha\cos E + \sqrt{1-e^2}\cos\alpha\sin E\right) \end{cases} \tag{88}$$

Since the equations above are 2π periodic functions, they can be expanded by trigonometric series, leaving the main problem of seeking the Fourier series' coefficients $b_n (n = 0, 1, 2, \ldots)$ of the following equation:

$$\frac{1}{1-e\cos E} = b_0 + \sum_{n=1}^{+\infty} b_n \cos nE \tag{89}$$

where

$$b_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1 - e\cos E}dE, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos nE}{1 - e\cos E}dE, \ \ n \geqslant 1 \tag{90}$$

Since the function expanded in Eq. 89 is even, there are no sine terms. In order to get the recurrence relation to calculate Eqs. 90, define a new definite integral

$$\bar{b}_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin E \sin nE}{1 - e\cos E}dE, \quad n \geqslant 1 \tag{91}$$

Integrating Eqs. 90 by parts and substituting Eq. 91 into it yields the following recurrence relation

$$\begin{cases} b_{n+1} = \dfrac{1}{e}b_n - \bar{b}_n \\ \bar{b}_{n+1} = \left(1 - \dfrac{1}{e^2}\right) b_n + \dfrac{1}{e}\bar{b}_n \end{cases}, \quad n \geqslant 1 \tag{92}$$

Combining Eqs. 92 leads to

$$b_{n+2} - \frac{2}{e}b_{n+1} + b_n = 0, \quad n \geqslant 1 \tag{93}$$

The recurrence formula above is easy to solve on the base of precalculating $b_0 = 1/\sqrt{1-e^2}$ and $b_1 = 2\lambda/\sqrt{1-e^2}$ (see the derivation in Appendix B1), noting that in Eqs. 90, b_0 is defined half the value of b_n when n tends to zero, so it should be doubled when employed in Eq. 93. The solutions of Eq. 93 are

$$b_0 = \frac{1}{\sqrt{1-e^2}}, \quad b_n = \frac{2}{\sqrt{1-e^2}}\lambda^n, \quad \bar{b}_n = \frac{2}{e}\lambda^n, \ \ n \geqslant 1 \tag{94}$$

where

$$\lambda = \frac{e}{1 + \sqrt{1-e^2}} \tag{95}$$

Substituting Eqs. 89 and 94 into Eqs. 88 and applying the product to sum formula of trigonometric functions (see the derivation in Appendix B2) results in the following equations

$$\begin{cases} \dfrac{x}{a} = X_0 + X_1 \cos E + U_1 \sin E + \sum\limits_{n=2}^{+\infty} (X_n \cos nE + U_n \sin nE) \\ \quad = X_0 + F \sum\limits_{n=1}^{+\infty} \lambda^n \sin(nE - \phi) \\ \dfrac{y}{a} = Y_0 + Y_1 \cos E + V_1 \sin E + \sum\limits_{n=2}^{+\infty} (Y_n \cos nE + V_n \sin nE) \\ \quad = Y_0 + Y_1 \cos E + V_1 \sin E + F \sum\limits_{n=2}^{+\infty} \lambda^n \cos(nE - \phi) \end{cases} \tag{96}$$

where

$$\begin{cases} X_0 = \dfrac{e\Delta e}{\sqrt{1-e^2}}, \quad X_1 = -2\dfrac{\sqrt{1-e^2}}{e}\lambda\Delta e, \quad U_1 = 2\lambda\Delta M \\ Y_0 = \Delta\beta + \Delta M, \quad Y_1 = 2\lambda\Delta M - e\Delta\beta, \quad V_1 = 2\dfrac{\sqrt{1-e^2}}{e}\lambda\Delta e + \dfrac{1}{\sqrt{1-e^2}}\Delta e \end{cases} \tag{97}$$

$$\begin{cases} X_n = -2\dfrac{\sqrt{1-e^2}}{e}\lambda^n \Delta e, U_n = Y_n \\ Y_n = 2\lambda^n \Delta M, \quad V_n = -X_n \end{cases}, \quad n \geqslant 2$$
$$F = \frac{2}{e}\sqrt{(e\Delta M)^2 + \left(\sqrt{1-e^2}\Delta e\right)^2}, \quad \phi = \arctan 2\left(\sqrt{1-e^2}\Delta e, e\Delta M\right) \tag{98}$$

The function arctan2(y, x), known as the four quadrant inverse tangent function, is defined as

$$\arctan 2\,(y,x) = \begin{cases} \arctan \dfrac{y}{x}, \text{ if } x \geqslant 0 \text{ and } y \geqslant 0 \\ \pi + \arctan \dfrac{y}{x}, \text{ if } x < 0 \\ 2\pi + \arctan \dfrac{y}{x}, \text{ if } x \geqslant 0 \text{ and } y < 0 \end{cases} \tag{99}$$

The z-component of relative motion in Eqs. 88 is not taken into account because of its simplicity when E is the only variable. Summarizing Eqs. 96 leads to the following characteristic of relative motion:

(1) Except the terms of order one, the trigonometric series' amplitudes are geometric series and corresponding phases are constant both in the x- and y-axes. The common ratio is λ and the phase angle in the y-axis is $\pi/2$ more than in the x-axis.
(2) When $e = 0$, the equations of relative motion expressed by trigonometric series become not singular but simple. The trajectory of relative motion in the x–y plane is an ellipse, of which the semi-major axis parallel to the x-axis is two times as long as the semi-minor axis parallel to the y-axis.

The true trajectory of the first-order relative motion in the x–y plane is not an ellipse, unless the trigonometric series of order higher than one are truncated. The error caused by truncation is decided by the common ratio $\lambda = e/\left(1+\sqrt{1-e^2}\right)$, which is approximately $0.5e$ when e is close to zero. For example, when $e = 0.01$, $\lambda = 0.005$, therefore, for the case of e not more than 0.01, truncating the terms of order higher than one in Eqs. 96 is acceptable.

5.2 Trigonometric series truncating the terms of order higher than one

Truncating the terms of order higher than one in Eqs. 96 yields

$$\begin{cases} \dfrac{x^{(1)}}{a} = X_0 + X_1 \cos E + U_1 \sin E = X_0 + \sqrt{X_1^2 + U_1^2}\sin(E-\phi) \\ \dfrac{y^{(1)}}{a} = Y_0 + Y_1 \cos E + V_1 \sin E = Y_0 + \sqrt{Y_1^2 + V_1^2}\cos\left(E-\phi'\right) \end{cases} \tag{100}$$

where ϕ is defined as Eq. 98, and $\phi' = \arctan 2\,(V_1, Y_1)$.

