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ORIGINAL ARTICLE

Investigation of equilibria of a satellite subjected to gravitational and aerodynamic torques

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Abstract Attitude motion of a satellite subjected to gravitational and aerodynamic torques in a circular orbit is investigated. In special case, when the center of pressure of aerodynamic forces is located on one of the principal central axes of inertia of the satellite, all equilibrium orientations are determined. Necessary and (or) sufficient conditions of stability are obtained for each equilibrium orientation. Evolution of domains where stability conditions take place is investigated. All bifurcation values of parameters corresponding to qualitative change of domains of stability are determined.

Keywords Satellite · Attitude motion · Gravitational torque · Aerodynamic torque · Equilibrium orientation · Conditions of stability

1 Introduction

The practical elaboration of attitude control systems for the Earth's artificial satellites is one of the key problems in the development of space technology. In accordance with the purpose of specific space missions the orientation of a satellite can be accomplished by means of active or passive methods. In developing passive attitude control systems

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it is possible to use the properties of gravitational and magnetic fields, the influence of aerodynamic drag and solar pressure, gyroscopic properties of rotating bodies, etc. An important advantage of passive attitude control systems consists in their capability to operate without any fuel or energy consumption.

On circular or low-elliptic orbits, at altitudes from 250 km up to 500 km, it is possible to use aerodynamic torque to orient a satellite's symmetry axis along the incident air flow which will be in a direction close to the orbit tangent. In the case of an aerodynamically stable satellite, any disturbance of the desired orientation produces a restoring aerodynamic torque along the axes perpendicular to the flow. As a result, this torque can force the longitudinal axis of the satellite to align with the incident air flow.

Dynamics of a satellite subjected to gravitational and aerodynamic torques are considered in many papers. The essential idea of the satellite's orientation by means of aerodynamic torque and elementary results of investigations are presented in (Roberson 1958; DeBra 1959; Wall 1959; Schrello 1961, 1962). More in-depth studies of that subject are described in (Beletskii 1967; Meirovitch and Wallace Jr. 1966; Modi and Schrivastava 1972; Sarychev and Ovchinnikov 1994). Positive and negative effects of aerodynamic drag influence on dynamics of the gravitational system satellite—stabilizer are analyzed in (Sarychev 1964, 1965a).

The first successful realization of an aerodynamic attitude control system was performed in the Soviet Union on the satellite Cosmos-149 (the "Space Arrow") launched in 1967. The attitude control system included an aerodynamic stabilizer and two single-axis gyroscopes. The aerodynamic stabilizer in the form of a truncated conical shell was mounted on four long (4–6 m) tubes (Fig. 1). The stabilizer of this construction provides enough aerodynamic restoring torque in pitch and yaw to achieve stability. The gyrodamper (V-yaw scheme composed of two integrating gyroscopes connected with the satellite body through a viscous-spring restraint) provides a damping of the satellite's natural oscillations (the satellite oscillations cause precession of the gyro rotor linked to the vehicle through a damping device, thus producing dissipation of energy) and restoring torques in pitch and yaw axes. The basic problems of the satellite's dynamics with an aerodynamic attitude control system have been

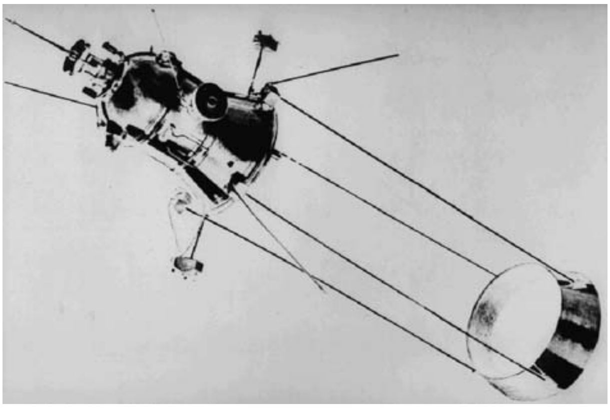


Fig. 1 Soviet artificial satellite Cosmos-149

discussed in (Sarychev 1968, 1969, 1978; Dranovsky et al. 1974; Sarychev and Sadov 1974; Sarychev et al. 1984).

The small American satellite PAMS (project GAMES) (see Pacini and Skillman 1995; Kumar et al. 1995, 1996) is another example of aerodynamic torque usage in attitude control of satellites. Here damping of natural oscillations of the satellite was performed with the help of hysteresis magnetic rods. This satellite was launched from the Space Shuttle board in May, 1996.

A number of works analyze the influence of aerodynamic torque on the dynamics of the orbital stations Salyut-6 and Salyut-7 with batteries of large solar arrays, when the orbital station was in a mode of gravitational orientation (see Sarychev and Sazonov 1981, 1982, 1984; Grechko et al. 1984; Sarychev et al. 1987).

Nurre (1968) and Frik (1970) study the influence of nonconservative components of the aerodynamic torque on the satellite's equilibria stability. This problem is considered in more detail in (Sazonov 1989) in case of attitude motion of orbital station Salyut-7 subjected to gravitational, aerodynamic, and damping torques. The original numerical-analytical research explains the basic properties of the mode of gravitational orientation of the orbital station.

The satellite's attitude motion in a circular orbit under the action of gravitational and aerodynamic torques is considered in (Sarychev and Mirer 2000). The effect of the atmosphere on a satellite is reduced to the drag force applied to the center of pressure and directed against the velocity of the satellite's center of mass relative to the air. The center of pressure is assumed to be at a fixed point in the satellite body. Note that the last assumption is valid if the shape of the satellite body is close to a sphere. In the case when the center of pressure is located on the satellite's principal central axis of inertia, all isolated equilibrium positions are determined in the orbital reference frame, and sufficient conditions of their stability are obtained. Moreover, the existence of eight one-parameter families of stationary solutions is proven.

The present work pushes forward the study started in (Sarychev and Mirer 2000). The main attention is devoted to detailed examination of the equilibria stability of the satellite. Sufficient conditions of stability obtained in Sarychev and Mirer (2000) are simplified and represented in a form convenient for further analysis. Along with sufficient conditions of stability, necessary conditions are discussed as well, keeping in mind their great importance for the general analysis of system stability (De Bra and Delp 1961; Longman et al. 1981). The point is that necessary conditions breakdown guarantees the instability, while the necessary conditions existence results, as a rule, in stability of equilibriums. Common analysis of sufficient and necessary conditions allows separating full space of a system's parameters into three domains: domain of stability (where sufficient and necessary condition hold true at once), unstable domain (all types of stability conditions are violated) and domain of possible stability (only necessary conditions hold true). Of course, to make the final conclusion about stability or instability of specific equilibrium we need to carry out an additional and rather laborious investigation of nonlinear equations of motion using, for example, an approach described in (Markeev and Sokolskii 1975). Necessary conditions of stability are determined from consideration of the linearized equations of motion and from the requirement that all the roots of the characteristic equation should be purely imaginary. Symmetry properties proved in the problem being considered allow one to restrict the investigation to only three of the six groups of equilibrium orientations. The results of a numerical-analytical research are presented in a series of figures. Here the domains where necessary and (or) sufficient conditions hold true are shown in a plane of two dimensionless inertial parameters at various values of dimensionless aerodynamic parameter.

2 Equations of motion

Consider the motion of a satellite subjected to gravitational and aerodynamic torques in a circular orbit. We introduce two right-handed Cartesian coordinate systems with origin in the satellite's center of mass O. $OX_1X_2X_3$ is the orbital reference frame. The axis OX_3 is directed along the radius vector of the satellite's center of mass; the axis OX_1 is in direction of the satellite's orbital motion. $Ox_1x_2x_3$ is the satellite's body reference frame; $Ox_i (i = 1, 2, 3)$ are the principal central axes of inertia of the satellite.

The orientation of the satellite's body reference frame with respect to the orbital reference frame is determined by the angles α, β and γ (see Fig. 2) and the direction cosines of the axes Ox_i in the orbital reference frame $a_{ij} = \cos(X_i, x_j)$ can be written as

$$\begin{aligned}
&a_{11} = \cos\alpha\cos\beta && a_{23} = -\cos\beta\sin\gamma\\
&a_{12} = \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma, && a_{31} = -\sin\alpha\cos\beta,\\
&a_{13} = \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma, && a_{32} = \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma,\\
&a_{21} = \sin\beta, && a_{33} = \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma.\\
&a_{22} = \cos\beta\cos\gamma, &&
\end{aligned} \quad (1)$$

Then equations of the satellite's attitude motion take the form (see Sarychev and Mirer 2000)

$$\begin{aligned}
&A\dot{p} + (C-B)qr - 3\omega_0^2(C-B)a_{32}a_{33} - \omega_0^2(h_2a_{13} - h_3a_{12}) = 0,\\
&B\dot{q} + (A-C)rp - 3\omega_0^2(A-C)a_{33}a_{31} - \omega_0^2(h_3a_{11} - h_1a_{13}) = 0,\\
&C\dot{r} + (B-A)pq - 3\omega_0^2(B-A)a_{31}a_{32} - \omega_0^2(h_1a_{12} - h_2a_{11}) = 0;
\end{aligned} \quad (2)$$

