RELATIVE EQUILIBRIA FOR GENERAL GRAVITY FIELDS IN THE SPHERE-RESTRICTED FULL 2-BODY PROBLEM

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Abstract. Equilibrium conditions for a mutually attracting general mass distribution and point mass are derived and their stability computed. The equilibrium conditions can be reduced to six equations in six unknowns, plus the existence of four integrals of motion consisting of the total angular momentum and energy of the system. The equilibrium conditions are further reduced to two independent equations, and their theoretical properties are studied. We derive three distinct conditions for a relative equilibrium which can be used to derive robust algorithms for solving these problems for non-symmetric gravity fields: a set of necessary conditions, a set of sufficient conditions, and a set of necessary and sufficient conditions. Each of these conditions is well suited for the computation of certain classes of equilibria. These equations are solved for non-symmetric gravity fields of interest, using a real asteroid shape model for the general gravity fields. Explicit conditions for the spectral and energetic stability of the resulting equilibria are also derived and computed for the shape of interest.

1. Introduction

The study of relative equilibria between massive bodies has received much interest over the years as many solar system bodies have been found to lie in or close to relative equilibria. Examples include the moon relative to the Earth, the Galilean satellites relative to Jupiter, and more recently asteroid binary secondaries with respect to their primaries. The study of this problem requires one to solve for and analyze the mutual gravitational attractions and torques between two non-spherical bodies. Usually, one of the bodies (generally the larger) is not in a synchronous rotation state, giving rise to periodic perturbations to the system. The study of these general systems in a quasi-equilibrium, where one of the bodies is synchronous but the other is not, is quite complex and is taxing to current analytical and numerical methods. A detailed discussion of this problem is given in a number of papers (Simo et al., 1991; Wang et al., 1991; Maciejewski, 1995; Fahnestock et al., 2005).

In this paper we do not consider that most general problem, what we have termed the Full 2-Body Problem (Scheeres, 2002b), but instead study a one-stage simplification of it, the Sphere-Restricted Full 2-Body Problem (SRF2BP), which arises when one of the bodies is a sphere (i.e., a point mass). Even for such a system, however, there are only limited dynamical results, mostly specialized to particular systems. In order to motivate future studies of the more complete system we perform a study of the SRF2BP that goes beyond previous work in that a set of practical algorithms for the computation of relative equilibria and their stability are found, analyzed and implemented. One key assumption of nearly all previous studies is that the gravity field of the non-spherical body can be approximated with its 2nd degree and order gravity field. Exceptions that exist generally only describe the process of determining the equilibrium configurations of a more general system without specific results for specific bodies (Wang et al., 1991). In this paper we address this limitation by deriving conditions that can be used to compute relative equilibrium in the SRF2BP for general gravity fields, and show its applicability by computing a range of relative equilibria for a complex asteroid shape.

The motivation for this work comes from a desire to understand and study the dynamics of binary asteroid systems during their formation process. Specifically, we suspect that most relative equilibria between two general asteroids will be unstable when the distance is close, and only become stable as their distance increases, consistent with ellipsoidal models (Scheeres, 2004). These questions are of interest for understanding the dynamics and initial evolution of binary asteroid systems after their formation. The work reported here will also enable the study and analysis of real asteroid shapes, as determined from *in situ* and ground-based observations. Current methodologies are not easily generalized to such non-symmetric bodies. As an additional result of our study we also shed light on previous results derived under a symmetric gravity field assumption.

In this paper we specifically derive equilibrium conditions for a mutually attracting general mass distribution and point mass. The full dimensionality of this system has 9 degrees of freedom and hence 18 seperate differential equations. The equilibrium conditions can be easily reduced to 6 equations in six unknowns, plus the existence of four integrals of motion consisting of the total angular momentum and energy of the system. We further reduce the equilibrium conditions to two independent equations and study their theoretical properties. We solve these equations for a non-symmetric gravity field of interest and determine their spectral and energetic stability.

There has been much previous work in this area (Kinoshita, 1972; Simo et al., 1991; Wang et al., 1991; Maciejewski, 1995; Scheeres, 2004). Our contribution is to perform an in-depth analysis of the equilibrium

equations themselves and reduce them to a minimal set of two equations. We find three different forms of these equations, a set that defines the necessary conditions for equilibrium, a set that defines sufficient conditions for equilibrium, and a set that defines necessary and sufficient conditions. Using these equations we study theoretical properties of these solutions and compute relative equilibria for a general mass distribution. A secondary contribution is the derivation of explicit formula for the computation of the spectral and energetic stability of such relative equilibria.

2. Fundamentals

We skip a complete derivation of the equations of motion and only state them in an appropriate form. For in-depth derivations of this problem see (Wang et al., 1991; Maciejewski, 1995; Koon et al., 2004; Scheeres, 2004). In the following we are concerned with stating the equations of motion for a general mass distribution and a sphere. These equations can be reduced to the relative motion between the bodies and the rotational dynamics of the general body. In fact, the equations of motion for the body's spin vector can be decoupled from its orientation, which is then solved after the fact. Our problem is analyzed most simply if we transform to a coordinate frame fixed to the rotating, general body. The main advantage of stating the equations of motion in this frame is that they remove the attitude of the body from consideration and allow for a set of "internal" variables decoupled from the inertial attitude of the system (Figure 1).
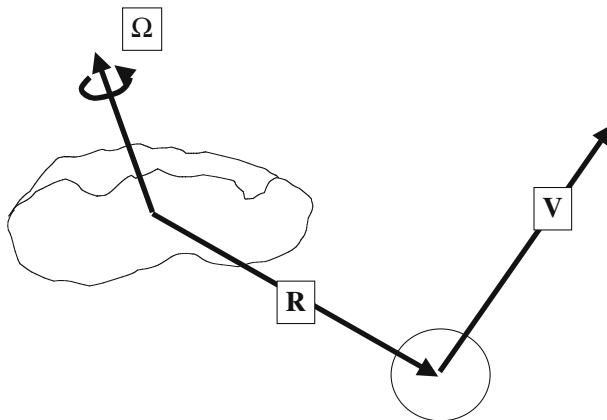

Figure 1. Representation of a SRF2BP.

2.1. EQUATIONS OF MOTION IN A ROTATING FRAME

The general equations of motion for our system are stated as:

$$\ddot{\mathbf{R}} + 2\boldsymbol{\Omega} \times \dot{\mathbf{R}} + \dot{\boldsymbol{\Omega}} \times \mathbf{R} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R}) = \mathcal{G}\left(M_c + M_s\right)\frac{\partial U}{\partial \mathbf{R}}, \tag{1}$$

$$\mathbf{I} \cdot \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathbf{I} \cdot \boldsymbol{\Omega} = -\mathcal{G}M_c M_s \mathbf{R} \times \frac{\partial U}{\partial \mathbf{R}}, \tag{2}$$

where $\mathbf{R}$ is the relative position vector between the bodies and is centered at the non-spherical body, $\boldsymbol{\Omega}$ is the angular velocity vector of the non-spherical body, M_c is the mass of the non-spherical body, M_s is the mass of the sphere, and $\mathcal{G}$ is the gravitational constant. The unit mass gravity potential is U and the inertia dyad is $\mathbf{I}$, defined as:

$$U(\mathbf{R}) = \frac{1}{M_c} \int_{\beta_c} \frac{dm(\boldsymbol{\rho})}{|\mathbf{R} + \boldsymbol{\rho}|}, \tag{3}$$

$$\mathbf{I} = -\int_{\beta_c} \tilde{\boldsymbol{\rho}} \cdot \tilde{\boldsymbol{\rho}} dm(\boldsymbol{\rho}), \tag{4}$$

where $\boldsymbol{\rho}$ is the position vector of the mass element dm in the general body, β_c represents the mass distribution of this body, and $\tilde{\boldsymbol{\rho}}$ denotes the cross-product dyad (see the Appendix A for a brief definition and discussion of dyads). This study uses dyad notation as it allows all analytical derivations to be performed independent of coordinate frame. Thus, the choice of reference frame can be delayed until the resultant formulae are implemented numerically.

The relative vector $\mathbf{R}$ can be expressed in terms of the two body's inertial centers of mass:

$$\mathbf{R} = \mathbf{A}^T \cdot \mathbf{r}_I, \tag{5}$$

$$= \mathbf{A}^T \cdot (\mathbf{r}_s - \mathbf{r}_c), \tag{6}$$

where $\mathbf{A}$ is the attitude matrix of the general body (i.e., the transformation matrix that takes the general body frame to the inertial frame), $\mathbf{A}^T$ is the transformation matrix that goes from the inertial frame to the general body frame, $\mathbf{r}_I$ is the relative vector between the bodies in inertial space, and $\mathbf{r}_s$ and $\mathbf{r}_c$ are the inertial position vectors of the two bodies.