The parametric equations above express an ellipse, termed truncated ellipse, of which E is the variable. Because the trigonometric series' amplitudes in the x-axis and y-axis are both the same geometrical series after the first order, the maximum relative error (true error divided by a) caused by truncation, denoted by err_x and err_y, can be evaluated as

$$\begin{cases} err_x = \max\left(\left|\dfrac{x - x^{(1)}}{a}\right|\right) = \max\left(\left|F\sum\limits_{n=2}^{+\infty}\lambda^n \sin(nE - \phi) + O\left(10^{-2k}\right)\right|\right) \\ \qquad \leqslant F\sum\limits_{n=2}^{+\infty}\lambda^n + \left|O\left(10^{-2k}\right)\right| = \dfrac{F\lambda^2}{1-\lambda} + \left|O\left(10^{-2k}\right)\right| \\ err_y = \max\left(\left|\dfrac{y - y^{(1)}}{a}\right|\right) = \max\left(\left|F\sum\limits_{n=2}^{+\infty}\lambda^n \cos(nE - \phi) + O\left(10^{-2k}\right)\right|\right) \\ \qquad \leqslant F\sum\limits_{n=2}^{+\infty}\lambda^n + \left|O\left(10^{-2k}\right)\right| = \dfrac{F\lambda^2}{1-\lambda} + \left|O\left(10^{-2k}\right)\right| \end{cases} \tag{101}$$

Equations 101 give the upper bound of the relative error when Eqs. 100 are used to approximate relative motion. Define two indexes as follows:

$$\begin{cases} erx = F\lambda^2/(1-\lambda) - err_x \\ ery = F\lambda^2/(1-\lambda) - err_y \end{cases} \tag{102}$$

The two indexes should be both not less than $-|O(10^{-2k})|$ according to Eqs. 101.

Numerical computation can examine whether Eqs. 101 are reasonable by examining whether the new defined two indexes are both not less than $-|O(10^{-2k})|$, where err_x and err_y are obtained by numerical computation.

Example 1: The leader orbital elements and corresponding differences are given as: $(a, i, \Omega, M_0, \omega) = (3 \times 10^7\ \text{m}, 1.1\ \text{rad}, 0, 0, 0)$, e varies from 0.01 to 0.8; $(\Delta a, \Delta e, \Delta i, \Delta\Omega, \Delta M_0, \Delta\omega) = (0, 0.001, 0.001\ \text{rad}, 0.001\text{rad}, 0.001\text{rad}, 0.001\text{rad})$. The graphs of *erx* and *ery* with respect to *e* are shown as Fig. 3, where the solid line, dash line, upper dash-dot and lower dash-dot line denote *erx*, *ery*, 5×10^{-6} and -5×10^{-6}, respectively.

As shown in Fig. 3, the graphs of *erx* and *ery* are always on the top of the graph of -5×10^{-6}, so Eqs. 101 are reasonable. The closer to the zero line the graphs of *erx* and

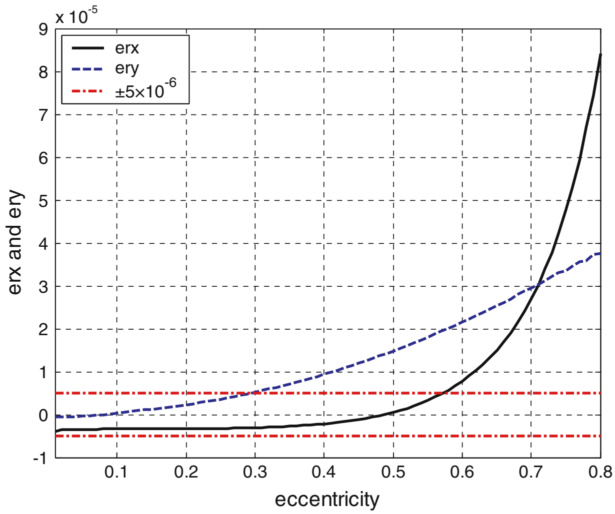

Fig. 3 The graphs of *erx* and *ery* with respect to *e*

ery lie, the more accurate the formula $F\lambda^2/(1-\lambda)$ is, when used to evaluate relative error caused by truncating the terms of order higher than one in Eqs. 96. Fig. 3 shows that the evaluation is suitable for the case of $e \leqslant 0.3$.

It is known from analytic geometry that Eqs. 100 generally represents the trajectory of an ellipse, except when $X_1V_1 = Y_1U_1$, in which case it represents a line segment. Define

$$\begin{cases} \lambda_1 = \dfrac{1}{2}\left[X_1^2 + U_1^2 + Y_1^2 + V_1^2 + \sqrt{\left(X_1^2 + U_1^2 + Y_1^2 + V_1^2\right)^2 - 4\left(X_1V_1 - Y_1U_1\right)^2}\right] \\ \lambda_2 = \dfrac{1}{2}\left[X_1^2 + U_1^2 + Y_1^2 + V_1^2 - \sqrt{\left(X_1^2 + U_1^2 + Y_1^2 + V_1^2\right)^2 - 4\left(X_1V_1 - Y_1U_1\right)^2}\right] \end{cases} \tag{103}$$

In the LVLH frame's x–y plane, using $\left(x_0^{(1)}, y_0^{(1)}\right)$, $A^{(1)}$, $B^{(1)}$ and $\vartheta^{(1)}$ to denote the ellipse's center coordinates, semi-major axis, semi-minor axis and the included angle between the semi-major axis and x-axis, respectively, it can be shown that (see the derivation in Appendix B3)

$$\begin{cases} \left(x_0^{(1)}, y_0^{(1)}\right) = (X_0, Y_0), \quad A^{(1)} = \sqrt{\lambda_1}, \quad B^{(1)} = \sqrt{\lambda_2} \\ \vartheta^{(1)} = \dfrac{1}{2}\arctan 2\left[2\left(X_1Y_1 + U_1V_1\right), X_1^2 + U_1^2 - Y_1^2 - V_1^2\right] \end{cases} \tag{104}$$

The ellipse expressed in the preceding equations is just a first-order approximation to the trigonometric series in the x–y plane. At this point it is not certain whether this is the most approximate ellipse, which will be addressed below.

5.3 Ellipse approximation to the first-order in-plane relative motion

By assuming that there is an approximate ellipse with its parameters expressed by $(x_0, y_0), A, B$ and ϑ, and defining a new function $\chi(E)$ to describe the phase variation in the x–y plane, the new ellipse's parametric equations, denoted by $\hat{x}/\hat{y}$ in the x/y-axis, can be written as

$$\begin{cases} \hat{x} = x_0 + A\cos\vartheta\cos\left(\chi\left(E\right)\right) - B\sin\vartheta\sin\left(\chi\left(E\right)\right) \\ \hat{y} = y_0 + A\sin\vartheta\cos\left(\chi\left(E\right)\right) + B\cos\vartheta\sin\left(\chi\left(E\right)\right) \end{cases} \tag{105}$$

It should be emphasized that the assumption made above is reasonable only if the real trajectory of relative motion in the x–y plane is not similar to a figure-eight shape or other non-convex figure, but rather an ellipse.