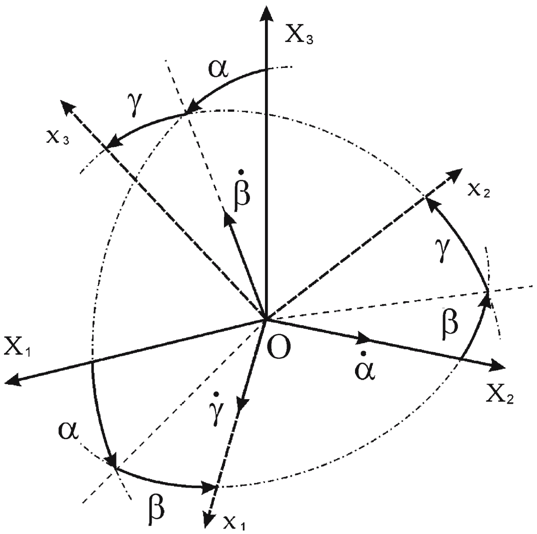


Fig. 2 Orientation of body-fixed axes with respect to the orbital reference frame

$$
\begin{aligned}
p &= (\dot\alpha + \omega_0)a_{21} + \dot\gamma = \bar p + \omega_0 a_{21},\\
q &= (\dot\alpha + \omega_0)a_{22} + \dot\beta \sin\gamma = \bar q + \omega_0 a_{22},\\
r &= (\dot\alpha + \omega_0)a_{23} + \dot\beta \cos\gamma = \bar r + \omega_0 a_{23}.
\end{aligned} \tag{3}
$$

Here

$$
h_1 = -\frac{Qa}{\omega_0^2}, \quad h_2 = -\frac{Qb}{\omega_0^2}, \quad h_3 = -\frac{Qc}{\omega_0^2},
$$

$A,\ B,\ C$ are the principal central moments of inertia of the satellite; $p,\ q,\ r$ are the projections of the satellite's angular velocity in the axes Ox_i; ω_0 is the angular velocity of the orbital motion of the satellite's center of mass; Q is the drag force acting on the satellite; $a,\ b,\ c$ are the coordinates of the satellite's center of pressure in the reference frame $Ox_1x_2x_3$. The dot designates differentiation with respect to time t.

For systems of equations (2) and (3) the generalized integral of energy (Beletskii 1966; Pars) reads as

$$
\begin{aligned}
&\frac{1}{2}\left(A\bar p^2 + B\bar q^2 + C\bar r^2\right) + \frac{3}{2}\omega_0^2\left[(A-C)\,a_{31}^2 + (B-C)\,a_{32}^2\right]\\
&\quad + \frac{1}{2}\omega_0^2\left[(B-A)\,a_{21}^2 + (B-C)\,a_{23}^2\right] - \omega_0^2\,(h_1a_{11} + h_2a_{12} + h_3a_{13}) = \text{const.}
\end{aligned} \tag{4}
$$

3 Equilibrium orientations of satellite

Putting in (2) and (3) $\alpha = \alpha_0 = \text{const}$, $\beta = \beta_0 = \text{const}$, $\gamma = \gamma_0 = \text{const}$, we get the equations

$$
\begin{aligned}
&(C-B)(a_{22}a_{23} - 3a_{32}a_{33}) - h_2a_{13} + h_3a_{12} = 0,\\
&(A-C)(a_{23}a_{21} - 3a_{33}a_{31}) - h_3a_{11} + h_1a_{13} = 0,\\
&(B-A)(a_{21}a_{22} - 3a_{31}a_{32}) - h_1a_{12} + h_2a_{11} = 0,
\end{aligned} \tag{5}
$$

which allow us to determine the satellite's equilibria in the orbital reference frame. Instead of (5) it is more convenient to use the equivalent system

$$
\begin{aligned}
&Aa_{21}a_{31} + Ba_{22}a_{32} + Ca_{23}a_{33} = 0,\\
&3(Aa_{11}a_{31} + Ba_{12}a_{32} + Ca_{13}a_{33}) + h_1a_{31} + h_2a_{32} + h_3a_{33} = 0,\\
&(Aa_{11}a_{21} + Ba_{12}a_{22} + Ca_{13}a_{23}) - h_1a_{21} - h_2a_{22} - h_3a_{23} = 0.
\end{aligned} \tag{6}
$$

Taking into account expressions (1), the system (6) can be considered as a system of three equations in unknowns $\alpha_0,\beta_0,\gamma_0$. Another way of closing equations (6) is to add the following orthonormality conditions for the direction cosines:

$$
\begin{aligned}
&a_{11}^2 + a_{12}^2 + a_{13}^2 = 1, \quad a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0,\\
&a_{21}^2 + a_{22}^2 + a_{23}^2 = 1, \quad a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0,\\
&a_{31}^2 + a_{32}^2 + a_{33}^2 = 1, \quad a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0.
\end{aligned} \tag{7}
$$

Next we will use systems (6) and (7) to investigate the satellite's equilibria.

As shown in (Sarychev and Mirer 2000), systems (6), (7) at $A \neq B \neq C$ can be resolved with respect to $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$. Then we get

$$
\begin{aligned}
&a_{11} = 3\,(I_3 - A)\,a_{31}/F, \quad a_{12} = 3\,(I_3 - B)\,a_{32}/F, \quad a_{13} = 3\,(I_3 - C)\,a_{33}/F,\\
&a_{21} = 3(B-C)a_{32}a_{33}/F, \quad a_{22} = 3(C-A)a_{33}a_{31}/F, \quad a_{23} = 3(A-B)a_{31}a_{32}/F,
\end{aligned} \tag{8}
$$

where $F = h_1a_{31} + h_2a_{32} + h_3a_{33}, I_3 = Aa_{31}^2 + Ba_{32}^2 + Ca_{33}^2$, and the direction cosines a_{31}, a_{32}, a_{33} satisfy the equations

$$\begin{aligned} &9\left[(B-C)^2 a_{32}^2 a_{33}^2 + (C-A)^2 a_{33}^2 a_{31}^2 + (A-B)^2 a_{31}^2 a_{32}^2\right] = (h_1a_{31} + h_2a_{32} + h_3a_{33})^2, \\ &3(B-C)(C-A)(A-B)\,a_{31}a_{32}a_{33} - \left[h_1(B-C)\,a_{32}a_{33}\right. \\ &\quad \left.+ h_2(C-A)\,a_{33}a_{31} + h_3(A-B)\,a_{31}a_{32}\right] \times (h_1a_{31} + h_2a_{32} + h_3a_{33}) = 0, \\ &a_{31}^2 + a_{32}^2 + a_{33}^2 = 1. \end{aligned} \tag{9}$$

After solving (9) the formulas (8) allow us to determine the remaining six direction cosines. Note that solutions (8) exist only in the case when any two direction cosines of the a_{31}, a_{32}, a_{33} set do not vanish simultaneously. Specific cases $a_{31} = a_{32} = 0, a_{32} = a_{33} = 0$, $a_{33} = a_{31} = 0$ must be examined by the direct investigation of systems (6) and (7).

Introducing the new variables $x_1 = a_{31}/a_{33}, y_1 = a_{32}/a_{33}$ and using the approach given in Sarychev and Gutnik (1984) it is possible to reduce the first two equations (9) to a single polynomial equation of the 12th order (for example, with respect to x_1) with real coefficients. This equation has at most 12 real roots, moreover, each root x_1 determines the only y_1 satisfying the first two equations of system (9). Next, using the last equation (9), we find values of direction cosinea_{33} for each solution x_1, y_1. So, two sets of values a_{31}, a_{32}, a_{33} correspond to each real root of polynomial equation. Each set a_{31}, a_{32}, a_{33} in turn, by virtue of (8), uniquely determines the remaining direction cosinesa_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}. Therefore, we conclude that the satellite subjected to gravitational and aerodynamic torques can have no more than 24 equilibrium positions in a circular orbit.

Further we consider a particular case $h_1 \neq 0,\ h_2 = h_3 = 0$ where the pressure center locates on the axis Ox_1 of the satellite (see Sarychev and Mirer 2000). Introducing dimensionless parameters $\theta_B = B/A, \theta_C = C/A$, and $H_1 = h_1/A$, the system (9) takes the form

$$\begin{aligned} &9\left[(\theta_B - \theta_C)^2 a_{32}^2 a_{33}^2 + (\theta_C - 1)^2 a_{33}^2 a_{31}^2 + (1-\theta_B)^2 a_{31}^2 a_{32}^2\right] = H_1^2 a_{31}^2, \\ &(\theta_B - \theta_C)\left[H_1^2 - 3(1-\theta_B)(\theta_C - 1)\right] a_{31}a_{32}a_{33} = 0, \\ &a_{31}^2 + a_{32}^2 + a_{33}^2 = 1. \end{aligned} \tag{10}$$

Note, that it is sufficient to consider only positive H_1, since the case $H_1 < 0$ can be reduced to the case $H_1 > 0$ by rotating the satellite-body reference frame on π radians about any degree-of-freedom perpendicular to the Ox_1 axis.

It follows from the second equation of system (10) that

$$H_1^2 = 3(1-\theta_B)(\theta_C - 1) \tag{11}$$

or

$$a_{31}a_{32}a_{33} = 0. \tag{12}$$

It is possible to show that in the case (11) the system (10) has 8 one-parameter families of solutions. Their properties are investigated in (Sarychev and Mirer 2000).