The evolution of the attitude matrix follows a simple equation:

$$\dot{\mathbf{A}} = \mathbf{A} \times \boldsymbol{\Omega}. \tag{7}$$

However, we find it convenient to use the more fundamental definition of attitude from the axis-angle coordinates, which describe the attitude of a

rigid body by a rotation of angle ϕ about an axis $\hat{\mathbf{a}}$, where the hat nota-tion denotes a unit vector. In these coordinates, the attitude matrix can be specified as:

$$\mathbf{A} = \cos\phi\,\mathbf{U} + (1 - \cos\phi)\hat{\mathbf{a}}\hat{\mathbf{a}} - \sin\phi\tilde{\hat{\mathbf{a}}}, \tag{8}$$

where $\mathbf{U}$ is the identity dyad. In a body-fixed frame, the axis-angle coordi-nates follow their own equations of motion:

$$\dot{\hat{\mathbf{a}}} = \frac{1}{2}\left[\tilde{\hat{\mathbf{a}}} - \cot\left(\frac{\phi}{2}\right)\tilde{\hat{\mathbf{a}}}\cdot\tilde{\hat{\mathbf{a}}}\right]\cdot\mathbf{\Omega}, \tag{9}$$

$$\dot{\phi} = \hat{\mathbf{a}}\cdot\mathbf{\Omega}. \tag{10}$$

This system, as stated above, has four integrals of motion defined for it, the total energy and the total angular momentum. In the body-fixed frame these are specifically give as (Scheeres, 2004b):

$$E = \frac{1}{2}\frac{M_s M_c}{M_s + M_c}\left(\dot{\mathbf{R}} + \mathbf{\Omega}\times\mathbf{R}\right)\cdot\left(\dot{\mathbf{R}} + \mathbf{\Omega}\times\mathbf{R}\right)$$
$$+ \frac{1}{2}\mathbf{\Omega}\cdot\mathbf{I}\cdot\mathbf{\Omega} - \mathcal{G}M_s M_c U(\mathbf{R}), \tag{11}$$

$$\mathbf{K} = \mathbf{A}\cdot\left[\mathbf{I}\cdot\mathbf{\Omega} + \frac{M_c M_s}{M_c + M_s}\left(\mathbf{R}\times\dot{\mathbf{R}} + \mathbf{R}\times(\mathbf{\Omega}\times\mathbf{R})\right)\right]. \tag{12}$$

2.2. EQUATIONS OF MOTION IN A NORMALIZED FRAME

The equations can be simplified by introducing three normalizations: a length scale, a time scale, and normalizing the inertia $\mathbf{I}$ by the total mass of the non-spherical body, M_c, as well as the length scale squared. For the length scale we take an arbitrary radius (often the maximum radius of the non-spherical body), denoted as α. To define the time scale we take the mean motion of the total system at this radius: $n = \sqrt{\mathcal{G}(M_c + M_s)/\alpha^3}$, and define the normalized position vectors as $\mathbf{r} = \mathbf{R}/\alpha$ and the normalized rotational velocity vectors as $\boldsymbol{\omega} = \mathbf{\Omega}/n$. Finally, we rescale the inertia matrix by divid-ing it by $M_c\alpha^2$. In the following, we define the new time derivatives by $()'$. This yields the following set of equations:

$$\mathbf{r}'' + 2\boldsymbol{\omega}\times\mathbf{r}' + \boldsymbol{\omega}'\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) = \frac{\partial\mathcal{U}}{\partial\mathbf{r}}, \tag{13}$$

$$\mathcal{I}\cdot\boldsymbol{\omega}' + \boldsymbol{\omega}\times\mathcal{I}\cdot\boldsymbol{\omega} = -\nu\mathbf{r}\times\frac{\partial\mathcal{U}}{\partial\mathbf{r}}, \tag{14}$$

where the normalized inertia matrix and gravitational force potential are, respectively,

$$\mathcal{I} = -\frac{1}{M_c \alpha} \int_{\beta_c} \tilde{\boldsymbol{\rho}} \cdot \tilde{\boldsymbol{\rho}} \, dm, \tag{15}$$

$$\mathcal{U} = \frac{\alpha}{M_c} \int_{\beta_c} \frac{dm}{|\mathbf{r} + \boldsymbol{\rho}|}. \tag{16}$$

The equations for the axis-angle variables are:

$$\hat{\mathbf{a}}' = \frac{1}{2} \left[\tilde{\tilde{\mathbf{a}}} - \cot\left(\frac{\phi}{2}\right) \tilde{\mathbf{a}} \cdot \tilde{\mathbf{a}} \right] \cdot \boldsymbol{\omega}, \tag{17}$$

$$\phi' = \hat{\mathbf{a}} \cdot \boldsymbol{\omega} \tag{18}$$

and the transformed integrals of motion are

$$\boldsymbol{k} = \mathbf{A} \cdot \left[\mathcal{I} \cdot \boldsymbol{\omega} + \nu \mathbf{r} \times (\mathbf{r}' + \boldsymbol{\omega} \times \mathbf{r}) \right], \tag{19}$$

$$\mathcal{E} = \frac{\nu}{2} (\mathbf{r}' + \boldsymbol{\omega} \times \mathbf{r}) \cdot (\mathbf{r}' + \boldsymbol{\omega} \times \mathbf{r}) + \frac{1}{2} \boldsymbol{\omega} \cdot \mathcal{I} \cdot \boldsymbol{\omega} - \nu \mathcal{U}. \tag{20}$$

We note that the "internal" variables of the system, the relative position $\mathbf{r}$, the velocity $\mathbf{r}'$, and the angular velocity of the system $\boldsymbol{\omega}$, are decoupled from the attitude of the system. We also note that the energy is only a function of these internal variables, while the angular momentum is a function of both the internal variables and the attitude of the system. One important observation, however, is that the angular momentum magnitude does not depend on the attitude and only depends on the internal variables. On the other hand, the unit vector along the angular momentum, $\hat{\boldsymbol{k}}$, is a function of the internal variables and the attitude dyad $\mathbf{A}$. Thus, if we only consider the internal variables of the system and their stability, we should only concern ourselves with the energy and angular momentum magnitude integrals.

One of the free parameters is the system's mass fraction between the spherical and non-spherical bodies:

$$\nu = \frac{M_s}{M_s + M_c}, \tag{21}$$

which is the same parameter as is found in the restricted three body problem. The case $\nu \to 0$ corresponds to the motion of a material point in the gravity field of the non-spherical body, with main application to orbital dynamics of a particle about a general gravity field. In this case, we see that the energy and angular momentum integrals are dominated by the rotational dynamics of the non-spherical body, and that the contribution of the spherical body's motion decouples from these integrals. The case $\nu \to 1$ corresponds to the motion of a massless non-spherical body about a point-mass, with an application to a large satellite in orbit about a planet. It is

important to note that the angular momentum and energy integrals still apply to this problem. There is no singularity as $M_c \to 0$ as the inertia dyad and gravity field of the non-spherical body are defined relative to the geometry of their mass distribution only. This indicates that analysis of gravity gradient satellites should not neglect translational motion, as it is coupled at this fundamental level to the rotational motion. To simplify our later computations of equilibria for a real asteroid model we leave the length unit unchanged (i.e., $\alpha = 1$) but apply the time normalization to eliminate the total mass from the problem.

2.3. ENERGY AT A CONSTANT ANGULAR MOMENTUM

Of specific interest later in this paper is the form of the energy when a constant angular momentum vector is imposed. We first note that the angular momentum integral can be rewritten in the form:

$$k = \mathbf{A} \cdot \left\{ [\mathcal{I} - \nu \tilde{\mathbf{r}} \cdot \tilde{\mathbf{r}}] \cdot \boldsymbol{\omega} + \nu \tilde{\mathbf{r}} \cdot \mathbf{r}' \right\} \tag{22}$$

and that $\boldsymbol{\omega}$ can be solved for as:

$$\boldsymbol{\omega} = \mathcal{J}^{-1} \cdot \left[\mathbf{A}^T \cdot k - \nu \tilde{\mathbf{r}} \cdot \mathbf{r}' \right], \tag{23}$$
$$\mathcal{J} = \mathcal{I} - \nu \tilde{\mathbf{r}} \cdot \tilde{\mathbf{r}}. \tag{24}$$

We note that the dyad $\mathcal{J}$ is never singular (assuming that $\mathcal{I}$ is not singular) and will play an important role later in our analysis. Physically, $\mathcal{J}$ is just the normalized inertia tensor of the entire binary system relative to the non-spherical body center of mass, and is also known as the "locked inertia" (Simo et al., 1991).