Defining a new quantity ε, we now analyze the following expressions

$$\begin{aligned} \varepsilon &= \frac{\left[(x/a - x_0)\cos\vartheta + (y/a - y_0)\sin\vartheta\right]^2}{A^2} + \frac{\left[-(x/a - x_0)\sin\vartheta + (y/a - y_0)\cos\vartheta\right]^2}{B^2} \\ &= C_1\left(x/a - x_0\right)^2 + 2C_2\left(x/a - x_0\right)\left(y/a - y_0\right) + C_3\left(y/a - y_0\right)^2 \end{aligned} \tag{106}$$

where

$$\begin{cases} C_1 = \left(\dfrac{\cos^2\vartheta}{A^2} + \dfrac{\sin^2\vartheta}{B^2}\right) = \dfrac{1}{2}\left(\dfrac{1}{A^2} + \dfrac{1}{B^2}\right) + \dfrac{1}{2}\left(\dfrac{1}{A^2} - \dfrac{1}{B^2}\right)\cos 2\vartheta \\ C_2 = \dfrac{1}{2}\left(\dfrac{1}{A^2} - \dfrac{1}{B^2}\right)\sin 2\vartheta \\ C_3 = \left(\dfrac{\sin^2\vartheta}{A^2} + \dfrac{\cos^2\vartheta}{B^2}\right) = \dfrac{1}{2}\left(\dfrac{1}{A^2} + \dfrac{1}{B^2}\right) - \dfrac{1}{2}\left(\dfrac{1}{A^2} - \dfrac{1}{B^2}\right)\cos 2\vartheta \end{cases} \tag{107}$$

In Eq. 106, the quantity ε is an index that indicates the "closeness" of the x–y trajectory to an ellipse. If the trajectory of the relative motion expressed by Eqs. 96 is exactly an ellipse, there exist five parameters A, B, x_0, y_0 and ϑ to make $\varepsilon = 1$. If not, it is worth discussing how to select these five parameters to make the trajectory close to an ellipse, i.e. to make ε as close to 1 as possible.

In order to obtain the trigonometric series of ε, the quadratic terms $x^2/a^2, y^2/a^2$ and xy/a^2 should be expanded first, while the most significant problem encountered is seeking the Fourier series' coefficients $c_n (n = 0, 1, 2, \ldots)$ of the following equation:

$$\frac{1}{(1 - e\cos E)^2} = c_0 + \sum_{n=1}^{+\infty} c_n \cos nE \tag{108}$$

where

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{(1 - e\cos E)^2} dE, \quad c_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos nE}{(1 - e\cos E)^2} dE, \ \ n \geqslant 1 \tag{109}$$

In order to get the recurrence relation to calculate Eqs. 109, define a new definite integral as follows

$$\bar{c}_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin E \sin nE}{(1 - e\cos E)^2} dE, \quad n \geqslant 1 \tag{110}$$

Integrating Eqs. 109 by parts and substituting Eqs. 90, 91, and 110 into it, yields the following recurrence relation

$$\begin{cases} c_{n+1} = \dfrac{1}{e}c_n - \bar{c}_n - \dfrac{1}{e}b_n \\ \bar{c}_{n+1} = \left(1 - \dfrac{1}{e^2}\right)c_n + \dfrac{1}{e}\bar{c}_n + \dfrac{2}{e^2}b_n - \dfrac{1}{e}\bar{b}_n \end{cases}, \quad n \geqslant 1 \tag{111}$$

Substituting for $b_n, \bar{b}_n$ from Eq. 94 and combining the two equations yields

$$\left[c_{n+2} - \frac{2(n+2)}{1 - e^2}\lambda^{n+2}\right] - \frac{2}{e}\left[c_{n+1} - \frac{2(n+1)}{1 - e^2}\lambda^{n+1}\right] + c_n - \frac{2n}{1 - e^2}\lambda^n = 0 \tag{112}$$

Similar to the solution of b_n above, by precalculating c_0=(1-e^2)$^{-1.5}$ and c_1=2e(1−e^2)$^{-1.5}$ (see the derivation in Appendix B1), the solution of c_n is

$$c_0 = \left(1 - e^2\right)^{-\frac{3}{2}}, \quad c_n = \frac{2}{1 - e^2}\left(n + \frac{1}{\sqrt{1 - e^2}}\right)\lambda^n, \ \ n \geqslant 1 \tag{113}$$

Squaring the x- and y-components of Eqs. 88 and substituting Eqs. 108 and 113 into them, after employing the product to sum formula of trigonometric function (see the derivation in Appendix B4), yields

$$\begin{cases} (x/a)^2 = G_0 + \sum\limits_{n=1}^{+\infty} (G_n \cos nE + H_n \sin nE) \\ (y/a)^2 = J_0 + \sum\limits_{n=1}^{+\infty} (J_n \cos nE + K_n \sin nE) \\ xy/a^2 = P_0 + \sum\limits_{n=1}^{+\infty} (P_n \cos nE + Q_n \sin nE) \\ (x/a)^2 + (y/a)^2 = R_0 + \sum\limits_{n=1}^{+\infty} (R_n \cos nE + S_n \sin nE) \end{cases} \tag{114}$$

It is worth pointing out that the amplitude of $(x/a)^2 + (y/a)^2$ is a geometric series with the common ratio λ after the second order. Though the amplitudes of $x^2/a^2, y^2/a^2$ and xy/a^2 are not geometric series, when n becomes big enough, they are approximate to geometric series and tend to zero asymptotically when $n \to \infty$. Based on Eqs. B19, B23, B26 and B30, these amplitudes are given by the following expressions.

$$\begin{cases} \sqrt{G_n^2 + H_n^2} = 2\lambda^n \left| n - \dfrac{1}{\sqrt{1-e^2}} \right| \left[(\Delta M)^2 + \dfrac{1-e^2}{e^2} (\Delta e)^2 \right], \quad n \geqslant 1 \\ \sqrt{J_n^2 + K_n^2} = 2\lambda^n \left(n + \dfrac{1}{\sqrt{1-e^2}} \right) \left[(\Delta M)^2 + \dfrac{1-e^2}{e^2} (\Delta e)^2 \right], \quad n \geqslant 3 \\ \sqrt{P_n^2 + Q_n^2} = 2n\lambda^n \left[(\Delta M)^2 + \dfrac{1-e^2}{e^2} (\Delta e)^2 \right], \quad n \geqslant 2 \\ \sqrt{R_n^2 + S_n^2} = \dfrac{4\lambda^n}{\sqrt{1-e^2}} \left[(\Delta M)^2 + \dfrac{1-e^2}{e^2} (\Delta e)^2 \right], \quad n \geqslant 3 \end{cases} \tag{115}$$

The Fourier series of $x/a, y/a, x^2/a^2, y^2/a^2$ and xy/a^2 are obtained as shown in Eqs. 96 and 114, so the Fourier series of ε in Eq. 106 can also be derived.

$$\varepsilon = \varepsilon_0 + \sum_{n=1}^{+\infty} (\varepsilon_n \cos nE + \upsilon_n \sin nE) \tag{116}$$

where the coefficients of ε are calculated by combining the coefficients of component terms $x/a, y/a, x^2/a^2, y^2/a^2$ and xy/a^2.