Now let $H_1^2 \neq 3(1-\theta_B)(\theta_C - 1)$. Then the second equation of system (10) takes the form (12). Successive consideration of the cases $a_{31} = 0$, $a_{32} = 0$, $a_{33} = 0$ results in the following six groups of isolated solutions (only nonzero direction cosines are presented) (see Sarychev and Mirer 2000):

$$a_{11} = a_{22}a_{33}, \quad a_{22} = \pm 1, \quad a_{33} = \pm 1; \tag{13}$$

$$a_{11} = -a_{23}a_{32}, \quad a_{23} = \pm 1, \quad a_{32} = \pm 1; \tag{14}$$

$$a_{11} = -x, \quad a_{13} = -a_{22}a_{31}, \quad a_{22} = \pm 1 \quad a_{31} = \pm\sqrt{1 - x^2}, \quad a_{33} = -xa_{22}; \tag{15}$$

$$a_{11} = -y, \quad a_{12} = a_{23}a_{31}, \quad a_{23} = \pm 1, \quad a_{31} = \pm\sqrt{1-y^2}, \quad a_{32} = ya_{23}; \tag{16}$$

$$a_{11} = 3y, \quad a_{12} = \pm\sqrt{1-9y^2}, \quad a_{21} = -a_{12}a_{33}, \quad a_{22} = 3ya_{33}, \quad a_{33} = \pm 1; \tag{17}$$

$$a_{11} = 3x, \quad a_{13} = \pm\sqrt{1-9x^2}, \quad a_{21} = a_{13}a_{32}, \quad a_{23} = -3xa_{32}, \quad a_{32} = \pm 1. \tag{18}$$

Here $x = H_1/3\,(1-\theta_C)$, $y = H_1/3\,(1-\theta_B)$. Note that each group (13)–(18) consists of four solutions corresponding to four specific sets of signs + and −. Solutions (13) and (14) exist for arbitrary values of systems parameters, solutions (15) exist at $x^2 \leq 1$, solutions (16) at $y^2 \leq 1$, solutions (17) at $9y^2 \leq 1$, and solutions (18) exist at $9x^2 \leq 1$. Therefore, four straight lines decompose the plane (x^2, y^2) into nine domains with a fixed number of equilibria (Fig. 3). Of course, at $H_1 = 0 \quad (x = y = 0)$ the solutions (13)–(18) coincide with the well known (Sarychev 1965b; Likins and Roberson 1966) 24 equilibrium orientations of a rigid body in a circular orbit.

Note that solutions (13), (15), and (17) transform to solutions (14), (16), and (18) correspondingly if we define the body reference frame in a different manner. Indeed, introduce a new reference frame $O\tilde{x}_1\tilde{x}_2\tilde{x}_3$ which can be obtained from $Ox_1x_2x_3$ by rotation around the axis Ox_1 through the angle of $\pi/2$. It is evident, that the axes $O\tilde{x}_i$ are also the principal central axes of inertia of the satellite, but their indexes and directions differ from Ox_i. Actually, the change-over from the axes Ox_i to the axes $O\tilde{x}_i$ can be accomplished by substitution $\theta_B \to \theta_C, \theta_C \to \theta_B$ (and, as a consequence, $x \to y, y \to x$).

Consider, for example, solutions (15). In the reference frame $O\tilde{x}_1\tilde{x}_2\tilde{x}_3$ it takes the form

$$\|\tilde{a}_{ij}\| = \begin{Vmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{Vmatrix} \cdot \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{Vmatrix} = \begin{Vmatrix} a_{11} & a_{13} & 0 \\ 0 & 0 & -a_{22} \\ a_{31} & a_{33} & 0 \end{Vmatrix},$$

from which

$$\tilde{a}_{12} = a_{13} = -a_{22}a_{31} = \tilde{a}_{23}\tilde{a}_{31}, \quad \tilde{a}_{11} = a_{11} = -x, \quad \tilde{a}_{23} = -a_{22} = \pm 1,$$
$$\tilde{a}_{31} = a_{31} = \pm\sqrt{1-x^2}, \quad \tilde{a}_{32} = a_{33} = -xa_{22} = x\tilde{a}_{23}.$$

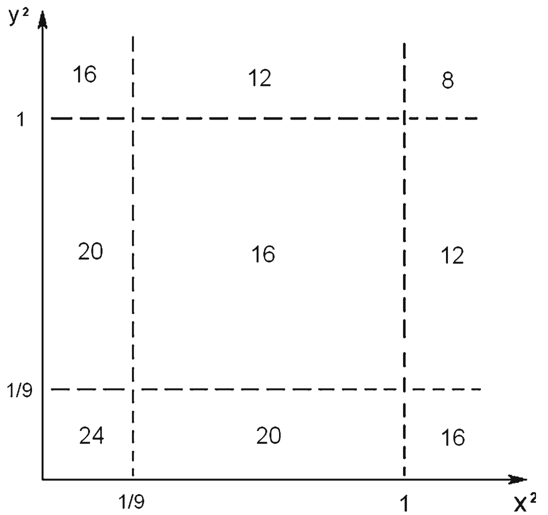


Fig. 3 Domains with fixed number of equilibrium orientations

So, taking into account that $x \to y$, the obtained expressions really coincide with (16).

Therefore, we can restrict ourselves to analyze only solutions (13), (15), and (17). The results for solutions (14), (16), and (18) can be obtained after corresponding substitution.

As for geometrical meaning of the solutions, note that for solutions (13) the similar axes of the orbital and satellite-body reference frames are collinear to each other. So the axis Ox_1 is directed along the axis OX_1 if $\bar{a}_{11} = 1$ and it is oppositely directed at $\bar{a}_{11} = -1$.

Solutions (15) and (17) can be obtained from (13) by rotation around the axes OX_2 and OX_3 through the angles α_0 and β_0 correspondingly, where $\cos^2\alpha_0 = x^2 = H_1^2/9\,(1-\theta_C)^2$ and $\cos^2\beta_0 = 9y^2 = H_1^2/(1-\theta_B)^2$.

4 Sufficient conditions of stability

To derive the sufficient conditions of stability (hereinafter called S-conditions) of equilibrium orientations (13)–(18) it is possible to use the energy integral (4). Designating

$$\alpha = \alpha_0 + \bar{\alpha}, \quad \beta = \beta_0 + \bar{\beta}, \quad \gamma = \gamma_0 + \bar{\gamma}, \tag{19}$$

where $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are small deviations from the satellite's equilibrium $\alpha = \alpha_0 = \text{const}$, $\beta = \beta_0 = \text{const}$, $\gamma = \gamma_0 = \text{const}$, the energy integral takes the form (see Sarychev and Mirer 2000)

$$\begin{aligned}
&\bar{p}^2 + \theta_B\bar{q}^2 + \theta_C\bar{r}^2 \\
&\quad + \omega_0^2\left(A_{\alpha\alpha}\bar{\alpha}^2 + A_{\beta\beta}\bar{\beta}^2 + A_{\gamma\gamma}\bar{\gamma}^2 + 2A_{\alpha\beta}\bar{\alpha}\bar{\beta} + 2A_{\beta\gamma}\bar{\beta}\bar{\gamma} + 2A_{\gamma\alpha}\bar{\gamma}\bar{\alpha}\right) \\
&\quad + \Sigma = \text{const.}
\end{aligned} \tag{20}$$

Here the symbol Σ designates the terms of the third and higher order with respect to $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$,

$$\begin{aligned}
A_{\alpha\alpha} &= 3\left[(1-\theta_C)\left(\bar{a}_{11}^2 - \bar{a}_{31}^2\right) + (\theta_B - \theta_C)\left(\bar{a}_{12}^2 - \bar{a}_{32}^2\right)\right] + H_1\bar{a}_{11}, \\
A_{\beta\beta} &= \left[(\theta_B - 1) - (\theta_B - \theta_C)\sin^2\gamma_0\right]\left(1 + 3\sin^2\alpha_0\right)\cos 2\beta_0 \\
&\quad - \frac{3}{4}(\theta_B - \theta_C)\sin 2\alpha_0 \sin\beta_0 \sin 2\gamma_0 + H_1\bar{a}_{11}, \\
A_{\gamma\gamma} &= (\theta_B - \theta_C)\left[\left(\bar{a}_{22}^2 - \bar{a}_{23}^2\right) - 3\left(\bar{a}_{32}^2 - \bar{a}_{33}^2\right)\right], \\
A_{\alpha\beta} &= -\frac{3}{2}(1-\theta_C)\sin 2\alpha_0 \sin 2\beta_0 + 3(\theta_B - \theta_C)\left(\bar{a}_{32}\cos\alpha_0 - \bar{a}_{12}\sin\alpha_0\right)\bar{a}_{22} \\
&\quad - H_1 \sin\alpha_0 \sin\beta_0, \\
A_{\beta\gamma} &= -\frac{1}{2}(\theta_B - \theta_C)\sin 2\beta_0 \sin 2\gamma_0 - 3(\theta_B - \theta_C)(\bar{a}_{33}\cos\gamma_0 - \bar{a}_{32}\sin\gamma_0)\bar{a}_{31}, \\
A_{\gamma\alpha} &= -3(\theta_B - \theta_C)\left(\bar{a}_{12}\bar{a}_{33} + \bar{a}_{13}\bar{a}_{32}\right); \\
\bar{a}_{ij} &= a_{ij}(\alpha_0,\ \beta_0,\ \gamma_0).
\end{aligned}$$

It follows from the Lyapunov theorem that solution $\alpha = \alpha_0, \beta = \beta_0, \gamma = \gamma_0$ is stable if the quadratic form

$$A_{\alpha\alpha}\bar{\alpha}^2 + A_{\beta\beta}\bar{\beta}^2 + A_{\gamma\gamma}\bar{\gamma}^2 + 2A_{\alpha\beta}\bar{\alpha}\bar{\beta} + 2A_{\beta\gamma}\bar{\beta}\bar{\gamma} + 2A_{\gamma\alpha}\bar{\gamma}\bar{\alpha}$$

is positive definite, that is the following inequalities take place

$$A_{\alpha\alpha} > 0, \quad A_{\alpha\alpha}A_{\beta\beta} - A_{\alpha\beta}^2 > 0,$$
$$A_{\alpha\alpha}A_{\beta\beta}A_{\gamma\gamma} + 2A_{\alpha\beta}A_{\beta\gamma}A_{\alpha\gamma} - A_{\alpha\alpha}A_{\beta\gamma}^2 - A_{\beta\beta}A_{\alpha\gamma}^2 - A_{\gamma\gamma}A_{\alpha\beta}^2 > 0.$$