Taking this result, we can eliminate $\boldsymbol{\omega}$ from explicitly appearing in the energy equation to find:

$$\mathcal{E} = \frac{\nu}{2} \mathbf{r}' \cdot \left[\mathbf{U} + \tilde{\mathbf{r}} \cdot \mathcal{J}^{-1} \cdot \tilde{\mathbf{r}} \right] \cdot \mathbf{r}' + \frac{1}{2} k \cdot \mathbf{A} \cdot \mathcal{J}^{-1} \cdot \mathbf{A}^T \cdot k - \nu \mathcal{U}. \tag{25}$$

Even though the explicit dependence of $\boldsymbol{\omega}$ is removed, we note that this occurs at the cost of introducing the attitude matrix $\mathbf{A}$ into the energy.

3. Equilibrium Conditions

In the following, starting from the equations of motion, we derive a number of different versions of the equilibrium conditions. Our main focus is to reduce the conditions to the minimum number possible, which is two. Such a reduction makes it feasible to solve for the equilibrium conditions

numerically for a specified gravity field. We find, however, that different versions of the conditions are more suited to finding particular types of equilibria.

3.1. FUNDAMENTAL EQUATIONS

In the above form, the analysis of equilibrium solutions is relatively easy. The conditions for equilibrium (i.e., that all time derivatives be zero) are:

$$\hat{\mathbf{a}} = \hat{\boldsymbol{\omega}}, \tag{26}$$

$$\tilde{\boldsymbol{\omega}} \cdot \tilde{\boldsymbol{\omega}} \cdot \mathbf{r} = \frac{\partial \mathcal{U}}{\partial \mathbf{r}}, \tag{27}$$

$$\tilde{\boldsymbol{\omega}} \cdot \mathcal{I} \cdot \boldsymbol{\omega} + \nu \tilde{\mathbf{r}} \cdot \frac{\partial \mathcal{U}}{\partial \mathbf{r}} = 0. \tag{28}$$

The first equation just places constraints on the axis of rotation. Equations (27) and (28) constitute a set of six equations for six unknowns, the relative position $\mathbf{r}$ and the angular velocity $\boldsymbol{\omega}$. The solution of these equations is non-trivial, especially for a non-symmetric mass distribution for the gravitational potential. It is important to note that for $\nu = 0$ the solution for Equation (28) reduces to the classic rigid body rotation results, equilibria are rotations about the principal axes of inertia. Given this, solutions to Equation (27) are then equilibrium points of a particle in the field of the uniformly rotating body, a subject studied previously for irregular mass distributions (Scheeres et al., 1996).

By definition, a system in equilibrium will have a constant uniform rotation vector $\boldsymbol{\omega}$ defined by its rotation rate $\omega = |\boldsymbol{\omega}|$ and its unit vector $\hat{\boldsymbol{\omega}}$, and thus the transformation matrix $\mathbf{A}$ will have the specific form:

$$\mathbf{A} = \cos(\omega t)\mathbf{I} + (1 - \cos(\omega t))\,\hat{\boldsymbol{\omega}}\hat{\boldsymbol{\omega}} + \tilde{\hat{\boldsymbol{\omega}}}\sin(\omega t). \tag{29}$$

The presence of a uniformly increasing angle, ωt, signifies that this is a relative equilibrium. We also note that the spin axis is a unity eigenvector of this transformation matrix, or $\mathbf{A} \cdot \boldsymbol{\omega} = \boldsymbol{\omega}$.

At an equilibrium the integrals of motion reduce to:

$$\boldsymbol{k} = \mathbf{A} \cdot \mathcal{J} \cdot \boldsymbol{\omega}, \tag{30}$$

$$\mathcal{E} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathcal{J} \cdot \boldsymbol{\omega} - \nu \mathcal{U}. \tag{31}$$

3.2. IMPLICATIONS OF THE EQUILIBRIA

There are a number of fundamental observations that we can make about the relative geometry of the system and conditions on the system when

it satisfies the equilibrium solutions. These observations are not new, and many were noted previously in Wang et al. (1991) and Maciejewski (1995). We restate them here as we can establish these results in a very simple manner, and to allow us to use them in our later discussion concerning the reduction of the equilibrium conditions.

PROPOSITION 1. *The spin axis is perpendicular to the gravitational acceleration.*

Consider Equation (27). Taking the dot product of $\boldsymbol{\omega}$ and $\mathcal{U}_{\mathbf{r}}$ we find: $\boldsymbol{\omega} \cdot \mathcal{U}_{\mathbf{r}} = \boldsymbol{\omega} \cdot \tilde{\boldsymbol{\omega}} \cdot \tilde{\boldsymbol{\omega}} \cdot \mathbf{r} \equiv 0$.

PROPOSITION 2. *The spin axis is an eigenvector of* $[\mathcal{I} - \nu\mathbf{rr}]$.

Insert Equation (27) for the gravitational acceleration into Equation (28) and simplify to find:

$$\boldsymbol{\omega} \times \mathcal{I} \cdot \boldsymbol{\omega} + \nu(\mathbf{r} \cdot \boldsymbol{\omega})\mathbf{r} \times \boldsymbol{\omega} = 0, \tag{32}$$

which can be re-written as:

$$\boldsymbol{\omega} \times [\mathcal{I} - \nu\mathbf{rr}] \cdot \boldsymbol{\omega} = 0, \tag{33}$$

where $\mathbf{rr}$ is a dyad. Assuming that $|\boldsymbol{\omega}| \neq 0$, this will be true if and only if $[\mathcal{I} - \nu\mathbf{rr}] \cdot \boldsymbol{\omega}$ is parallel to $\boldsymbol{\omega}$, or:

$$[\mathcal{I} - \nu\mathbf{rr}] \cdot \boldsymbol{\omega} = \sigma\boldsymbol{\omega}, \tag{34}$$

where σ is an eigenvalue of the dyadic $\mathcal{I} - \nu\mathbf{rr}$ and $\boldsymbol{\omega}$ is its eigenvector.

COROLLARY 3. *The spin axis is an eigenvector of* $\mathcal{J} = [\mathcal{I} - \nu\tilde{\mathbf{r}} \cdot \tilde{\mathbf{r}}]$.

Note that $-\tilde{\mathbf{r}} \cdot \tilde{\mathbf{r}} = r^2\mathbf{U} - \mathbf{rr}$. Thus this is rewritten as $[\mathcal{I} - \nu\mathbf{rr}] + \nu r^2\mathbf{U}$. From Proposition 2 the spin axis is an eigenvector of the first term, and every vector is an eigenvector of the identity dyad.

COROLLARY 4. *The angular momentum vector is parallel to the spin axis.*

Note that the angular momentum equation can be written as: $\boldsymbol{k} = \mathbf{A} \cdot \mathcal{J} \cdot \boldsymbol{\omega}$. The spin vector is an eigenvector of both dyads, and thus $\boldsymbol{k}$ must be parallel to it, and will be an eigenvector to these dyads as well.

COROLLARY 5. *The eigenvalue of the $\mathcal{J}$ dyad along the angular velocity is the ratio of the angular momentum magnitude over the angular velocity magnitude.*

Note that $\boldsymbol{k}=k\hat{\boldsymbol{k}}=\mathbf{A}\cdot\mathcal{J}\cdot\boldsymbol{\omega}=\lambda\mathbf{A}\cdot\boldsymbol{\omega}=\lambda\boldsymbol{\omega}=\lambda\omega\hat{\boldsymbol{\omega}}$. But $\hat{\boldsymbol{k}}=\hat{\boldsymbol{\omega}}$ from Corollary 4, leading to $\lambda=k/\omega$.

PROPOSITION 6. *The relative position vector lies in the plane defined by the gravitational acceleration and the spin axis.*

This can be directly shown by rearranging Equation (27) into the form:

$$\mathbf{r}=\frac{1}{\omega^2}(\boldsymbol{\omega}\cdot\mathbf{r})\boldsymbol{\omega}-\frac{1}{\omega^2}\mathcal{U}_{\mathbf{r}}. \tag{35}$$

PROPOSITION 7. *Stationary values of the energy at a constant value of angular momentum are a necessary condition for equilibrium.*

To establish this, we assume that the system is in relative equilibrium and then show that the energy is at a stationary point. When we take the variation of energy, we note that the angular momentum must be held constant. We can explicitly do this by taking the variation of the energy given in Equation (25), where the angular momentum appears explicitly. Starting from Equation (25), we take a first variation of the energy and then substitute the equilibrium conditions. In the following we note that the only requirements we use are that $\mathbf{r}'=0$, Equations (26) and (27) and Corollaries 3–5.