$$\begin{cases} \varepsilon_0 = C_1 \left(G_0 - 2x_0 X_0 + x_0^2\right) + 2\left(C_2 P_0 - x_0 Y_0 - y_0 X_0 + x_0 y_0\right) \\ \qquad + C_3 \left(J_0 - 2y_0 Y_0 + y_0^2\right) \\ \varepsilon_n = C_1 \left(G_n - 2x_0 X_n\right) + 2C_2 \left(P_n - x_0 Y_n - y_0 X_n\right) + C_3 \left(J_n - 2y_0 Y_n\right) \\ \upsilon_n = C_1 \left(H_n - 2x_0 U_n\right) + 2C_2 \left(Q_n - x_0 V_n - y_0 U_n\right) + C_3 \left(K_n - 2y_0 V_n\right) \end{cases}, n \geqslant 1 \tag{117}$$

Note that, when $n \geqslant 3$, there are general expressions as follows:

$$\begin{cases} \varepsilon_n = 2\lambda^n \left\{ \left[n\left(C_3 - C_1\right) + \dfrac{C_3 + C_1}{\sqrt{1-e^2}} \right] \left[(\Delta M)^2 - \dfrac{1-e^2}{e^2} (\Delta e)^2 \right] \right. \\ \qquad \left. - 4nC_2 \dfrac{\sqrt{1-e^2}}{e} \Delta e \Delta M + 2 \dfrac{\sqrt{1-e^2}}{e} \Delta e \left(C_1 x_0 + C_2 y_0\right) - 2\Delta M \left(C_2 x_0 + C_3 y_0\right) \right\} \\ \upsilon_n = 2\lambda^n \left\{ \left[n\left(C_3 - C_1\right) + \dfrac{C_3 + C_1}{\sqrt{1-e^2}} \right] \dfrac{2\sqrt{1-e^2}}{e} \Delta e \Delta M \right. \\ \qquad + 2nC_2 \left[(\Delta M)^2 - \dfrac{1-e^2}{e^2} (\Delta e)^2 \right] - 2\Delta M \left(C_1 x_0 + C_2 y_0\right) \\ \qquad \left. - 2 \dfrac{\sqrt{1-e^2}}{e} \Delta e \left(C_2 x_0 + C_3 y_0\right) \right\} \end{cases} \tag{118}$$

Even when $e = 0.9$, the function $n\lambda^n$ is a decreasing function for n larger than one, and tends to zero asymptotically as n tends to infinity, so only the first few terms of the series are significant.

As shown in Eq. 116, the quantity ε is expressed by a Fourier series. There are five parameters we can choose freely to make ε close to 1. Since the amplitudes of the first few terms of Fourier series are primary, one method is to regard $(x_0, y_0, C_1, C_2, C_3)$ as given and solve the equations

$$\begin{cases} \varepsilon_0 = 1 \\ \varepsilon_j = 0, \quad j = 1, 2 \\ \upsilon_j = 0, \quad j = 1, 2 \end{cases} \tag{119}$$

The solutions cause Eq. 116 collapse to

$$\varepsilon = 1 + \sum_{n=3}^{+\infty} \left(\varepsilon_n \cos nE + \upsilon_n \sin nE\right) \tag{120}$$

After obtaining the solutions of Eqs. 119, inverse the Eqs. 107 to yield

$$\begin{cases} A = \sqrt{2} \Big/ \sqrt{C_1 + C_3 - \sqrt{(C_1 - C_3)^2 + 4C_2^2}} \\ B = \sqrt{2} \Big/ \sqrt{C_1 + C_3 + \sqrt{(C_1 - C_3)^2 + 4C_2^2}} \\ \vartheta = \dfrac{1}{2} \arctan 2\left(-2C_2, C_3 - C_1\right) \end{cases} \tag{121}$$

Then the five parameters of the approximate ellipse are all given, leaving the only unknown in Eqs. 105 as $\chi(E)$. It can be solved approximately by substituting x/a and y/a from Eq. 88 for $\hat{x}$ and $\hat{y}$ from Eq. 105

$$\begin{cases} \cos\vartheta\,(x/a - x_0) + \sin\vartheta\,(y/a - y_0) \approx A\cos\left(\chi\,(E)\right) \\ -\sin\vartheta\,(x/a - x_0) + \cos\vartheta\,(y/a - y_0) \approx B\sin\left(\chi\,(E)\right) \end{cases} \tag{122}$$

Hence

$$\chi\,(E) \approx \arctan 2\left\{\left[-\sin\vartheta\,(x/a - x_0) + \cos\vartheta\,(y/a - y_0)\right]\big/B, \left[\cos\vartheta\,(x/a - x_0) + \sin\vartheta\,(y/a - y_0)\right]\big/A\right\} \tag{123}$$

Hereto the six unknowns of the approximate ellipse in Eqs. 105 are given in Eqs. 119, 121 and 123. Though solving the nonlinear Eqs. 119 analytically seems impossible, obtaining computational solutions is very easy. In order to illustrate the advantage of Eqs. 105 over Eqs. 100 to approximate first-order in-plane relative motion, an example is given as follows:

Example 2: The leader's COE and corresponding differences are given as

$$\begin{cases} (a, e, i, \Omega, M_0, \omega) = \left(1.5 \times 10^7 \text{ m}, 0.5, \pi/3 \text{ rad}, \pi/6 \text{ rad}, 0, \pi/4 \text{ rad}\right) \\ \Delta\,(a, e, i, \Omega, M_0, \omega) = (0, 0.001, 0.002 \text{ rad}, 0.003 \text{ rad}, 0.004 \text{ rad}, -0.001 \text{ rad}) \end{cases}$$

Then use Eqs. 104 to calculate the five parameters of the truncated ellipse Eqs. 100 as follows:

$$\begin{cases} x_0^{(1)} = 0.267949 \times 10^{-3} \\ y_0^{(1)} = 4.497403 \times 10^{-3} \end{cases}, \begin{cases} A^{(1)} = 3.109326 \times 10^{-3} \\ B^{(1)} = 1.928146 \times 10^{-3} \end{cases}, \vartheta^{(1)} = 0.999668 = 57.28^\circ$$

Use Eqs. 119 and 121 to calculate the five parameters of the approximate ellipse Eqs. 105 as follows:

$$\begin{cases} x_0 = -0.816504 \times 10^{-4} \\ y_0 = 4.926381 \times 10^{-3} \end{cases}, \begin{cases} A = 3.432228 \times 10^{-3} \\ B = 2.029528 \times 10^{-3} \end{cases}, \vartheta = 1.031120 = 59.08^\circ$$

As shown, these parameters are somewhat different between the truncated ellipse and the approximate ellipse. Substituting them into Eq. 117, respectively, yields

$$\begin{aligned} \varepsilon^{(1)} &= 1.078592 + 0.566989\cos(E + 0.214960) + 0.151924\cos(2E + 0.214960) \\ &\quad + 0.155922\cos(3E + 0.774023) + 0.0726920\cos(4E + 0.951973) \\ &\quad + 0.0280078\cos(5E + 1.022784) + 0.00981032\cos(6E + 1.060432) \\ &\quad + 0.00324877\cos(7E + 1.083732) + 0.00103698\cos(8E + 1.099557) + \cdots \\ \varepsilon &= 1.000000 + 0.115523\cos(3E + 1.039709) + 0.0615564\cos(4E + 1.142292) \\ &\quad + 0.0247520\cos(5E + 1.176613) + 0.00884890\cos(6E + 1.193752) \\ &\quad + 0.00296542\cos(7E + 1.204023) + 0.000953895\cos(8E + 1.210863) + \cdots \end{aligned}$$