Taking into account expressions (1) for direction cosines, we get for solutions (13) $\sin\alpha_0 = 0$, $\sin\beta_0 = 0$, $\sin\gamma_0 = 0$, whence follows $A_{\alpha\beta} = A_{\alpha\gamma} = A_{\beta\gamma} = 0$ and stability conditions take the form

$$A_{\alpha\alpha} > 0, \quad A_{\beta\beta} > 0, \quad A_{\gamma\gamma} > 0,$$

or

$$3(1-\theta_C) + H_1\bar{a}_{11} > 0, \quad (\theta_B - 1) + H_1\bar{a}_{11} > 0, \quad \theta_B - \theta_C > 0; \tag{21}$$

for solution (15), which exist at $H_1^2 \le 9(\theta_C - 1)^2$, $\sin\beta_0 = \sin\gamma_0 = 0$, $A_{\alpha\beta} = A_{\alpha\gamma} = 0$ and stability conditions take the form

$$A_{\alpha\alpha} > 0, \quad A_{\beta\beta} > 0, \quad A_{\beta\beta}A_{\gamma\gamma} - A_{\beta\gamma}^2 > 0,$$

or

$$\theta_C - 1 > 0, \quad \theta_B - \theta_C > 0, \quad H_1^2 < 12(\theta_B - 1)(\theta_C - 1)^2/(\theta_B - \theta_C); \tag{22}$$

for solutions (17), which exist at $H_1^2 \le (\theta_B - 1)^2$, $\sin\alpha_0 = \sin\gamma_0 = 0$, $A_{\alpha\beta} = A_{\beta\gamma} = 0$ and stability conditions take the form

$$A_{\alpha\alpha} > 0, \quad A_{\beta\beta} > 0, \quad A_{\alpha\alpha}A_{\gamma\gamma} - A_{\alpha\gamma}^2 > 0,$$

or

$$1 - \theta_B > 0, \quad \theta_B - \theta_C > 0. \tag{23}$$

At substitution $\theta_B \to \theta_C$, $\theta_C \to \theta_B$ conditions (21) for solutions (13) go over into conditions for solutions (14), conditions (22) for solutions (15) transform to conditions for solutions (16), and conditions (23) for solutions (17) transform to conditions for solutions (18). Note that the last inequality (22) follows from the first two inequalities and condition $9(\theta_C - 1)^2 \ge H_1^2$, and therefore can be omitted. Indeed

$$12\frac{(\theta_B - 1)(\theta_C - 1)^2}{\theta_B - \theta_C} = 9(\theta_C - 1)^2\,\frac{4\theta_B - 4}{3(\theta_B - \theta_C)} =$$
$$= 9(\theta_C - 1)^2\left[1 + \frac{(\theta_B - \theta_C) + 4(\theta_C - 1)}{3(\theta_B - \theta_C)}\right] > 9(\theta_C - 1)^2 \ge H_1^2.$$

Remember also, that it is necessary to take into account the conditions of physical realization of a rigid body, i.e., we must take into account inequalities

$$\theta_B + 1 \ge \theta_C, \quad \theta_B + \theta_C \ge 1, \quad 1 + \theta_C \ge \theta_B. \tag{24}$$

Further the region where (24) are valid will be referred to as the working region.

Now we consider domains in the plane (θ_B, θ_C) where S-conditions are valid (hereinafter called S-domains) and analyze their evolution depending on dimensionless parameter H_1 (Fig. 4). Eight straight lines generally take part in forming S-domains. It is convenient to enumerate these lines as following:

$$\begin{array}{ll} 1.\ \theta_B = 1 - H_1, & 5.\ \theta_C = 1 - H_1, \\ 2.\ \theta_B = 1 - H_1/3, & 6.\ \theta_C = 1 - H_1/3, \\ 3.\ \theta_B = 1 + H_1/3, & 7.\ \theta_C = 1 + H_1/3, \\ 4.\ \theta_B = 1 + H_1, & 8.\ \theta_C = 1 + H_1. \end{array} \tag{25}$$

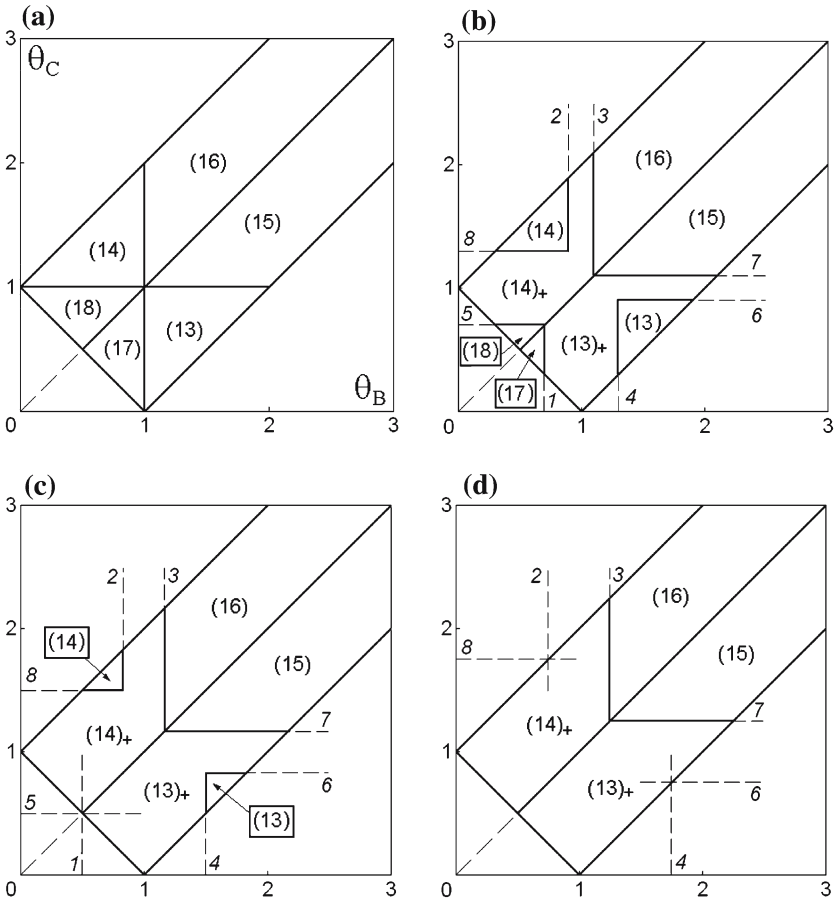


Fig. 4 Evolution of S-domains: (**a**) $H_1 = 0$, (**b**) $H_1 = 0.3$, (**c**) $H_1 = 0.5$, (**d**) $H_1 = 0.75$

If $H_1 = 0$, the satellite is a rigid body subjected only to gravitational torque. In this case solutions (13)–(18) are stable in the domains shown in Fig. 4a (here and below the horizontal and vertical axes correspond to dimensionless parameters θ_B and θ_C, respectively). The whole plane (θ_B, θ_C) is partitioned by the straight lines $\theta_B = 1$, $\theta_C = 1$, and $\theta_B = \theta_C$ into six domains. In each domain S-conditions hold true only for a single specific group of solutions (the domains in the figure are marked by the numbers of corresponding solutions).

When $H_1 > 0$, the character of domain boundaries changes. Boundaries of S-domains are defined now by the straight lines (25). Moreover, for solutions (13) and (14) the following peculiarity takes place: S-domains in cases $\bar{a}_{11} = 1$ and $\bar{a}_{11} = -1$ do not coincide (see Fig. 4b). S-conditions hold true for $\bar{a}_{11} = \pm 1$ in domains (13) and (14), while in domains designated $(13)_+$ and $(14)_+$ S-conditions are satisfied only for $\bar{a}_{11} = 1$.

When H_1 increases, the mutual positions of the partition boundaries change, and the form of S-domains changes qualitatively at two bifurcation values of H_1 (1/2 and 3/4). Domains (17) and (18) degenerate at the first bifurcation value, and domains (13) and (14) also degenerate at the second value.

5 Necessary conditions of stability

To investigate necessary conditions of stability (hereinafter called N-conditions) of the satellite's equilibrium orientations we must study the stability properties of the linearized equations of motion in the vicinity of the specific solution $\alpha = \alpha_0 = \text{const}$, $\beta = \beta_0 = \text{const}$, $\gamma = \gamma_0 = \text{const}$. To derive linearized equations of motion it is necessary to substitute

$$\alpha = \alpha_0 + \bar{\alpha}, \quad \beta = \beta_0 + \bar{\beta}, \quad \gamma = \gamma_0 + \bar{\gamma},$$

where $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are small deviations from the satellite's equilibrium α_0, β_0, γ_0, into (2)–(3) and after Taylor series expansion of obtained equations drop second and higher order terms.

First of all it is convenient to obtain linearized expressions of direction cosines. So we have

$$\begin{aligned}
a_{11} &= \cos(\alpha_0 + \bar{\alpha})\cos\left(\beta_0 + \bar{\beta}\right)\\
&= (\cos\alpha_0 - \bar{\alpha}\sin\alpha_0)\left(\cos\beta_0 - \bar{\beta}\sin\beta_0\right)\\
&= \cos\alpha_0\cos\beta_0 - \bar{\alpha}\sin\alpha_0\cos\beta_0 - \bar{\beta}\cos\alpha_0\sin\beta_0\\
&= \bar{a}_{11} + \bar{\alpha}\bar{a}_{31} - \bar{\beta}\bar{a}_{21}\cos\alpha_0,\\
a_{12} &= \bar{a}_{12} + \bar{\alpha}\bar{a}_{32} - \bar{\beta}\bar{a}_{11}\cos\gamma_0 + \bar{\gamma}\bar{a}_{13},\\
a_{13} &= \bar{a}_{13} + \bar{\alpha}\bar{a}_{33} + \bar{\beta}\bar{a}_{11}\sin\gamma_0 - \bar{\gamma}\bar{a}_{12},\\
a_{31} &= \bar{a}_{31} - \bar{\alpha}\bar{a}_{11} + \bar{\beta}\bar{a}_{21}\sin\alpha_0,\\
a_{32} &= \bar{a}_{32} - \bar{\alpha}\bar{a}_{12} - \bar{\beta}\bar{a}_{31}\cos\gamma_0 + \bar{\gamma}\bar{a}_{33},\\
a_{33} &= \bar{a}_{33} - \bar{\beta}\bar{a}_{13} + \bar{\beta}\bar{a}_{31}\sin\gamma_0 - \bar{\gamma}\bar{a}_{32}.
\end{aligned}$$

Here $\bar{a}_{ij} = a_{ij}(\alpha_0, \beta_0, \gamma_0)$. Linearization procedure of the direction cosine a_{11} described maximum in detail.