Note that the first variation of the first term of $\mathcal{E}$ will be identically zero, given the equilibrium condition $\mathbf{r}'=0$. Carrying out the variations for the remaining terms we find:

$$\delta\mathcal{E}=\frac{1}{2}\boldsymbol{k}\cdot\left[\delta\mathbf{A}\cdot\mathcal{J}^{-1}\cdot\mathbf{A}^T+\mathbf{A}\cdot\delta\mathcal{J}^{-1}\cdot\mathbf{A}^T+\mathbf{A}\cdot\mathcal{J}^{-1}\cdot\delta\mathbf{A}^T\right]\cdot\boldsymbol{k}-\nu\delta\mathcal{U}. \tag{36}$$

The variations $\delta\mathbf{A}$, $\delta\mathcal{J}^{-1}$, and $\delta\mathcal{U}$ will be considered in a moment, but first we make the following observations. From Corollaries 4 and 5 we know that $\boldsymbol{k}$ is an eigenvector of $\mathcal{J}^{-1}$ with an eigenvalue equal to $1/\lambda$. Also, from Equation (26) we know that $\boldsymbol{k}$ is an eigenvector of $\mathbf{A}$ with unity eigenvalue. Thus we make the following simplifications: $\mathcal{J}^{-1}\cdot\mathbf{A}^T\cdot\boldsymbol{k}=\boldsymbol{k}\cdot\mathbf{A}\cdot\mathcal{J}^{-1}=\boldsymbol{k}/\lambda$ and $\boldsymbol{k}\cdot\mathbf{A}=\mathbf{A}^T\cdot\boldsymbol{k}=\boldsymbol{k}$. This reduces the equation to

$$\delta\mathcal{E}=\frac{1}{2\nu}\left[\frac{1}{\lambda}\boldsymbol{k}\cdot\delta\mathbf{A}\cdot\boldsymbol{k}+\boldsymbol{k}\cdot\delta\mathcal{J}^{-1}\cdot\boldsymbol{k}+\frac{1}{\lambda}\boldsymbol{k}\cdot\delta\mathbf{A}^T\cdot\boldsymbol{k}\right]-\delta\mathcal{U}. \tag{37}$$

First consider $k \cdot \delta \mathbf{A} \cdot k$. The general form of $\mathbf{A}$, prior to evaluating it at the equilibrium conditions, is $\mathbf{A} = \cos \phi \mathbf{U} + (1 - \cos \phi) \hat{\mathbf{a}} \hat{\mathbf{a}} - \sin \phi \tilde{\hat{\mathbf{a}}}$. Since k is kept constant we can rewrite this as $k \cdot \delta \mathbf{A} \cdot k = \delta(k \cdot \mathbf{A} \cdot k)$. The expression becomes $k \cdot \mathbf{A} \cdot k = \cos \phi |k|^2 + (1 - \cos \phi)(\hat{\mathbf{a}} \cdot k)^2 - \sin \phi k \cdot \tilde{\hat{\mathbf{a}}} \cdot k$ and the last term is identically zero. Taking variations of this expression then gives us:

$$\delta(k \cdot \mathbf{A} \cdot k) = \sin \phi \left[(\hat{\mathbf{a}} \cdot k)^2 - |k|^2 \right] \delta \phi + 2(1 - \cos \phi)(\hat{\mathbf{a}} \cdot k)(k \cdot \delta \hat{\mathbf{a}}). \qquad (38)$$

Now apply Corollary 4 and Equation (26) to note that $\hat{\mathbf{a}}$ and k are parallel to each other. This immediately zeros out the $\delta \phi$ variation term. Next note that the variation $\delta \hat{\mathbf{a}}$ must be perpendicular to itself, and hence must be perpendicular to k, making the second variation term zero as well.

Now consider the term $k \cdot \delta \mathcal{J}^{-1} \cdot k$. First note the identity: $\delta \mathcal{J}^{-1} = -\mathcal{J}^{-1} \cdot \delta \mathcal{J} \cdot \mathcal{J}^{-1}$. This, along with Corollary 4, tells us that $k \cdot \delta \mathcal{J}^{-1} \cdot k = -k \cdot \delta \mathcal{J} \cdot k / \lambda^2$. Again, since the angular momentum is a constant, this reduces to $-\delta(k \cdot \mathcal{J} \cdot k) / \lambda^2$. Consider the quantity $k \cdot \mathcal{J} \cdot k = k \cdot \mathcal{I} \cdot k - \nu k \cdot \tilde{\mathbf{r}} \cdot \tilde{\mathbf{r}} \cdot k$. The first term is identically constant and will disappear under the variation, the second term can be rewritten using the following identity: $k \cdot \tilde{\mathbf{r}} \cdot \tilde{\mathbf{r}} \cdot k = \mathbf{r} \cdot \tilde{k} \cdot \tilde{k} \cdot \mathbf{r}$. Thus we find that $k \cdot \delta \mathcal{J}^{-1} \cdot k = 2 \frac{\nu}{\lambda^2} \mathbf{r} \cdot \tilde{k} \cdot \tilde{k} \cdot \delta \mathbf{r}$.

Finally, consider the term $\delta \mathcal{U}$. Since $\mathcal{U}$ is only a function of position, when expressed in the body-fixed frame, we have the simple result: $\delta \mathcal{U} = \mathcal{U}_{\mathbf{r}} \cdot \delta \mathbf{r}$, where $\mathcal{U}_{\mathbf{r}}$ signifies the gradient of $\mathcal{U}$ with respect to the position vector $\mathbf{r}$. Putting all of these elements together, we find a simplified expression for the energy variation:

$$\delta \mathcal{E} = \left[\frac{1}{\lambda^2} \mathbf{r} \cdot \tilde{k} \cdot \tilde{k} - \mathcal{U}_{\mathbf{r}} \right] \cdot \delta \mathbf{r}. \qquad (39)$$

Now we can apply Corollaries 4 and 5 to make the substitution $k/\lambda = \boldsymbol{\omega}$ when at the equilibrium conditions.

$$\delta \mathcal{E} = [\mathbf{r} \cdot \tilde{\boldsymbol{\omega}} \cdot \tilde{\boldsymbol{\omega}} - \mathcal{U}_{\mathbf{r}}] \cdot \delta \mathbf{r}. \qquad (40)$$

The term multiplied by the variation $\delta \mathbf{r}$ is precisely Equation (27), which is satisfied under the equilibrium hypothesis, and thus we establish that $\delta \mathcal{E} = 0$ to first order, i.e., the energy is stationary at an equilibrium point.

PROPOSITION 8. *A necessary and sufficient condition for a relative equilibrium is that the energy is at a stationary value and the angular momentum vector is an eigenvector of the dyad* $\mathcal{J} = \mathcal{I} - \nu \tilde{\mathbf{r}} \cdot \tilde{\mathbf{r}}$.

That it is necessary arises from Corollary 4 and Proposition 7. To prove sufficiency, we first need to note that we can derive Equation (39) just using the fact that k is an eigenvector of $\mathcal{J}$ and a unity eigenvector of $\mathbf{A}$ (the

latter can be enforced trivially by choosing $\mathbf{A} = \mathbf{U}$ initially). Then, the condition that the energy is at a stationary value is reduced to a condition on the eigenvalue λ. If this is true, we can identify the angular rate ω with k/λ causing the equilibrium conditions to be identically satisfied.

3.3. LOCALLY CENTRAL GRAVITY FIELDS

Before we consider the general case of relative equilibria, we first discuss a special case that has been analyzed extensively in the past, albeit in a less general way. If the relative position vector $\mathbf{r}$ and the local gravitational acceleration $\mathcal{U}_{\mathbf{r}}$ are parallel to each other, we consider this to be a "locally central" point in the gravity field. For a spherical mass distribution, we note that all points are locally central, hence these gravity fields are often called central gravity fields. For a symmetric mass distribution such as an ellipsoid, we note that there are only isolated points at which the gravity field is locally central, at the six semi-axes of the body. For the ellipsoid we note that these regions of locally central gravity coincide with the principal axes of inertia, a situation that allows for the easy identification of equilibrium configurations, which indeed most past research has focused on. Similarly, for any gravity field only consisting of even degrees and orders we will find a similar result. For the more general cases, however, we find that locally central points in the gravity field do not coincide with the principal axes of the inertia ellipsoid. This has important consequences for the existence of relative equilibria which we formalize as follows:

PROPOSITION 9. *A locally central point in a gravity field for a SRF2BP is a relative equilibrium if and only if the position vector is perpendicular to a principal axis.*

To show this, let

$$\mathbf{g} = \mathcal{U}_{\mathbf{r}}. \tag{41}$$

Thus, at a locally central point we have $\mathbf{g} = -g\hat{\mathbf{r}}$ (we note that the vectors are anti-parallel in general). Now we can rewrite Equations (27) and (28) and evaluate Proposition 1 to find:

$$(g - \omega^2 r)\hat{\mathbf{r}} = 0, \tag{42}$$

$$\boldsymbol{\omega} \times \mathcal{I} \cdot \boldsymbol{\omega} = 0, \tag{43}$$

$$\boldsymbol{\omega} \cdot \hat{\mathbf{r}} = 0. \tag{44}$$

All of these equations, taken together, place constraints on the angular velocity vector. From Equation (43) we see that $\boldsymbol{\omega}$ must lie along a

principal axis of the inertia ellipsoid. From Equation (44) we see that $\boldsymbol{\omega}$ must also be perpendicular to the position vector. Equation (42) just provides us with the necessary magnitude of the rotation rate once the other equations are satisfied. Then, if the position vector is not perpendicular to a principal axis Equation (43) cannot be satisfied, as $\omega \neq 0$ in general and the gravity (and position vector in this case) must be perpendicular to the spin axis from Proposition 1. To establish the neccessary conditions it is trivial to note that if the position is perpendicular to a principal axis, a relative equilibrium may be found.