As predicted, the parameters of the truncated ellipse $\varepsilon^{(1)}$ do not satisfy Eqs. 119. The numerical differences between the truncated and approximate ellipses are unable to show the advantage of the latter intuitively, unless shown in Fig. 4. The solid line represents the accurate trajectory achieved by computing the x–y plane relative motion equations (15) and (16), the dashed line represents the trajectory of the approximate ellipse with the asterisk as its center and the dot line represents the trajectory of the truncated ellipse with a small circle as its center. It is obvious that the trajectory of the approximate ellipse is closer to the accurate trajectory than the truncated ellipse. The similarity in trajectory does not necessarily imply similarity in motion because the phase may not synchronize. Figure 5, where the various lines represent the same quantities as in Fig. 4, shows that the motion of the approximate ellipse is also closer than the truncated ellipse in both the x- and y-axes. Note that in Figs. 4 and 5, x and y are both nondimensionalized by the semi-major axis of the leader. Thus the approximate ellipse expressed by Eqs. 105, with parameters given by Eqs. 119 and

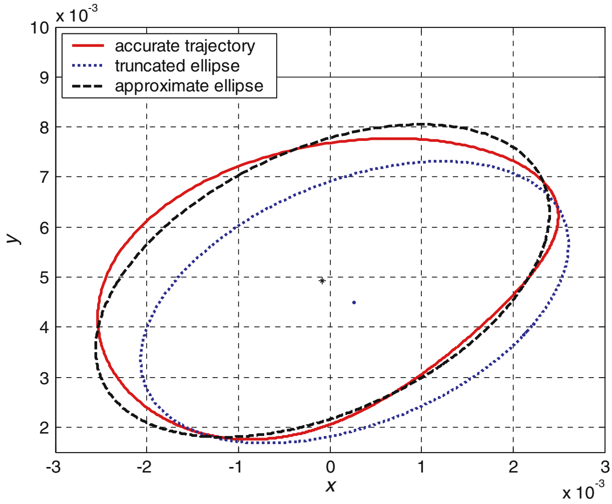

Fig. 4 Comparison of relative motion trajectories

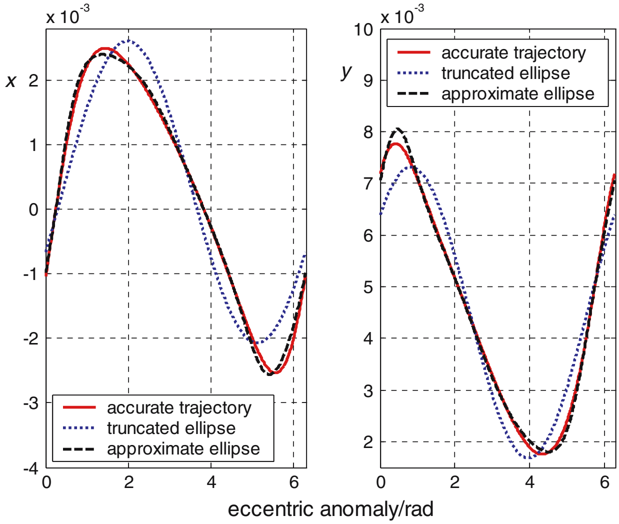

Fig. 5 Comparison of relative motion in the x–y plane

121, and the phase given by Eq. 123, is a useful approach to the real first-order relative motion.

6 Conclusions

The reference orbital element approach, which describes the relative motion on the celestial sphere, is applied to study the relative motion of satellite formation flying. In close formation, the dimensionless distance, defined as the ratio of the maximal distance between the leader and follower satellites to the leader semi-major axis, is a small quantity. In order to keep a formation close over time, the semi-major axis of the follower should equal to that of the leader. The differences of orbital elements such as eccentricity, orbit inclination, right ascension of the ascending node, mean anomaly, eccentric anomaly, argument of latitude, true anomaly and argument of perigee should all be small quantities with the same order of magnitude as the dimensionless distance.

The second-order relative motion equations are developed, of which the only variable is the true anomaly and all other quantities are constant. Setting relative position and velocity as fundamental variables, and expanding the semi-major axis of the follower around that of the leader by Taylor series, it is shown that the periodicity condition is equivalent to the Taylor series of order one being zero. It is then explained why the periodicity condition derived from Lawden's equations cannot be equivalent to the equality of the semi-major axes absolutely. The integration constants of the periodic solutions of Lawden's equations formulate by the variations of orbit elements instead of initial relative position and velocity. These periodic solutions possess similar forms as the first-order relative motion equations derived in this paper. The former can be considered as further first-order approximations of the latter. It is explained that the former is suitable for researching the problem of spacecraft rendezvous and docking, and the latter is suitable for researching the problem of relative orbit configurations.

Using eccentric anomaly as the angle variable, the first-order equations of in-plane relative motion are expanded as trigonometric series. Except the terms of order one, the trigonometric series' amplitudes are geometric series and corresponding phases are constant both in the x- and y-axes. When truncating the series of order higher than one, the trajectory expressed by the retained series is an ellipse. An index is defined to evaluate the upper bound of the error caused by truncation. When the trajectory of the in-plane relative motion is similar to an ellipse, a method to seek this ellipse is presented. An example is given to show that the approximate ellipse is useful to approximate the real first-order relative motion, at least more accurately than the truncated ellipse.

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Appendix A

In order to derive the second-order relative motion equations from the order-of-magnitude relations, express the quantities of the follower as that of the leader adding the corresponding differences, such as $r_f = r + \Delta r$ and so on. Then Eqs. 15–17 can be written as

$$x = (r + \Delta r)\left[\sin^2 \tfrac{i^r}{2} \cos(\varphi + 2f + \Delta f) + \cos^2 \tfrac{i^r}{2} \cos(\Delta\beta + \Delta f) - 1\right] + \Delta r \quad \text{(A1)}$$

$$y = (r + \Delta r)\left[-\sin^2 \tfrac{i^r}{2} \sin(\varphi + 2f + \Delta f) + \cos^2 \tfrac{i^r}{2} \sin(\Delta\beta + \Delta f)\right] \quad \text{(A2)}$$

$$z = (r + \Delta r)\sin i^r \sin(\alpha + f + \Delta f) \quad \text{(A3)}$$

where the expressions in Eqs. 38 are used.