Next, linearization of kinematical equations (3) results in

$$\begin{aligned}
p &= \dot{\bar{\alpha}}\bar{a}_{21} + \dot{\bar{\gamma}} + \bar{\beta}\omega_0\cos\beta_0 + \omega_0\bar{a}_{21},\\
q &= \dot{\bar{\alpha}}\bar{a}_{22} + \dot{\bar{\beta}}\sin\gamma_0 - \bar{\beta}\omega_0\bar{a}_{21}\cos\gamma_0 + \bar{\gamma}\omega_0\bar{a}_{23} + \omega_0\bar{a}_{22},\\
r &= \dot{\bar{\alpha}}\bar{a}_{23} + \dot{\bar{\beta}}\cos\gamma_0 + \bar{\beta}\omega_0\bar{a}_{21}\sin\gamma_0 - \bar{\gamma}\omega_0\bar{a}_{22} + \omega_0\bar{a}_{23}.
\end{aligned}$$

Substituting linearized direction cosines and angular velocity components in the Eq. (2) and turning to dimensionless parameters after simple manipulation we get

$$\begin{aligned}
&\theta_B\ddot{\bar{\alpha}}\bar{a}_{22} + 2(1-\theta_C)\dot{\bar{\alpha}}\bar{a}_{21}\bar{a}_{23} + \left[3(1-\theta_C)\left(\bar{a}_{11}\bar{a}_{33} + \bar{a}_{13}\bar{a}_{31}\right) + H_1\bar{a}_{33}\right]\bar{\alpha}\\
&\quad + \theta_B\ddot{\bar{\beta}}\sin\gamma_0 - (\theta_B + \theta_C - 1)\dot{\bar{\beta}}\sin\beta_0\cos\gamma_0\\
&\quad - \left\{(1-\theta_C)\left[\left(1 + 3\sin^2\alpha_0\right)\cos 2\beta_0\sin\gamma_0 + \tfrac{3}{2}\sin 2\alpha_0\sin\beta_0\cos\gamma_0\right] + H_1\bar{a}_{23}\cos\alpha_0\right\}\bar{\beta}\\
&\quad + (\theta_B - \theta_C + 1)\dot{\bar{\gamma}}\bar{a}_{23} - \left[(1-\theta_C)\left(\bar{a}_{21}\bar{a}_{22} - 3\bar{a}_{31}\bar{a}_{32}\right) + H_1\bar{a}_{12}\right]\bar{\gamma} = 0,
\end{aligned}$$

$$\begin{aligned}
&\theta_C\ddot{\bar{\alpha}}\bar{a}_{23} + 2(\theta_B - 1)\dot{\bar{\alpha}}\bar{a}_{21}\bar{a}_{22} + \left[3(\theta_B - 1)\left(\bar{a}_{11}\bar{a}_{32} + \bar{a}_{12}\bar{a}_{31}\right) - H_1\bar{a}_{32}\right]\bar{\alpha}\\
&\quad + \theta_C\ddot{\bar{\beta}}\cos\gamma_0 + (\theta_B + \theta_C - 1)\dot{\bar{\beta}}\sin\beta_0\sin\gamma_0\\
&\quad + \left\{(\theta_B - 1)\left[\left(1 + 3\sin^2\alpha_0\right)\cos 2\beta_0\cos\gamma_0 - \frac{3}{2}\sin 2\alpha_0\sin\beta_0\sin\gamma_0\right] + H_1\bar{a}_{22}\cos\alpha_0\right\}\bar{\beta}\\
&\quad + (\theta_B - \theta_C - 1)\dot{\bar{\gamma}}\bar{a}_{22} + \left[(\theta_B - 1)\left(\bar{a}_{21}\bar{a}_{23} - 3\bar{a}_{31}\bar{a}_{33}\right) - H_1\bar{a}_{13}\right]\bar{\gamma} = 0, \qquad (26)
\end{aligned}$$

$$\ddot{\bar{\alpha}}\bar{a}_{21} - 2\left(\theta_B - \theta_C\right)\dot{\bar{\alpha}}\bar{a}_{22}\bar{a}_{23} - 3\left(\theta_B - \theta_C\right)\left(\bar{a}_{12}\bar{a}_{33} + \bar{a}_{13}\bar{a}_{32}\right)\bar{\alpha}$$
$$+ \left[1 - \left(\theta_B - \theta_C\right)\cos 2\gamma_0\right]\cos\beta_0\dot{\bar{\beta}}$$
$$- \left(\theta_B - \theta_C\right)\left[\left(1 + 3\sin^2\alpha_0\right)\sin\beta_0\sin 2\gamma_0 - \frac{3}{2}\sin 2\alpha_0\cos 2\gamma_0\right]\cos\beta_0\bar{\beta}$$
$$+ \ddot{\bar{\gamma}} + \left(\theta_B - \theta_C\right)\left[\left(\bar{a}_{22}^2 - \bar{a}_{23}^2\right) - 3\left(\bar{a}_{32}^2 - \bar{a}_{33}^2\right)\right]\bar{\gamma} = 0,$$

Here the dot designates differentiation with respect to $\tau = \omega_0 t$. Note, that zero order terms with respect to $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ vanish due to (5)–(6).

Now consider a specific group of the satellite's equilibria. Taking into account expressions (1) for direction cosines, we get for solutions (13) $\sin\alpha_0 = 0$, $\sin\beta_0 = 0$, $\sin\gamma_0 = 0$, and linearized equations (26) become

$$\begin{aligned} &\theta_B\ddot{\bar{\alpha}} + \bar{a}_1\bar{\alpha} = 0, \\ &\theta_C\ddot{\bar{\beta}} + b_2\bar{\beta} + \left(b_1 - 1\right)\dot{\bar{\gamma}}\cos\beta_0 = 0, \\ &\ddot{\bar{\gamma}} + 4b_1\bar{\gamma} + \left(1 - b_1\right)\dot{\bar{\beta}}\cos\beta_0 = 0, \end{aligned} \tag{27}$$

where

$$\bar{a}_1 = 3\left(1 - \theta_C\right) + H_1\bar{a}_{11}, \quad b_1 = \theta_B - \theta_C, \quad b_2 = \theta_B - 1 + H_1\bar{a}_{11}, \quad \cos\beta_0 = \pm 1,$$
$$\bar{a}_{11} = \pm 1.$$

The characteristic equation of system (27)

$$\left(\theta_B\lambda^2 + \bar{a}_1\right)\left(\theta_C\lambda^4 + a_1\lambda^2 + a_2\right) = 0$$

decomposes into quadratic and biquadratic equations. Here the following new designations are introduced:

$$a_1 = b_2 + 4\theta_C b_1 + \left(1 - b_1\right)^2, \quad a_2 = 4b_1b_2.$$

N-conditions imply that there are no roots of characteristic equation with positive real parts. Hence, in the case when there are only even terms in the polynomial, one should require all roots to be purely imaginary. Therefore, λ^2 should be real and negative and we can write down the following conditions:

$$\bar{a}_1 > 0\,, \quad a_1 > 0, \quad a_2 > 0, \quad D_2 = a_1^2 - 4\theta_C a_2 > 0. \tag{28}$$

Of course, physical constraints (24) must be satisfied together with conditions (28).

Note, that inequality $a_1 > 0$ can not become the equality on the boundary of the domain where N-conditions take place (below N-domain), because at $a_1 = 0$ the condition $D_2 > 0$ results in $a_2 < 0$ and violates (28).

The condition $a_2 > 0$ holds true either at $b_1 > 0$ and $b_2 > 0$, or at $b_1 < 0$ and $b_2 < 0$. In the first case all sufficient conditions of stability (21) are satisfied. It is evident that in this case all necessary conditions (28) are also satisfied, i.e. $a_1 > 0$ and $D_2 > 0$. In that way, we get the domain with all necessary and sufficient conditions of stability of solution (13) satisfied which was already obtained in the previous section.

Let now $b_1 < 0$, $b_2 < 0$. Then S-conditions are violated, and N-conditions take the form

$$\bar{a}_1 > 0\,, \quad a_1 > 0, \quad D_2 > 0\,, \quad \theta_B < \theta_C\,, \quad \theta_B < 1 - H_1\bar{a}_{11}. \tag{29}$$

We can disregard the third inequality of the system (24), since it holds true automatically at $\theta_B < \theta_C$.