We see that for ellipsoids, where the locally central points are aligned with the principal axes, there are four distinct rotation directions that are equilibria at each locally central point, two associated with each of the other principal axes. For a general mass distribution, however, we find that this situation is not true. In fact, for any body with non-zero coefficients in the odd degrees or orders of their harmonic gravity fields this situation will not occur generically.

3.4. REDUCTION OF THE EQUILIBRIUM CONDITIONS FOR NON-LOCALLY CENTRAL POINTS

To determine an equilibrium condition, we must solve six equations, Equations (27) and (28), for six unknowns $\mathbf{r}$ and $\boldsymbol{\omega}$. We note, however, that we expect there to be 4-fold degeneracies in these solutions due to the existence of four integrals of motion, stated in Equations (30) and (31). Thus, we expect to be able to reduce these six conditions to just two independent conditions for an equilibrium. Formally, we could perform these reductions directly from the integrals, but this is somewhat tedious and does not take advantage of certain symmetries that are present in the physical problem we are considering. In the following we make the tacit assumption that $\mathbf{r}$ and $\mathbf{g}$ are not collinear, i.e., are non-locally central, as that case is analyzed above.

The equilibrium conditions we find below provide two equations for the three components of the relative position $\mathbf{r}$. In practice, we can specify one element of the position vector as a free parameter, and solve for the two other components as functions of this. A natural parameter to choose for the free parameter is the radius of the position vector, as we expect solutions to exist at all values of radius. Thus, we will represent these equilibrium conditions as a function of $\hat{\mathbf{r}}$, which can also be represented as the two angles that define the relative position in the body-fixed frame. Then these two equations are solved for the angular location of the relative position vector for a given value of radius. For each value of radius there will be multiple solutions.

In our work we have also found three different forms of the equilibrium conditions that provide necessary, sufficient, and necessary and sufficient conditions for an equilibrium. These are all described in detail in the current analysis. We state all three conditions as the necessary and sufficient condition suffers from some disadvantages as an algorithm that cause us to not use it to completely replace the other two conditions. Specifically, we have found poor convergence in solving for relative equilibria where the system spins in the vicinity of a non-maximum moment of inertia. In this situation, we find that the sufficient condition works well. The necessary condition is given as its derivation provides a useful introduction to a derivation of the sufficient condition.

3.4.1. *A necessary condition for equilibrium*

We first note that it is possible to completely eliminate the angular velocity $\boldsymbol{\omega}$ from the equilibrium conditions, in essence solving for $\boldsymbol{\omega}$ as a function of the relative position.

We first derive a formula for the spin rate as a function of $\mathbf{r}$ and $\hat{\boldsymbol{\omega}}$. Note that Equation (27) can be rewritten as:

$$(\boldsymbol{\omega} \cdot \mathbf{r})\boldsymbol{\omega} - \omega^2 \mathbf{r} = \mathcal{U}_{\mathbf{r}}. \tag{45}$$

Taking the dot product of $\mathbf{r}$ with this equation and solving for ω^2 yields:

$$\omega^2 = \frac{-\mathbf{r} \cdot \mathcal{U}_{\mathbf{r}}}{r^2 \left[1 - \left(\hat{\boldsymbol{\omega}} \cdot \hat{\mathbf{r}} \right)^2 \right]}. \tag{46}$$

Next we note that the direction of the angular velocity, $\hat{\boldsymbol{\omega}}$, can be determined once a position vector $\mathbf{r}$ is specified. To do this we note from Propositions 1 and 6 that $\boldsymbol{\omega}$ is perpendicular to $\mathcal{U}_{\mathbf{r}}$ and lies in the plane defined by $\mathcal{U}_{\mathbf{r}} = \mathbf{g}$ and $\mathbf{r}$. Define the dyadic:

$$\mathbf{U}_{gg} = (\mathbf{g} \cdot \mathbf{g})\mathbf{U} - \mathbf{gg}, \tag{47}$$

which extracts the component of a vector orthogonal to $\mathbf{g}$. Thus, we can state that

$$\mathbf{u} = \mathbf{U}_{gg} \cdot \mathbf{r}, \tag{48}$$
$$\hat{\boldsymbol{\omega}} = \hat{\mathbf{u}}. \tag{49}$$

Note that this is not well defined if we are at a locally central point, as then $\mathbf{U}_{gg} \cdot \mathbf{r} = 0$, and the unit vector $\hat{\mathbf{u}}$ is undefined. Under our assumption of a non-locally central point, however, we are guaranteed that $\mathbf{U}_{gg} \cdot \mathbf{r} \neq 0$. The vector $\mathbf{u}$ plays an important role in the following discussion, so we will give an explicit formula for it and its magnitude:

$$\mathbf{u} = g^2\mathbf{r} - (\mathbf{r}\cdot\mathbf{g})\mathbf{g}, \tag{50}$$

$$u = g\sqrt{g^2r^2 - (\mathbf{r}\cdot\mathbf{g})^2}. \tag{51}$$

We note that this condition is independent of the magnitude of $\boldsymbol{\omega}$, which can be solved for after the fact from Equation (46). Thus, without loss of generality we can substitute $\mathbf{u}$ into Equation (33) to find the function:

$$\mathbf{F}(\mathbf{r}) = \tilde{\mathbf{u}}\cdot[\mathcal{I} - \nu\mathbf{r}\mathbf{r}]\cdot\mathbf{u} \tag{52}$$

and the equilibrium condition:

$$\mathbf{F}(\mathbf{r}) = 0, \tag{53}$$

where we recall that $\mathbf{u}$ is an explicit function of $\mathbf{r}$. In general, we note that this is only a necessary condition for a position $\mathbf{r}$ to be an equilibrium, as $\mathbf{u} = 0$ will also satisfy this equation and may not be an equilibrium.

Although $\mathbf{F}$ is a general 3-dimensional vector, there are only two independent directions defined for it, implying that this represents only two independent conditions. This is expected as it represents the culmination of the elimination of the four integrals of motion from the six initial conditions for equilibrium. A general discussion of these two independent directions is simple to carry out. First, we recall that the vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{g}}$ are orthogonal to each other. Thus, we can use them to define a mutually orthogonal vector $\hat{\mathbf{n}} = \hat{\mathbf{g}}\times\hat{\mathbf{u}} = \tilde{\hat{\mathbf{g}}}\cdot\hat{\mathbf{u}} = \tilde{\hat{\mathbf{g}}}\cdot\tilde{\hat{\mathbf{u}}}$. Then we can establish the following:

PROPOSITION 10. *Necessary conditions for the system to be in relative equilibrium are that*

$$0 = g\mathbf{g}\cdot\tilde{\mathbf{r}}\cdot\mathcal{I}\cdot\mathbf{u}, \tag{54}$$

$$0 = \frac{u}{g}\left[\nu(\mathbf{r}\cdot\mathbf{g})\frac{u^2}{g^2} - \mathbf{g}\cdot\mathcal{I}\cdot\mathbf{u}\right]. \tag{55}$$

To establish this, first consider the projection of $\mathbf{F}$ along $\hat{\mathbf{u}}$. Writing this out yields $\hat{\mathbf{u}}\cdot\mathbf{F} = \hat{\mathbf{u}}\cdot\tilde{\mathbf{u}}\cdot[\mathcal{I} - \nu\mathbf{r}\mathbf{r}]\cdot\mathbf{u} \equiv 0$, as $\hat{\mathbf{u}}\cdot\tilde{\mathbf{u}} \equiv 0$. Thus we see that the condition is trivially satisfied along this direction, and that the vector $\mathbf{F}$ must lie in the plane defined by $\hat{\mathbf{g}}$ and $\hat{\mathbf{n}}$. The condition that $\mathbf{F}$ equal zero along these two directions defines our necessary condition.

Next consider the projection of $\mathbf{F}$ along $\hat{\mathbf{g}}$, or $\hat{\mathbf{g}}\cdot\mathbf{F}$. We first note that $\hat{\mathbf{g}}\cdot\tilde{\mathbf{u}} = g^2\hat{\mathbf{g}}\cdot\tilde{\mathbf{r}}$. Thus, the term multiplied by ν is eliminated from this projection. Expanding the remaining terms we find:

$$\hat{\mathbf{g}}\cdot\mathbf{F} = g\mathbf{g}\cdot\tilde{\mathbf{r}}\cdot\mathcal{I}\cdot\mathbf{u}. \tag{56}$$

Equating this to zero represents one necessary condition for an equilibrium.