Several of the terms above can be approximated as follows:

$$\cos(\varphi + 2f + \Delta f) = \cos(\varphi + 2f) - \sin(\varphi + 2f)\,\Delta f + O\left((\Delta f)^2\right) \tag{A4}$$

$$\cos(\Delta\beta + \Delta f) = 1 - \frac{1}{2}(\Delta\beta + \Delta f)^2 + O\left((\Delta\beta + \Delta f)^4\right) \tag{A5}$$

$$\sin(\varphi + 2f + \Delta f) = \sin(\varphi + 2f) + \cos(\varphi + 2f)\,\Delta f + O\left((\Delta f)^2\right) \tag{A6}$$

$$\sin(\Delta\beta + \Delta f) = \Delta\beta + \Delta f + O\left((\Delta\beta + \Delta f)^3\right) \tag{A7}$$

$$\sin(\alpha + f + \Delta f) = \sin(\alpha + f) + \cos(\alpha + f)\,\Delta f + O\left((\Delta f)^2\right) \tag{A8}$$

Substituting the equations above into Eqs. A1–A3, and truncating the terms of order higher than second, yields

$$x = r\sin^2\frac{i^r}{2}\left[\cos(\varphi + 2f) - 1\right] + \cos^2\frac{i^r}{2}\left[\Delta r - \frac{r}{2}(\Delta\beta + \Delta f)^2\right] + aO\left(10^{-3k}\right) \tag{A9}$$

$$y = -r\sin^2\frac{i^r}{2}\sin(\varphi + 2f) + \cos^2\frac{i^r}{2}(r + \Delta r)(\Delta\beta + \Delta f) + aO\left(10^{-3k}\right) \tag{A10}$$

$$z = \sin i^r\left[(r + \Delta r)\sin(\alpha + f) + r\Delta f\cos(\alpha + f)\right] + aO\left(10^{-3k}\right) \tag{A11}$$

The problem left is to expand Δr and Δf to second order. As shown in Eqs. 2 and 4, both r and f are functions of e and E, and in Eq. 3 E is a function of e and M, so r and f can be treated as functions of e and M, namely $r = r(e,M)$ and $f = f(e,M)$. The increments of the two functions can be expanded as

$$\Delta r = \frac{\partial r}{\partial e}\Delta e + \frac{\partial r}{\partial M}\Delta M + \frac{1}{2}\frac{\partial^2 r}{\partial e^2}(\Delta e)^2 + \frac{\partial^2 r}{\partial e\partial M}\Delta e\Delta M + \frac{1}{2}\frac{\partial^2 r}{\partial M^2}(\Delta M)^2 + O\left(10^{-3k}\right) \tag{A12}$$

$$\Delta f = \frac{\partial f}{\partial e}\Delta e + \frac{\partial f}{\partial M}\Delta M + \frac{1}{2}\frac{\partial^2 f}{\partial e^2}(\Delta e)^2 + \frac{\partial^2 f}{\partial e\partial M}\Delta e\Delta M + \frac{1}{2}\frac{\partial^2 f}{\partial M^2}(\Delta M)^2 + O\left(10^{-3k}\right) \tag{A13}$$

where the partial derivatives can be derived from Eqs. 2–4.

$$\begin{cases} \dfrac{\partial r}{\partial e} = -a\cos f \\ \dfrac{\partial r}{\partial M} = \dfrac{ae}{\sqrt{1 - e^2}}\sin f \end{cases}, \quad \begin{cases} \dfrac{\partial f}{\partial e} = \dfrac{1}{1 - e^2}(2 + e\cos f)\sin f \\ \dfrac{\partial f}{\partial M} = \dfrac{(1 + e\cos f)^2}{\left(1 - e^2\right)^{\frac{3}{2}}} \end{cases} \tag{A14}$$

$$\begin{cases} \dfrac{\partial^2 r}{\partial e^2} = a \sin f \dfrac{\partial f}{\partial e} \\ \dfrac{\partial^2 r}{\partial e \partial M} = a \sin f \dfrac{\partial f}{\partial M} \\ \dfrac{\partial^2 r}{\partial M^2} = \dfrac{ae \cos f}{\sqrt{1-e^2}} \dfrac{\partial f}{\partial M} \end{cases}, \quad \begin{cases} \dfrac{\partial^2 f}{\partial e^2} = \dfrac{\left[4e + \left(1+e^2\right)\cos f\right]\sin f}{\left(1-e^2\right)^2} + \dfrac{(2\cos f + e\cos 2f)}{1-e^2}\dfrac{\partial f}{\partial e} \\ \dfrac{\partial^2 f}{\partial e \partial M} = \dfrac{(2\cos f + e\cos 2f)}{1-e^2}\dfrac{\partial f}{\partial M} \\ \dfrac{\partial^2 f}{\partial M^2} = \dfrac{-2e\sin f\,(1+e\cos f)}{\left(1-e^2\right)^{\frac{3}{2}}}\dfrac{\partial f}{\partial M} \end{cases} \tag{A15}$$

Note that because $\Delta a = 0$, the partial derivatives with respect to a are neglected. Finally, substituting Eqs. A12–A15 into Eqs. A9–A11 yields the second-order relative motion Equations 35–37.

Appendix B

B.1. In order to integrate the even trigonometric function $(1 - e\cos E)^{-1}$, change the variable E to s through the formula $E = 2\arctan(s)$.

$$b_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1-e\cos E}dE = \frac{1}{\pi}\int_0^{+\infty} \frac{2}{1-e+(1+e)\,s^2}ds \tag{B1}$$

Then change the variable s to l through $s = \sqrt{\frac{1-e}{1+e}}\tan l$

$$b_0 = \frac{1}{\pi}\int_0^{\pi/2} \frac{2}{1-e}\sqrt{\frac{1-e}{1+e}}dl = \frac{1}{\sqrt{1-e^2}} \tag{B2}$$

Finally, apply the result of b_0

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos E}{1-e\cos E}dE = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{e}\left(-1 + \frac{1}{1-e\cos E}\right)dE = \frac{2\lambda}{\sqrt{1-e^2}} \tag{B3}$$

where λ is defined in Eq. 95.

Similarly,

$$\begin{aligned} c_0 &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{(1-e\cos E)^2}dE = \frac{1}{\pi}\int_0^{+\infty} \frac{2\left(1+s^2\right)}{\left[1-e+(1+e)\,s^2\right]^2}ds \\ &= \frac{1}{\pi}\int_0^{\pi/2} \frac{2}{\left(1-e^2\right)^{3/2}}(1+e\cos 2l)dl = \left(1-e^2\right)^{-\frac{3}{2}} \end{aligned} \tag{B4}$$

$$\begin{aligned} c_1 &= \frac{1}{\pi}\int_0^{2\pi} \frac{\cos E}{(1-e\cos E)^2}dE = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{e}\left[-\frac{1}{1-e\cos E} + \frac{1}{(1-e\cos E)^2}\right]dE \\ &= -\frac{2b_0}{e} + \frac{2c_0}{e} = 2e\left(1-e^2\right)^{-\frac{3}{2}} \end{aligned} \tag{B5}$$

B.2. When substituting $(1-e\cos E)^{-1} = b_0 + \sum_{n=1}^{+\infty} b_n \cos nE$ into Eqs. 88, the difficulty is to calculate $\sin E\,(1-e\cos E)^{-1} = b_0 \sin E + \sin E \sum_{n=1}^{+\infty} b_n \cos nE$. Apply the product to sum formula,

$$b_0 \sin E + \sin E \sum_{n=1}^{+\infty} b_n \cos nE = b_0 \sin E + \sum_{n=1}^{+\infty} \frac{b_n}{2} \left[\sin (n+1) E - \sin (n-1) E\right]$$
$$= \left(b_0 - \frac{b_2}{2}\right) \sin E + \sum_{n=2}^{+\infty} \frac{b_{n-1} - b_{n+1}}{2} \sin nE \quad \text{(B6)}$$

Substitution of b_n from Eq. 94 into Eq. B6 yields

$$\sin E \,(1 - e \cos E)^{-1} = \frac{2}{e} \sum_{n=1}^{+\infty} \lambda^n \sin nE \quad \text{(B7)}$$

Applying Eq. B7 to Eqs. 88 and combining the coefficients yield Eqs. 96.