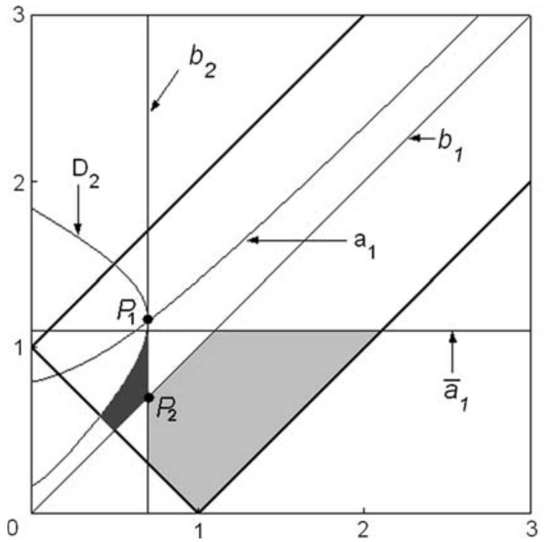
At least one of inequalities (29) becomes an equality on the boundary of the N-domain. At the same time the equality $a_1 = 0$ is possible only if $a_2 = 0$ and $D_2 = 0$simultaneously. In that way, the domain can be bounded by the straight lines $\bar{a}_1 = 0, \theta_C = 1 \pm \theta_B$, $b_1 = 0, b_2 = 0$and by the curve $D_2 = 0$. An example of such a domain and corresponding boundaries are given in Fig. 5 ($\bar{a}_{11} = 1$). The domain where the necessary and sufficient conditions of stability are satisfied is marked out by light-gray color, while the domain where only the necessary conditions of stability hold true are dark-gray. Note, that the curve $D_2 = 0$ is tangent to the straight line $b_2 = 0$ at point P_1. The curve $a_1 = 0$ also passes through this point. The straight lines $b_1 = 0$ and $b_2 = 0$ intersect at point P_2 where $\theta_B = \theta_C = 1 - H_1\bar{a}_{11}$. Further we analyze the cases $\bar{a}_{22} = 1$ and $\bar{a}_{22} = -1$ individually.

If $\bar{a}_{22} = 1$, then N-domains modify their form qualitatively at specific values of H_1 either when the point P_1 is located on the straight line $\theta_C = 1 + H_1/3$ or at the intersection of the straight lines $\theta_B = \theta_C - 1$ and $\theta_B = 1 - \theta_C$, or when the point P_2 is located on the straight line $\theta_B = 1 - \theta_C$. From here we get bifurcation values $H_1 = 3/8$, $H_1 = 1/2$, and $H_1 = 1$. N-domain degenerates at $H_1 = 1$ and vanishes at $H_1 > 1$. The evolution of the domain form is shown in Fig. 6 where N-domains for solutions (13) in case $\bar{a}_{11} = 1$ are presented.

Now let $\bar{a}_{11} = -1$. N-domain is formed by the straight lines $\theta_B = \theta_C$, $\theta_B + \theta_C = 1$, $\bar{a}_1 = 0$, and by the curve $D_2 = 0$. If the parameter H_1 increases then the curve $D_2 = 0$ approaches the straight line $\theta_B = \theta_C$ and, as a consequence, N-domain decreases. Moreover, a point P_3 (where $D_2 = 0$ and $b_1 = 0$ intersect) shifts. Note, that the condition $a_1 = 0$ also holds at point P_3. Qualitative change of the domain's form takes place at $H_1 = 1/2$ and $H_1 = 3/4$.

The point P_3 is located on the boundary $\theta_B + \theta_C = 1$ of the working region at $H_1 = 1/2$, and after that the N-domain is bounded only by the lines $\theta_B = \theta_C$, $\bar{a}_1 = 0$, and $D_2 = 0$. At $H_1 = 3/4$ the point P_3 is located on the line $\bar{a}_1 = 0$. As a result, the N-domain degenerates to a point and vanishes. Remember that the domain where sufficient conditions of stability exist also degenerates to a point and vanishes at $H_1 = 3/4$. Thus, in the case $\bar{a}_{11} = -1$ at $H_1 > 3/4$, a domain of sufficient and (or) necessary conditions of stability does not

Fig. 5 S-domain (light-gray) and N-domain (dark-gray) for solutions (13) at $H_1 = 0.3$



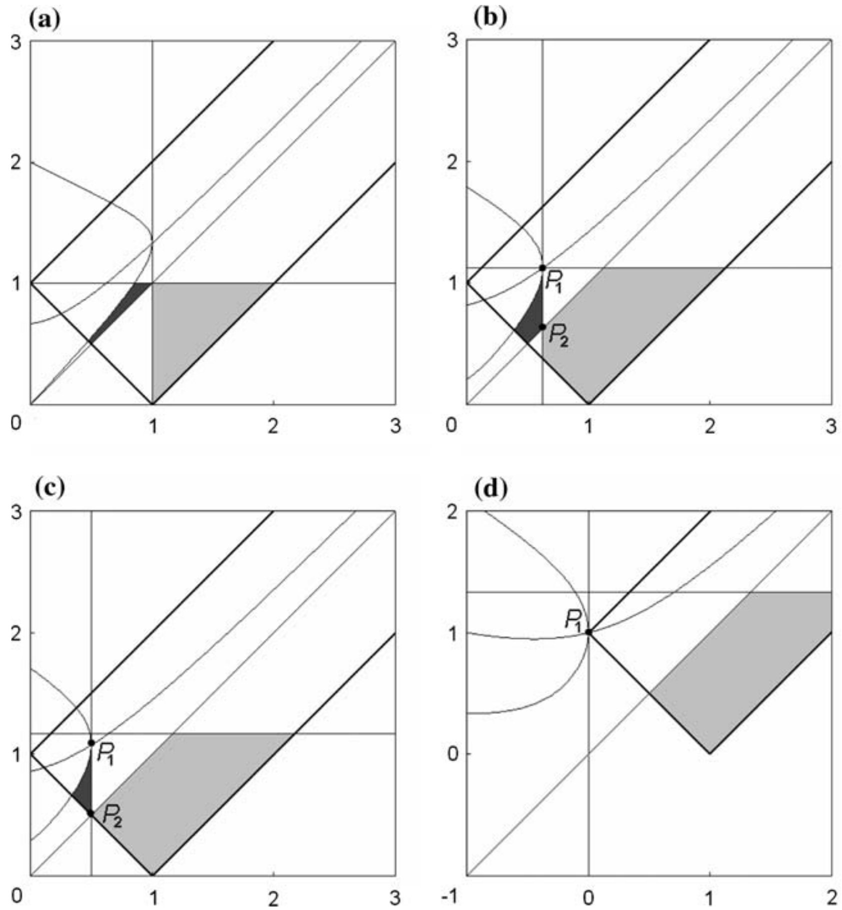


Fig. 6 Evolution of stability domains for solutions (13) in the case $\bar{a}_{22} = 1$: (**a**) $H_1 = 0$, (**b**) $H_1 = 0.375$, (**c**) $H_1 = 0.5$, (**d**) $H_1 = 1$

exist for solutions (13). Corresponding examples of domains for solutions (13) in the case $\bar{a}_{11} = -1$ are shown in Fig. 7.

Now let us consider solutions (15). In this case $\sin\beta_0 = 0$, $\sin\gamma_0 = 0$, $\cos^2\alpha_0 = x^2 = H_1^2/9\,(1-\theta_C)^2$, and the system (26) takes the form

$$\begin{aligned} &\theta_B\ddot{\bar{\alpha}} + \bar{a}_1\bar{\alpha} = 0,\\ &\theta_C\ddot{\bar{\beta}} + \left[4\,(\theta_B-1) - 3b_1\cos^2\alpha_0\right]\beta + (b_1-1)\,\dot{\bar{\gamma}}\cos\beta_0 + 3b_1x\bar{a}_{31}\bar{\gamma}\cos\beta_0 = 0,\\ &\ddot{\bar{\gamma}}\cos\beta_0 + b_1\left(1+3\cos^2\alpha_0\right)\bar{\gamma}\cos\beta_0 + (1-b_1)\,\dot{\bar{\beta}} + 3b_1x\bar{a}_{31}\bar{\beta} = 0, \end{aligned} \tag{30}$$

where

$$\bar{a}_1 = 3\,(1-\theta_C)\sin^2\alpha_0, \quad b_1 = \theta_B - \theta_C, \quad a_{31} = \pm\sqrt{1-x^2}, \quad \cos\beta_0 = \pm 1.$$

The characteristic equation of system (30)

$$\left(\theta_B\lambda^2 + \bar{a}_1\right)\left(\theta_C\lambda^4 + a_1\lambda^2 + a_2\right) = 0$$