Finally, consider the projection of $\mathbf{F}$ along $\hat{\mathbf{n}}$, or $\hat{\mathbf{n}} \cdot \mathbf{F}$. This does not result in as simple of a result,

$$\hat{\mathbf{n}} \cdot \mathbf{F} = \frac{u}{g} \left[\nu(\mathbf{r} \cdot \mathbf{g}) \frac{u^2}{g^2} - \mathbf{g} \cdot \mathcal{I} \cdot \mathbf{u} \right]. \tag{57}$$

Equating Equation (57) to zero results in the second independent equilibrium condition. Again, we see that $u = 0$ will satisfy these conditions without putting any constraints on the orientation of the angular velocity, implying that this is only a necessary condition.

3.4.2. *A sufficient condition for equilibrium*

Now we derive a different form of the equilibrium conditions, decomposing the $\mathbf{g}$ vector into two components. This form of the equations will allow us to define a set of sufficient conditions for an equilibrium to exist.

First, substitute Equation (48) into Equation (53), simplifying and expanding all terms results in the expression:

$$g^4 \tilde{\mathbf{r}} \cdot \mathcal{I} \cdot \mathbf{r} - g^2 (\mathbf{r} \cdot \mathbf{g}) \tilde{\mathbf{g}} \cdot \mathcal{I} \cdot \mathbf{r} + \nu g^2 r^2 (\mathbf{r} \cdot \mathbf{g}) \tilde{\mathbf{g}} \cdot \mathbf{r}$$
$$- g^2 (\mathbf{r} \cdot \mathbf{g}) \tilde{\mathbf{r}} \cdot \mathcal{I} \cdot \mathbf{g} + (\mathbf{r} \cdot \mathbf{g})^2 \tilde{\mathbf{g}} \cdot \mathcal{I} \cdot \mathbf{g} - \nu (\mathbf{r} \cdot \mathbf{g})^3 \tilde{\mathbf{g}} \cdot \mathbf{r} = 0. \tag{58}$$

Next we note that the gravitational acceleration can be split into two components, one parallel to the radius vector $\mathbf{r}$ and one perpendicular to it:

$$\mathbf{g} = g_\parallel \hat{\mathbf{r}} + g_\perp \hat{\mathbf{p}}, \tag{59}$$

where $g^2 = g_\parallel^2 + g_\perp^2$. We note that $\hat{\mathbf{r}} \cdot \hat{\mathbf{p}} = 0$, although $\hat{\mathbf{p}}$ is not uniquely defined when $g_\perp = 0$. Making these substitutions we find:

$$(g_\parallel^2 + g_\perp^2)^2 \tilde{\mathbf{r}} \cdot \mathcal{I} \cdot \mathbf{r} - r g_\parallel (g_\parallel^2 + g_\perp^2) (\tilde{\mathbf{g}}_\parallel + \tilde{\mathbf{g}}_\perp) \cdot \mathcal{I} \cdot \mathbf{r}$$
$$+ \nu r^3 g_\parallel (g_\parallel^2 + g_\perp^2) \tilde{\mathbf{g}}_\perp \cdot \mathbf{r} - r g_\parallel (g_\parallel^2 + g_\perp^2) \tilde{\mathbf{r}} \cdot \mathcal{I} \cdot (\mathbf{g}_\parallel + \mathbf{g}_\perp)$$
$$+ r^2 g_\parallel^2 (\tilde{\mathbf{g}}_\parallel + \tilde{\mathbf{g}}_\perp) \cdot \mathcal{I} \cdot (\mathbf{g}_\parallel + \mathbf{g}_\perp) - \nu r^3 g_\parallel^3 \tilde{\mathbf{g}}_\perp \cdot \mathbf{r} = 0. \tag{60}$$

A few observations for this system can be noted. First, $g_\parallel$, $g_\perp$, and $\hat{\mathbf{p}}$ are all functions of the relative position $\mathbf{r}$. Next, if we group all terms that are not factored by $g_\perp$, we find an equation which can be shown to be identically equal to zero:

$$g_\parallel^4 \tilde{\mathbf{r}} \cdot \mathcal{I} \cdot \mathbf{r} - r g_\parallel^3 \tilde{\mathbf{g}}_\parallel \cdot \mathcal{I} \cdot \mathbf{r} - r g_\parallel^3 \tilde{\mathbf{r}} \cdot \mathcal{I} \cdot \mathbf{g}_\parallel + r^2 g_\parallel^2 \tilde{\mathbf{g}}_\parallel \cdot \mathcal{I} \cdot \mathbf{g}_\parallel = 0. \tag{61}$$

To show that this is identically zero we note that $\hat{\mathbf{p}}$ is defined as being perpendicular to $\hat{\mathbf{r}}$. We see that this is true independent of whether $\hat{\mathbf{r}}$ is

along a principal axis or not. Thus, in Equation (60) we need only retain terms that are factored by $g_\perp$. Expanding this condition and simplifying we find:

$$\tilde{\mathbf{F}}(\mathbf{r}) = g_\perp^4 \tilde{\mathbf{r}} \cdot \mathcal{I} \cdot \mathbf{r} - r g_\parallel g_\perp^2 \tilde{\mathbf{g}}_\perp \cdot \mathcal{I} \cdot \mathbf{r} + v r^3 g_\parallel g_\perp^2 \tilde{\mathbf{g}}_\perp \cdot \mathbf{r}$$
$$- r g_\parallel g_\perp^2 \tilde{\mathbf{r}} \cdot \mathcal{I} \cdot \mathbf{g}_\perp + r^2 g_\parallel^2 \tilde{\mathbf{g}}_\perp \cdot \mathcal{I} \cdot \mathbf{g}_\perp \tag{62}$$

with the equilibrium condition being:

$$\tilde{\mathbf{F}}(\mathbf{r}) = 0. \tag{63}$$

Now we can state the following proposition.

PROPOSITION 11. *The following two conditions are sufficient for a relative equilibrium to exist*

$$g_\perp g_\parallel \hat{\mathbf{n}} \cdot \mathcal{I} \cdot \hat{\mathbf{p}} - g_\perp^2 \hat{\mathbf{n}} \cdot \mathcal{I} \cdot \hat{\mathbf{r}} = 0, \tag{64}$$
$$\left(g_\perp^2 - g_\parallel^2\right) \hat{\mathbf{p}} \cdot \mathcal{I} \cdot \hat{\mathbf{r}} + g_\parallel g_\perp \left(\hat{\mathbf{r}} \cdot \mathcal{I} \cdot \hat{\mathbf{r}} - \hat{\mathbf{p}} \cdot \mathcal{I} \cdot \hat{\mathbf{p}} - v r^2\right) = 0. \tag{65}$$

We have defined two basic orthogonal directions, $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$, and we can again form an additional direction $\hat{\mathbf{n}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$. As before, we expect the vector $\tilde{\mathbf{F}}$ to have a null value along the vector $\mathbf{u}$. To show this now, we project the vector equation $\tilde{\mathbf{F}} = 0$ into $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ to find:

$$\hat{\mathbf{r}} \cdot \tilde{\mathbf{F}} = r^2 g_\perp g_\parallel \left(g_\perp g_\parallel \hat{\mathbf{n}} \cdot \mathcal{I} \cdot \hat{\mathbf{p}} - g_\perp^2 \hat{\mathbf{n}} \cdot \mathcal{I} \cdot \hat{\mathbf{r}}\right), \tag{66}$$
$$\hat{\mathbf{p}} \cdot \tilde{\mathbf{F}} = r^2 g_\perp^2 \left(g_\parallel g_\perp \hat{\mathbf{n}} \cdot \mathcal{I} \cdot \hat{\mathbf{p}} - g_\perp^2 \hat{\mathbf{n}} \cdot \mathcal{I} \cdot \mathbf{r}\right) \tag{67}$$

both of which must equal zero. We note that $g_\parallel \neq 0$ in general and that $g_\perp \neq 0$ by assumption, and thus we see that these two conditions are identical to each other and only define one condition.