B.3. The center of the curve expressed by Eqs. 100 is obviously (X_0, Y_0), because for arbitrary E, the two points $[x^{(1)}(E), y^{(1)}(E)]$ and $[x^{(1)}(E+\pi), y^{(1)}\ (E+\pi)]$ are symmetric with respect to $(X_0,\ Y_0)$. When $X_1 V_1 = Y_1 U_1$, Eqs. 100 lead to

$$\frac{x^{(1)} - aX_0}{y^{(1)} - aY_0} = \frac{X_1}{Y_1} = \frac{U_1}{V_1} \quad \text{(B8)}$$

Equation B8 expresses a line. Since $x^{(1)}$ and $y^{(1)}$ are bounded, Eqs. 100 express a line segment, in fact. When $X_1 V_1 \neq Y_1 U_1$, by regarding $\cos E$ and $\sin E$ as given, solving Eqs. 100 yields

$$\cos E = \frac{V_1 x_* - U_1 y_*}{X_1 V_1 - Y_1 U_1}, \quad \sin E = \frac{-Y_1 x_* + X_1 y_*}{X_1 V_1 - Y_1 U_1} \quad \text{(B9)}$$

where $x^{(1)}/a - X_0$ and $y^{(1)}/a - Y_0$ are denoted by x_* and y_*. Because $\cos^2 E + \sin^2 E = 1$, Eqs. B9 yield

$$\left(Y_1^2 + V_1^2\right) x_*^2 - 2\left(X_1 Y_1 + U_1 V_1\right) x_* y_* + \left(X_1^2 + U_1^2\right) x_*^2 = \left(X_1 V_1 - Y_1 U_1\right)^2 \quad \text{(B10)}$$

If (x_*, y_*) is really on the ellipse assumed, it should satisfy

$$\frac{\left[x_* \cos \vartheta^{(1)} + y_* \sin \vartheta^{(1)}\right]^2}{\left(A^{(1)}\right)^2} + \frac{\left[-x_* \sin \vartheta^{(1)} + y_* \cos \vartheta^{(1)}\right]^2}{\left(B^{(1)}\right)^2} = 1 \quad \text{(B11)}$$

Compared with Eq. B10, we can show that

$$\begin{cases} \frac{1}{2}\left[\left(A^{(1)}\right)^{-2} + \left(B^{(1)}\right)^{-2}\right] - \frac{1}{2}\left[\left(B^{(1)}\right)^{-2} - \left(A^{(1)}\right)^{-2}\right] \cos 2\vartheta^{(1)} \\ \quad = \left(Y_1^2 + V_1^2\right)\left(X_1 V_1 - Y_1 U_1\right)^{-2} \\ \frac{1}{2}\left[\left(B^{(1)}\right)^{-2} - \left(A^{(1)}\right)^{-2}\right] \sin 2\vartheta^{(1)} = \left(X_1 Y_1 + U_1 V_1\right)\left(X_1 V_1 - Y_1 U_1\right)^{-2} \\ \frac{1}{2}\left[\left(A^{(1)}\right)^{-2} + \left(B^{(1)}\right)^{-2}\right] + \frac{1}{2}\left[\left(B^{(1)}\right)^{-2} - \left(A^{(1)}\right)^{-2}\right] \cos 2\vartheta^{(1)} \\ \quad = \left(X_1^2 + U_1^2\right)\left(X_1 V_1 - Y_1 U_1\right)^{-2} \end{cases} \quad \text{(B12)}$$

Solving Eqs. B12 leads to the solutions (104).

B.4. From Eqs. 88, we can derive that

$$\left(\frac{x}{a}\right)^2 = \left[\Delta e\left(e-\cos E\right)+e\Delta M\sin E\right]^2\frac{1}{\left(1-e\cos E\right)^2} \tag{B13}$$

$$\left(\frac{y}{a}\right)^2 = \left[\Delta\beta\left(1-e\cos E\right)+\frac{\Delta e}{\sqrt{1-e^2}}\sin E+\frac{\sqrt{1-e^2}\left(\Delta M+\Delta e\sin E\right)}{1-e\cos E}\right]^2 \tag{B14}$$

$$\frac{xy}{a^2} = \left[\frac{\Delta e\left(e-\cos E\right)+e\Delta M\sin E}{1-e\cos E}\right]\left[\Delta\beta\left(1-e\cos E\right)+\frac{\Delta e}{\sqrt{1-e^2}}\sin E\right.$$
$$\left.+\frac{\sqrt{1-e^2}\left(\Delta M+\Delta e\sin E\right)}{1-e\cos E}\right] \tag{B15}$$

The following relationships involving trigonometric functions of E can be obtained:

$$\begin{cases}\sin E\cos E\left(1-e\cos E\right)^{-1}=e^{-1}\sin E\left[\left(1-e\cos E\right)^{-1}-1\right]\\ \sin^2 E\left(1-e\cos E\right)^{-1}=e^{-2}+e^{-1}\cos E+\left(1-e^{-2}\right)\left(1-e\cos E\right)^{-1}\\ \cos E\left(1-e\cos E\right)^{-2}=e^{-1}\left[\left(1-e\cos E\right)^{-2}-\left(1-e\cos E\right)^{-1}\right]\\ \cos^2 E\left(1-e\cos E\right)^{-2}=e^{-2}\left[\left(1-e\cos E\right)^{-2}-2\left(1-e\cos E\right)^{-1}+1\right]\\ \sin E\cos E\left(1-e\cos E\right)^{-2}=e^{-1}\sin E\left[\left(1-e\cos E\right)^{-2}-\left(1-e\cos E\right)^{-1}\right]\\ \sin^2 E\left(1-e\cos E\right)^{-2}=\left(1-e^{-2}\right)\left(1-e\cos E\right)^{-2}+e^{-2}\left[2\left(1-e\cos E\right)^{-1}-1\right]\end{cases} \tag{B16}$$