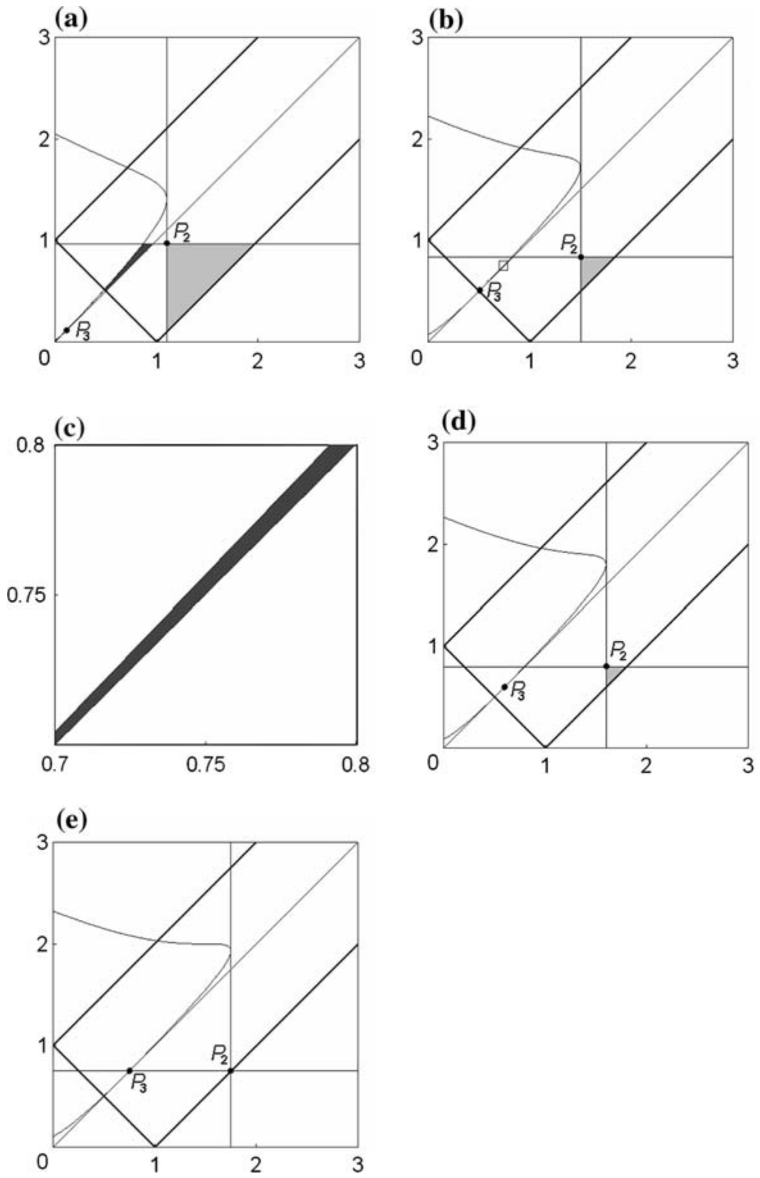


Fig. 7 Evolution of stability domains for solutions (13) in the case $\bar{a}_{22} = -1$: (**a**) $H_1 = 0.1$, (**b**) $H_1 = 0.5$, (**c**) enlarged fragment in (*b*), (**d**) $H_1 = 0.6$, (**e**) $H_1 = 0.75$

decomposes into quadratic and biquadratic equations. Here

$$a_1 = 3b_1 (\theta_C - 1)\cos^2 \alpha_0 + b_1\theta_C + 4(\theta_B - 1) + (1 - b_1)^2,$$
$$a_2 = 4b_1b_2, \quad b_2 = \theta_B - 1 + 3(\theta_C - 1)\cos^2 \alpha_0.$$

Necessary conditions of stability (N-conditions) have the form (28). Moreover, the conditions of existence of solutions (15) must be satisfied, i.e. $x^2 < 1$.

At least one of inequalities (28) becomes an equality on the boundary of the N-domain. It is possible to show that the line $a_1 = 0$ cannot be boundary. Indeed, if $a_1 = 0$ the condition $D_2 > 0$ results in $a_2 < 0$ and violates (28).

The equality $\bar{a}_1 = 0$ takes place on the straight lines $c_{1,2} = \theta_C - 1 \mp H_1/3 = 0$ and $c_3 = \theta_C - 1 = 0$. The first two straight lines are the boundaries of domains where solutions (15) exist, while the last straight line can be excluded, since solutions (15) do not exist at $1 - H_1/3 < \theta_C < 1 + H_1/3$.

The condition $a_2 > 0$ holds true either at $b_1 > 0$ and $b_2 > 0$, or at $b_1 < 0$ and $b_2 < 0$. In the first case all sufficient conditions of stability (22) are satisfied. It is evident that in this case all necessary conditions of stability (28) are also satisfied, i.e. $a_1 > 0$ and $D_2 > 0$. Therefore, we get the domain in which necessary and sufficient conditions of stability for solution (15), already obtained in the previous section, are satisfied.

Now let $b_1 < 0, b_2 < 0$. Then sufficient conditions of stability are not met, and necessary conditions take the form

$$\bar{a}_1 > 0\,, \quad a_1 > 0, \quad D_2 > 0\,, \quad \theta_B < \theta_C\,, \quad \theta_B < 1 - 3(\theta_C - 1)\cos^2\alpha_0. \tag{31}$$

We can exclude the third inequality of system (24) since it is automatically satisfied if $\theta_B < \theta_C$. An example of N-domain in the plane $(\theta_B,\ \theta_C)$ is shown in Fig. 8.

The curves $D_2 = 0$, $b_2 = 0$, and $a_1 = 0$ intersect in a common point P (more exactly, the curve $a_1 = 0$ passes through a tangency point of the curves $D_2 = 0$ and $b_2 = 0$). At small H_1 the point P is located between the straight lines $c_1 = 0$ and $c_2 = 0$.

The first bifurcation value $H_1 = 3/8$ occurs when the point P is situated on the line $c_1 = 0$. At $H_1 = \sqrt{3}/2$ the point P is on the boundary of the working region $\theta_C - \theta_B = 1$. As a result, the N-domain degenerates, and at $H_1 > \sqrt{3}/2 \approx 0.866$ it ceases to exist. The examples of N-domains for solutions (15) are presented in Fig. 9.

Finally, let us consider solutions (17). In this case $\sin\alpha_0 = 0$, $\sin\gamma_0 = 0$, $\cos^2\beta_0 = 9y^2 = H_1^2/(1-\theta_B)^2$, and the characteristic equation of the system (25) takes the form

$$a_0\lambda^6 + a_1\lambda^4 + a_2\lambda^2 + a_3 = 0, \tag{32}$$

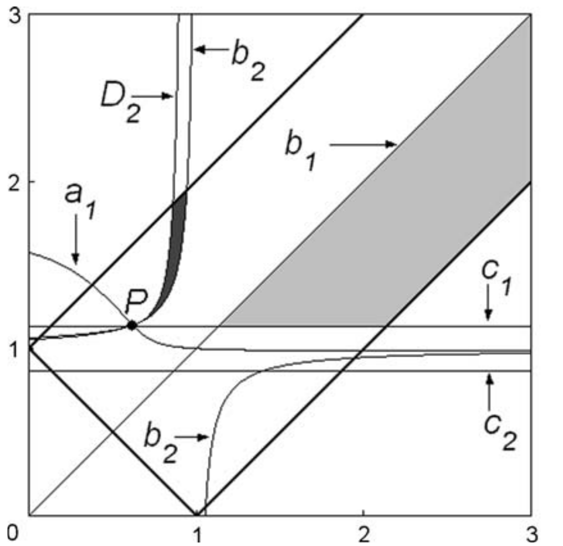


Fig. 8 N-domain for solutions (15) at $H_1 = 0.4$

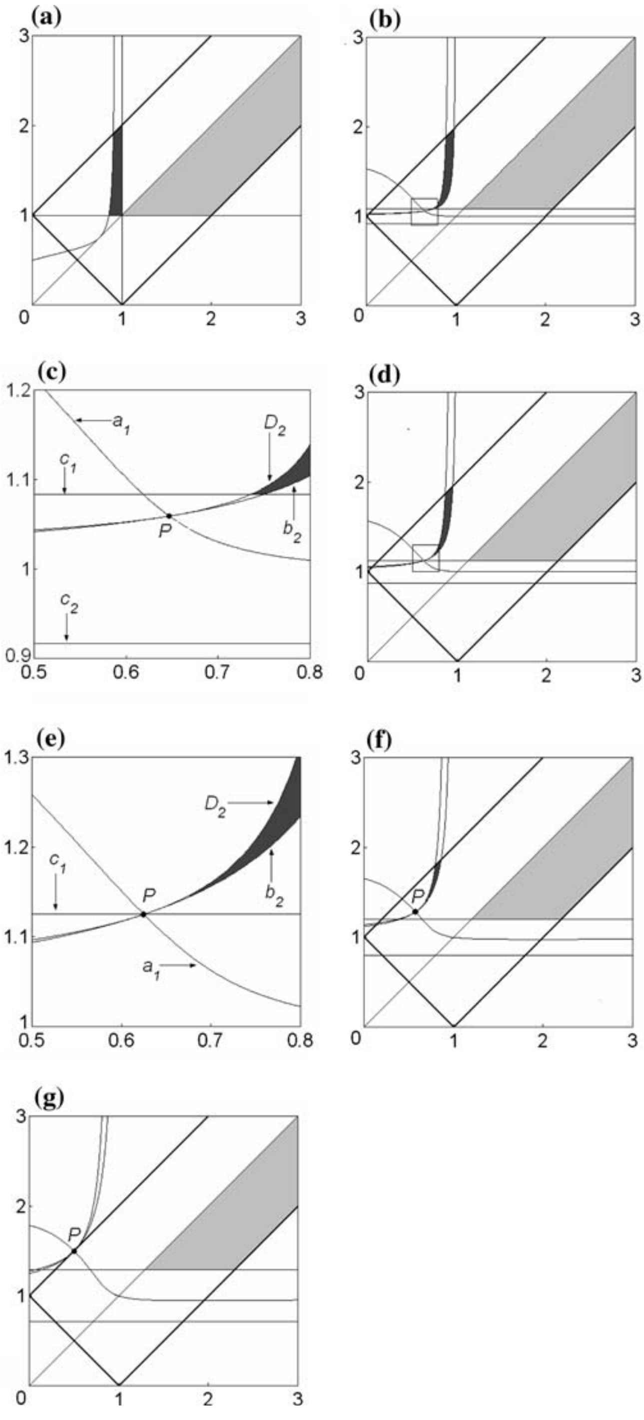


Fig. 9 Evolution of stability domains for solutions (15): (**a**) $H_1 = 0$, (**b**) $H_1 = 0.25$, (**c**) enlarged fragment in (**b**), (**d**) $H_1 = 0.375$, (**e**) enlarged fragment in (*d*), (**f**) $H_1 = 0.6$, (**g**) $H_1 = 0.866$

where

$$a_0 = \theta_B\theta_C,$$

$$a_1 = \left[\theta_B\theta_C + (1-\theta_C)(1-\theta_B+3\theta_C) + 3\theta_B\theta_C(\theta_B-\theta_C)\right] \\ -(1-\theta_B)\left(1+\theta_B^2-\theta_B-2\theta_C-\theta_B\theta_C\right)\cos^2\beta_0,$$

$$a_2 = \left[4(1-\theta_B)(1-\theta_C) + 3(\theta_B-\theta_C)\left(1-3\theta_C^2-\theta_B+2\theta_C+2\theta_B\theta_C\right)\right] \\ +(1-\theta_B)\left[4\theta_C-(\theta_B-\theta_C)(5+2\theta_B-12\theta_C)-3(1-\theta_B)(\theta_B-\theta_C)\cos^2\beta_0\right] \\ \times\cos^2\beta_0,$$

$$a_3 = 4(1-\theta_B)\ (\theta_B-\theta_C)\left[3(1-\theta_C)+(1-\theta_B)\cos^2\beta_0\right]\sin^2\beta_0\ .$$

The necessary conditions of stability are the conditions of purely imaginary roots of the polynomial (32), and they can be written as (see Kats 1951)

$$\begin{aligned} &a_1 > 0\,, \quad a_2 > 0\,, \quad a_3 > 0, \\ &D_2 = a_1^2 - 3a_0a_2 > 0\,, \quad D_3 = a_1^2a_2 - 4a_0a_2^2 + 3a_0a_1a_3 > 0, \\ &D_4 = a_1^2a_2^2 + 18a_0a_1a_2a_3 - 4a_0a_2^3 - 4a_1^3a_3 - 27a_0^2a_3^2 > 0. \end{aligned} \tag{33}$$

As in the previous cases, inequalities (24) should hold true together with (33). Moreover, the condition of existence of solutions (17) must be taken into account, i.e. $y^2 < 1/9$.