Then the final condition is represented by $\hat{\mathbf{n}} \cdot \tilde{\mathbf{F}}$:

$$\hat{\mathbf{n}} \cdot \tilde{\mathbf{F}} = r^2 g_\perp^2 \left[\left(g_\perp^2 - g_\parallel^2\right) \hat{\mathbf{p}} \cdot \mathcal{I} \cdot \hat{\mathbf{r}} + g_\parallel g_\perp \left(\hat{\mathbf{r}} \cdot \mathcal{I} \cdot \hat{\mathbf{r}} - v r^2 - \hat{\mathbf{p}} \cdot \mathcal{I} \cdot \hat{\mathbf{p}}\right)\right]. \tag{68}$$

In both of these independent conditions, we see that they are explicitly factored by the term $g_\perp$. This allows us to define a sufficiency condition for our system to be in equilibrium by removing this factor. Note, while we can remove this factor from both conditions, we find that it is necessary to only remove it from one, we choose to remove it from the second condition. Doing so, we have two independent equations that define a sufficient condition for the system to be in equilibrium:

$$g_\perp g_\parallel \hat{\mathbf{n}} \cdot \mathcal{I} \cdot \hat{\mathbf{p}} - g_\perp^2 \hat{\mathbf{n}} \cdot \mathcal{I} \cdot \hat{\mathbf{r}} = 0, \tag{69}$$
$$\left(g_\perp^2 - g_\parallel^2\right) \hat{\mathbf{p}} \cdot \mathcal{I} \cdot \hat{\mathbf{r}} + g_\parallel g_\perp \left(\hat{\mathbf{r}} \cdot \mathcal{I} \cdot \hat{\mathbf{r}} - \hat{\mathbf{p}} \cdot \mathcal{I} \cdot \hat{\mathbf{p}} - v r^2\right) = 0. \tag{70}$$

We should note that this set of equations is not necessary, in that if we set $g_\perp = 0$ in them they do not provide us the correct general condition for equilibrium in this situation, but only a sub-set of the general condition. This condition is useful in that it allows us to avoid the spurious solutions to the necessary conditions where $\mathbf{u} = 0$, yet which aren't an equilibrium.

3.4.3. *A necessary and sufficient condition for equilibrium*

Finally, using a much different approach, we find a necessary and sufficient condition. At the heart of this approach is Proposition 8. Given a trial value of $\mathbf{r}$ we can form the dyad $\mathcal{J}$ and delineate the three mutually orthogonal eigenvectors and their eigenvalues, denoted as $\mathbf{w}_i$ and λ_i, $i = 1, 2, 3$. Choose one of these eigenvectors to be aligned with the system angular momentum (we can choose this freely as the angular momentum is a free parameter of the system) and assume that the axis of rotation is initially aligned in this direction as well, denote this eigenvector and eigenvalue as $\mathbf{w}$ and λ. Given this, the remaining condition for an equilibrium is:

$$\frac{1}{\lambda^2}\mathbf{r}\cdot\tilde{\mathbf{k}}\cdot\tilde{\mathbf{k}} - \mathcal{U}_{\mathbf{r}} = 0, \tag{71}$$

where $\mathbf{w} \parallel \mathbf{k}$. If we assume a value of $|\mathbf{k}|$ we then have three equations for three unknowns, the elements of the position vector $\mathbf{r}$. Again, we find it convenient to decompose this equation into three transverse directions. We will consider the projection of this equation into the three directions $\mathbf{r}$, $\mathbf{w}$, and $\mathbf{w} \times \mathbf{g}$ where $\mathbf{g} = \mathcal{U}_{\mathbf{r}}$ again.

Projecting Equation (71) along $\mathbf{r}$ we find:

$$\frac{k^2}{\lambda^2}\mathbf{r}\cdot\tilde{\mathbf{w}}\cdot\tilde{\mathbf{w}}\cdot\mathbf{r} - \mathbf{r}\cdot\mathbf{g} = 0. \tag{72}$$

We can solve this equation for the magnitude of the angular momentum:

$$k^2 = -\lambda^2\frac{\mathbf{r}\cdot\mathbf{g}}{r^2\left[1 - (\hat{\mathbf{r}}\cdot\hat{\mathbf{w}})^2\right]}. \tag{73}$$

Instead of specifying the angular momentum magnitude, we can use this equation with a set value of the radius to define the angular momentum magnitude. In this way we can easily eliminate one of the unknowns.

PROPOSITION 12. *The following two conditions are necessary and sufficient for a relative equilibrium to exist.*

$$\mathbf{w}\cdot\mathbf{g} = 0, \tag{74}$$
$$\mathbf{w}\cdot\tilde{\mathbf{r}}\cdot\mathbf{g} = 0. \tag{75}$$

To establish this, project Equation (71) along $\mathbf{w}$. The first term is identically zero, as $\mathbf{w} \parallel k = 0$. The remaining condition is $\mathbf{w} \cdot \mathbf{g} = 0$. Next, project along $\mathbf{w} \times \mathbf{g}$, which is orthogonal to $\mathbf{g}$, to find $\mathbf{w} \cdot \tilde{\mathbf{g}} \cdot \mathbf{r} = \mathbf{w} \cdot \tilde{\mathbf{r}} \cdot \mathbf{g} = 0$. These two conditions constitute the necessary and sufficient conditions for equilibrium.

In practice we keep r constant and only vary $\hat{\mathbf{r}}$ to satisfy the two conditions. This method has the advantage of letting us control the placement of the angular velocity. A drawback of the sufficiency conditions derived previously is that there is no direct control over which possible eigenvector orientation the solution is drawn to. A drawback of this necessary and sufficient condition is that it only seems to converge upon solutions that take the maximum eigenvalue of the locked inertia matrix $\mathcal{J}$ as the angular momentum of choice. In all situations where we have chosen a non-maximum eigenvalue of $\mathcal{J}$ the solution procedure has not converged. We surmise that this is due to the strong saddle structure in the vicinity of the equilibrium in these situations.

4. Stability of the Equilibria

Given the existence of relative equilibria, the next question of interest is the stability of the equilibria. For dynamical systems involving both translational and rotational motion, it is not sufficient to establish spectral stability, but we must also determine energetic stability. This is well known in the field of attitude dynamics, where a rotating body is spectrally stable if it rotates about its maximum or minimum moment of inertia, but is only energetically stable if it rotates about its maximum moment of inertia. We will find similar situations here, when we consider the stability of our relative equilibrium.

We must note that it is still of interest to compute spectral stability, as this can provide important information about the motion of the system in the vicinity of the equilibrium, such as the frequencies of oscillation, the characteristic times of instability, and the orientation of these motions along the eigenvectors of the linear system. This information cannot be found using an energetic stability analysis. In the following we provide conditions for evaluating the stability of our relative equilibria. Theoretical approaches to determining the energetic stability of a system are detailed in (Simo et al., 1991), however we go through an explicit derivation and discussion as we are able to provide the general conditions that must be checked for stability of our relative equilibria.

4.1. SPECTRAL STABILITY

For our system, we are only concerned with the stability of the internal variables, those being the relative position and velocity of the bodies and the angular velocity of the non-spherical body. Thus, the equations of motion we must analyze are given by Equations (13) and (14). Now note that the energy integral is only a function of these internal variables, and thus a system that evolves in the neighborhood of a relative equilibrium must conserve this quantity. In general we note that the angular momentum integral is a function of the internal variables and the attitude of the system. However, as established earlier, the magnitude of the angular momentum is only a function of the internal variables, and thus must be conserved under their variation. The unit vector defining the angular momentum is not conserved under the evolution of the internal variables alone and thus is not considered in the following.

To determine spectral stability, we consider small deviations of the dynamical system from a relative equilibrium and study the characteristic exponents of its linearized dynamics. Thus, in the following, assume that we have found a relative equilibrium solution of the form $\mathbf{r} = \mathbf{r}^*$, $\mathbf{r}' = 0$, $\boldsymbol{\omega} = \boldsymbol{\omega}^*$ that satisfy Equations (27) and (28). Next, consider small deviations from these conditions, $\delta\mathbf{r}$, $\delta\mathbf{r}'$, $\delta\boldsymbol{\omega}$, and linearize the equations of motion to find:

$$
\begin{bmatrix} \delta\mathbf{r}' \\ \delta\mathbf{r}'' \\ \delta\boldsymbol{\omega}' \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{U} & \mathbf{0} \\ \begin{array}{c} -\tilde{\boldsymbol{\omega}}\cdot\tilde{\boldsymbol{\omega}}+\mathcal{U}_{\mathrm{rr}} \\ +\nu\tilde{\mathbf{r}}\cdot\mathcal{I}^{-1}\cdot\left[\tilde{\mathcal{U}}_{\mathbf{r}}-\tilde{\mathbf{r}}\cdot\mathcal{U}_{\mathrm{rr}}\right] \end{array} & -2\tilde{\boldsymbol{\omega}} & \begin{array}{c} \tilde{\boldsymbol{\omega}}\cdot\tilde{\mathbf{r}}+\widetilde{\boldsymbol{\omega}\times\mathbf{r}} \\ +\tilde{\mathbf{r}}\cdot\mathcal{I}^{-1}\cdot\left[\widetilde{\mathcal{I}\cdot\boldsymbol{\omega}}-\tilde{\boldsymbol{\omega}}\cdot\mathcal{I}\right] \end{array} \\ \nu\mathcal{I}^{-1}\cdot\left[\tilde{\mathcal{U}}_{\mathbf{r}}-\tilde{\mathbf{r}}\cdot\mathcal{U}_{\mathrm{rr}}\right] & \mathbf{0} & \mathcal{I}^{-1}\cdot\left[\widetilde{\mathcal{I}\cdot\boldsymbol{\omega}}-\tilde{\boldsymbol{\omega}}\cdot\mathcal{I}\right] \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\mathbf{r}' \\ \delta\boldsymbol{\omega} \end{bmatrix}. \tag{76}
$$