Expanding Eqs. B13–B15 is not difficult but somewhat complicated. All the terms involving $(1-e\cos E)^{-1}$ are of one the forms in Eqs. B16. The key is to expand the functions $(1-e\cos E)^{-1}$, $(1-e\cos E)^{-2}$, $\sin E(1-e\cos E)^{-1}$ and $\sin E(1-e\cos E)^{-2}$, of which the first three are expanded as Eqs. 89, 108 and B7, respectively, and the fourth is expanded as follows:

$$\begin{aligned}\sin E\left(1-e\cos E\right)^{-2} &= c_0\sin E+\sum_{n=1}^{+\infty}\frac{c_n}{2}\left[\sin\left(n+1\right)E-\sin\left(n-1\right)E\right]\\ &= \left(c_0-\frac{c_2}{2}\right)\sin E+\sum_{n=2}^{+\infty}\frac{c_{n-1}-c_{n+1}}{2}\sin nE\\ &= \frac{2}{e\sqrt{1-e^2}}\sum_{n=1}^{+\infty}n\lambda^n\sin nE\end{aligned} \tag{B17}$$

Substitute Eqs. 89, 108, B7 and B17 into Eqs. B13, B14, B15. After combining the coefficients, we can obtain Eqs. 114, where

$$G_0=\left(\frac{1}{\sqrt{1-e^2}}-1\right)\left(\Delta M\right)^2+\frac{1}{1+\sqrt{1-e^2}}\left(\Delta e\right)^2 \tag{B18}$$

$$\begin{cases} G_n = -2\left[(\Delta M)^2 - \dfrac{1-e^2}{e^2}(\Delta e)^2\right]\left(n - \dfrac{1}{\sqrt{1-e^2}}\right)\lambda^n \\ H_n = -4\dfrac{\sqrt{1-e^2}}{e}\Delta e\Delta M\left(n - \dfrac{1}{\sqrt{1-e^2}}\right)\lambda^n \end{cases}, \quad n \geqslant 1 \tag{B19}$$

$$J_0 = \left(1 + \frac{1}{2}e^2\right)(\Delta\beta)^2 + 2\sqrt{1-e^2}\Delta\beta\Delta M + \frac{(\Delta M)^2}{\sqrt{1-e^2}} + \left[1 + \frac{\lambda}{e} + \frac{1}{2\left(1-e^2\right)}\right](\Delta e)^2 \tag{B20}$$

$$\begin{cases} J_1 = -2e(\Delta\beta)^2 + \dfrac{2e}{\sqrt{1-e^2}}(\Delta M)^2 + 2\lambda(\Delta e)^2 \\ K_1 = 2\left(\dfrac{1}{\sqrt{1-e^2}} + \sqrt{1-e^2}\right)\Delta\beta\Delta e + 4\Delta e\Delta M \end{cases} \tag{B21}$$

$$\begin{cases} J_2 = \dfrac{1}{2}e^2(\Delta\beta)^2 + 2\left(2 + \dfrac{1}{\sqrt{1-e^2}}\right)\lambda^2(\Delta M)^2 \\ \qquad\qquad -\left[2\dfrac{1-e^2}{e^2}\left(2 + \dfrac{1}{\sqrt{1-e^2}}\right)\lambda^2 + \dfrac{1}{2}\dfrac{1}{1-e^2}\right](\Delta e)^2 \\ K_2 = -\dfrac{e}{\sqrt{1-e^2}}\Delta\beta\Delta e + 4\dfrac{\sqrt{1-e^2}}{e}\left(2 + \dfrac{1}{\sqrt{1-e^2}}\right)\lambda^2\Delta e\Delta M \end{cases} \tag{B22}$$

$$\begin{cases} J_n = 2\left[(\Delta M)^2 - \dfrac{1-e^2}{e^2}(\Delta e)\right]^2\left(n + \dfrac{1}{\sqrt{1-e^2}}\right)\lambda^n \\ K_n = 4\dfrac{\sqrt{1-e^2}}{e}\Delta e\Delta M\left(n + \dfrac{1}{\sqrt{1-e^2}}\right)\lambda^n \end{cases}, \quad n \geqslant 3 \tag{B23}$$

$$P_0 = e\Delta\beta\Delta e + \frac{e}{\sqrt{1-e^2}}\Delta e\Delta M \tag{B24}$$

$$\begin{cases} P_1 = -\Delta\beta\Delta e + \left(\dfrac{3\sqrt{1-e^2}+1}{\sqrt{1-e^2}}\lambda^2 - 1\right)\Delta e\Delta M \\ Q_1 = e\Delta\beta\Delta M + 2\lambda(\Delta M)^2 + \left(\dfrac{\sqrt{1-e^2}}{e}\lambda^2 + \dfrac{e}{\sqrt{1-e^2}}\right)(\Delta e)^2 \end{cases} \tag{B25}$$

$$\begin{cases} P_n = -4\dfrac{\sqrt{1-e^2}}{e}\Delta e\Delta M n\lambda^n \\ Q_n = 2\left[(\Delta M)^2 - \dfrac{1-e^2}{e^2}(\Delta e)^2\right]n\lambda^n \end{cases}, \quad n \geqslant 2 \tag{B26}$$

$$\begin{aligned} R_0 &= \left(1 + \frac{1}{2}e^2\right)(\Delta\beta)^2 + 2\sqrt{1-e^2}\Delta\beta\Delta M + \left(\frac{2}{\sqrt{1-e^2}} - 1\right)(\Delta M)^2 \\ &\quad + \left[1 + \frac{2\lambda}{e} + \frac{1}{2\left(1-e^2\right)}\right](\Delta e)^2 \end{aligned} \tag{B27}$$

$$\begin{cases} R_1 = -2e\,(\Delta\beta)^2 + \dfrac{4\lambda}{\sqrt{1-e^2}}\,(\Delta M)^2 + \dfrac{2\lambda^2}{e}\,(\Delta e)^2 \\ S_1 = 2\left(\dfrac{1}{\sqrt{1-e^2}} + \sqrt{1-e^2}\right)\Delta\beta\Delta e + \dfrac{8\lambda}{e}\Delta e\Delta M \end{cases} \tag{B28}$$

$$\begin{cases} R_2 = \dfrac{1}{2}e^2\,(\Delta\beta)^2 + \dfrac{4}{\sqrt{1-e^2}}\lambda^2\,(\Delta M)^2 - \left(4\dfrac{\sqrt{1-e^2}}{e^2}\lambda^2 + \dfrac{1}{2\left(1-e^2\right)}\right)(\Delta e)^2 \\ S_2 = -\dfrac{e}{\sqrt{1-e^2}}\Delta\beta\Delta e + \dfrac{8}{e}\lambda^2\Delta e\Delta M \end{cases} \tag{B29}$$

$$\begin{cases} R_n = \dfrac{4}{\sqrt{1-e^2}}\left[(\Delta M)^2 - \dfrac{1-e^2}{e^2}\,(\Delta e)^2\right]\lambda^n \\ S_n = \dfrac{8}{e}\Delta e\Delta M\lambda^n \end{cases}, \quad n \geqslant 3 \tag{B30}$$

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