At least one of inequalities (33) becomes the equality on the boundary of the N-domain. It is possible to show that lines $a_1 = 0$ and $a_2 = 0$ cannot be boundaries. Indeed, if $a_1 = 0$ (remind that $a_0 = \theta_B\theta_C \neq 0$) the condition $D_2 > 0$ results in $a_2 < 0$ and violates (33). If $a_2 = 0 (a_0 \neq 0,\ a_1 \neq 0)$ the condition $D_4 > 0$ results in $-4a_1^3a_3 - 27a_0^2a_3^2 > 0$ witch, in turn, results in $a_3 < 0$ and also violates (33).

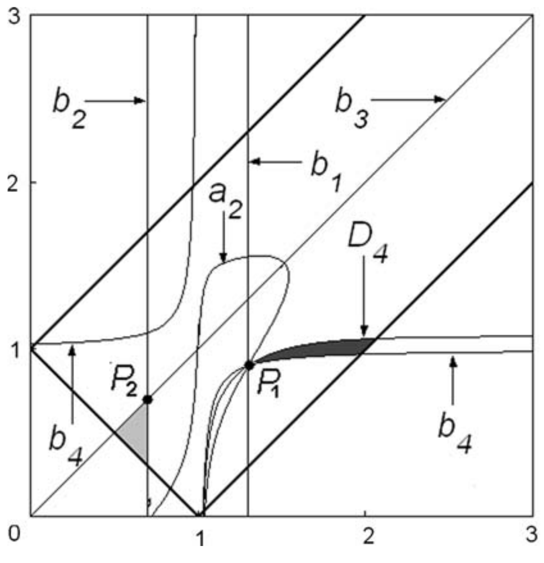


Fig. 10 Typical shape of domains for solutions (17) at $H_1 = 0.3$

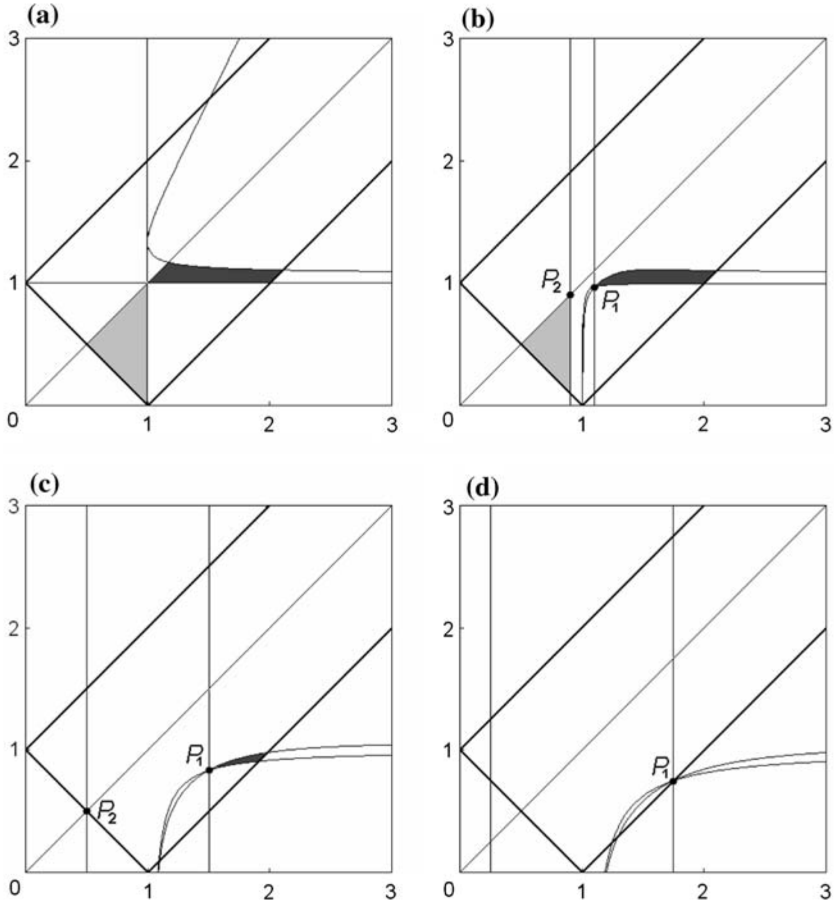


Fig. 11 Evolution of stability domains for solutions (17): (**a**) $H_1 = 0$, (**b**) $H_1 = 0.1$, (**c**) $H_1 = 0.5$, (**d**) $H_1 = 0.75$.

Thus, it is shown that the N-domain can be bounded by the lines $a_3 = 0, D_2 = 0, D_3 = 0$ and $D_4 = 0$. Moreover, numerical investigations demonstrate that only the lines $a_3 = 0$ or $D_4 = 0$ can be the boundaries of domain (33).

It follows from the form of a_3 that the equality $a_3 = 0$ takes place on the straight lines $b_{1,2} = 0$ $\left(\theta_B - 1 \mp H_1 = 0\right)$, $b_3 = 0$ $(\theta_B = \theta_C)$, and on the hyperbola $b_4 = 0$ $\left(3\,(1-\theta_B)\,(1-\theta_C) + H_1^2 = 0\right)$, while the straight line $b_5 = 0$ $(\theta_B = 1)$ can be excluded since solutions (17) do not exist at $1 - H_1 < \theta_B < 1 + H_1$. An example of N-domain in the plane $(\theta_B,\ \theta_C)$ is shown in Fig. 10. The N-domain is marked out by dark-gray color and is bounded by the straight line $\theta_C = \theta_B - 1$, and by the curves $b_4 = 0$ and $D_4 = 0$. The domain of fulfillment of the necessary and sufficient conditions of stability (S-domain) is marked out in the same figure in light-gray color. It is bounded by the straight lines $\theta_C = 1 - \theta_B$, $b_2 = 0$, and $b_3 = 0$. The curves $D_4 = 0$, $a_2 = 0$, and the straight lines $b_1 = 0$ and $b_4 = 0$ intersect in a common point $P_1 = \left(1 + H_1, 1 - H_1/3\right)$.

For varying H_1, the evolution of domains is presented in Fig. 11. Figure 11a corresponds to the case when the aerodynamic torque equals zero $(H_1 = 0)$. At $H_1 \neq 0$ the line $\theta_B = 1$

is split into two straight lines $\theta_B = 1 \pm H_1$ ($b_{1,2} = 0$). When H_1 increases, the band $1 - H_1 < \theta_B < 1 + H_1$, where solutions (17) do not exist, widens, and at $H_1 = 0.5$ the point P_2 is located on the boundary of the working region $\theta_C = 1 - \theta_B$. Thus, the S-domain degenerates into a point (Fig. 11c), and at $H_1 > 0.5$ ceases to exist.

The N-domain also reduces when H_1 increases, and degenerates into a point at $H_1 = 3/4$ when the point P_1 is situated on the boundary $\theta_C = \theta_B - 1$. At $H_1 > 3/4$ N-conditions are violated in any point of the working region.

6 Conclusion

In this work the attitude motion of a satellite under the action of gravitational and aerodynamic torques in a circular orbit has been investigated. The main attention was given to determination of a satellite equilibrium orientations in the orbital reference frame and to the analysis of their stability. The numerical method of determination of all satellite equilibria is suggested in the general case ($h_1 \neq 0, h_2 \neq 0, h_3 \neq 0$). The explicit expressions for direction cosines as functions of parameters θ_B, θ_C, H_1 for six groups (13)–(18) of equilibrium orientations are presented in particular case when $h_1 \neq 0, h_2 = 0, h_3 = 0$. It is proved that groups of solutions (13), (15), and (17) may be transformed to groups of solutions (14), (16), and (18), respectively, by the substitutions $\theta_B \to \theta_C, \theta_C \to \theta_B$. Using the Lyapunov theorem, the sufficient conditions of stability of the equilibrium orientations are obtained in the form of simple inequalities. In the final part of the work, the evolution of domains of the necessary conditions of stability is investigated in detail by numerical-analytical method in the plane of two dimensionless moments of inertia (θ_B, θ_C) at different values of parameter H_1. All bifurcation values of H_1 corresponding to the qualitative change of the stability domains are determined. Moreover, all types of domains, where sufficient and (or) necessary conditions of stability of solutions (13), (15), and (17) hold true, are given. Corresponding domains for solutions (14), (16), and (18) are symmetric to them with respect to the straight line $\theta_B = \theta_C$.

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