The resulting 9×9 matrix is degenerate in that it allows an additional integral of motion beyond the energy – the angular momentum magnitude. To correctly account for this we must remove this variation from the dynamical system. If we denote the angular momentum vector in the body-fixed frame as $\mathbf{k} = \mathcal{I}\cdot\boldsymbol{\omega} + \nu\mathbf{r}\times\left(\mathbf{r}'+\boldsymbol{\omega}\times\mathbf{r}\right)$, then the constraint on the linearized system about the relative equilibrium is

$$\mathbf{k}\cdot\delta\mathbf{k} = 0. \tag{77}$$

This can be reduced to an explicit condition on the variations in the internal variables

$$\boldsymbol{\omega} \cdot \mathcal{J} \cdot \left[\nu \left(\tilde{\mathbf{r}} \cdot \tilde{\boldsymbol{\omega}} - \widetilde{\boldsymbol{\omega} \times \mathbf{r}} \right) \Big| \nu \tilde{\mathbf{r}} \Big| \mathcal{J} \right] \cdot \begin{bmatrix} \delta \mathbf{r} \\ \delta \mathbf{r}' \\ \delta \boldsymbol{\omega} \end{bmatrix} = 0. \tag{78}$$

We note that at least one of the coefficients of the $\delta\boldsymbol{\omega}$ variable is always non-singular, which allows us to remove a component of the angular velocity vector variation from the dynamics matrix, yielding an 8×8 matrix, the eigenvalues of which determine spectral stability. In practice, once a relative equilibrium is found the 9×9 matrix is computed and the component of the angular velocity that has the largest eigenvalue is removed to reduce the matrix to an 8×8 matrix. Then the eigenvalues of this matrix are computed to determine the spectral stability.

The characteristic exponents of the equilibrium solution can be of three types in general, real, imaginary, and complex. Each pair of real solutions correspond to a pair of 1-dimensional exponentially stable and unstable manifolds, while a set of complex solutions consists of a pair of 2-dimensional exponentially stable and unstable manifolds. The presence of any characteristic exponent with a real part indicates that the system is unstable, as small deviations from the equilibrium will grow at an exponentially increasing rate in time. Each pair of imaginary solutions correspond to one set of 2-dimensional harmonic oscillations about the equilibrium point. For an equilibrium point to be spectrally stable, all 4 complex–conjugate pairs of the characteristic exponents must be purely imaginary. The test for stability is easy, once the characteristic exponents are found. If any of the characteristic exponents has a non-zero real value, the system is hyperbolically unstable, whereas if none of the characteristic exponents has a real component, the system is spectrally stable. We note that spectral stability only determines the linear stability of an equilibrium configuration. To determine the non-linear stability of an equilibrium condition one must in addition consider energetic stability.

4.2. ENERGETIC STABILITY

The computation of energetic stability allows one to identify a sufficient condition for the non-linear stability of an equilibrium point. At heart, energetic stability is conceptually much simpler than spectral stability, as it just provides conditions under which no allowable variation of the system state can result in a lower energy value, and hence defines the lowest energy state of the system. The computation of this condition is not as straight-forward, however, and is generally difficult to perform. A theoretical basis and algorithmic description for evaluating this stability is given in (Simo et al., 1991), however it is couched in terms of geometric mechanics, which

is sometimes difficult to implement for generic gravity fields. In the following we present a classical derivation of the energetic stability condition for our system, only using basic ideas taken from the calculus.

Specifically, to determine whether a relative equilibrium is energetically stable we must compute the second variation of the energy with respect to the internal system variables. This second variation can be described as a quadratic form, and stability is present if that quadratic form is positive definite. If it is not positive definite, then there exist pathways in which the system can evolve to a lower energy state, and hence the system is not stable at the current equilibrium point and energy value. In the current analysis the difficulty lies in performing this second variation while enforcing the condition that the angular momentum must be conserved. The following analysis is simplified, however, by the following insights, mentioned earlier. First, the energy integral is only a function of the internal variables, and does not involve the inertial attitude of the system. Second, we only need the internal system to conserve the angular momentum magnitude, meaning that we only have a single constraint on our energy variation computation.

To properly carry out our second variation, we must first discuss the condition for a relative equilibrium in terms of finding stationary points of the energy at a specified value of angular momentum. To do this we introduce the modified energy functional by appending the angular momentum value constraint to the energy with a Lagrange multiplier:

$$\Lambda = \mathcal{E} + \lambda \left(\mathbf{k} \cdot \mathbf{k} - k^2 \right). \tag{79}$$

Stationary values of Λ then correspond to stationary values of the energy at a specified value of the angular momentum magnitude, k^2. Now let us consider variations in the internal variables $\mathbf{r}$, $\mathbf{r}'$, and $\boldsymbol{\omega}$ as well as in the Lagrange multiplier λ.

The first variation yields:

$$\delta \Lambda = \delta \mathcal{E} + 2\lambda \mathbf{k} \cdot \delta \mathbf{k} + \left(\mathbf{k} \cdot \mathbf{k} - k^2 \right) \delta \lambda. \tag{80}$$

The difficulty is in establishing the proper value of λ to yield a stationary value of Λ. It is important to note that we do not yet enforce the condition that variations in angular momentum must be zero, only that the nominal angular momentum is equal to a certain value, thus we cannot set $\mathbf{k} \cdot \delta \mathbf{k} = 0$ yet. This is a dynamical constraint that the first variation need not verify to find a locally stationary value. We restate the above equation explicitly in terms of the variation of the internal states and note that the coefficient of $\delta \lambda$ is identically zero to find:

$$\delta \Lambda = [\mathcal{E}_\mathbf{x} + 2\lambda \mathbf{k} \cdot \mathbf{k_x}] \cdot \delta \mathbf{x}. \tag{81}$$

We know, however, that at a relative equilibrium we have the following necessary conditions: $\mathbf{r}' = 0$ and that the angular momentum is parallel to the angular velocity of the system. Using these facts we find the following relations for $\mathcal{E}_\mathbf{x}$ and $2\lambda\mathbf{k}\cdot\mathbf{k_x}$. We note that the gradient of the energy is not, by itself, identically zero at a relative equilibrium.

$$\mathcal{E}_\mathbf{x} = \begin{bmatrix} -\nu\tilde{\boldsymbol{\omega}}\cdot\tilde{\boldsymbol{\omega}}\cdot\mathbf{r} - \nu\mathcal{U}_\mathbf{r} \\ \nu\tilde{\boldsymbol{\omega}}\cdot\mathbf{r} \\ \mathcal{J}\cdot\boldsymbol{\omega} \end{bmatrix}, \tag{82}$$

where we have used our previous definition of $\mathcal{J}$. A similar analysis of the term $2\lambda\mathbf{k}\cdot\mathbf{k_x}$, where we rewrite this as $2\lambda\frac{k}{\omega}\boldsymbol{\omega}\cdot\mathbf{k_x}$ using Corollary 4 yields the following.

$$2\lambda\frac{k}{\omega}\boldsymbol{\omega}\cdot\mathbf{k_x} = 2\lambda k/\omega \begin{bmatrix} -2\nu\tilde{\boldsymbol{\omega}}\cdot\tilde{\boldsymbol{\omega}}\cdot\mathbf{r} \\ \nu\tilde{\boldsymbol{\omega}}\cdot\mathbf{r} \\ \mathcal{J}\cdot\boldsymbol{\omega} \end{bmatrix}. \tag{83}$$

The condition for Λ to be stationary independent of variations in the internal variables is now that $\mathcal{E}_\mathbf{x} + 2\lambda\mathbf{k}\cdot\mathbf{k_x} \equiv \mathbf{0}$. This condition applied to the last two blocks of the matrix equations yield the same condition on λ, that

$$\lambda = -\frac{\omega}{2k}. \tag{84}$$

Substituting this into the first block and simplifying yields $\tilde{\boldsymbol{\omega}}\cdot\tilde{\boldsymbol{\omega}}\cdot\mathbf{r} - \mathcal{U}_\mathbf{r} = \mathbf{0}$ which is a necessary condition for a relative equilibrium and hence must be identically satisfied. Thus we find the value of λ.

Now consider the second variation of Λ, which must be done prior to applying our constraints on the variation of the angular momentum.

$$\delta^2\Lambda = \delta^2\mathcal{E} + 4\mathbf{k}\cdot\delta\mathbf{k}\delta\lambda + 2\lambda\left[\delta\mathbf{k}\cdot\delta\mathbf{k} + \mathbf{k}\cdot\delta^2\mathbf{k}\right]. \tag{85}$$

We note again that the second order variation $\delta^2\lambda$ is again multiplied by a term identically zero, and thus we do not show it. Now, at this second order variation, we wish to restrict ourselves to a constant value of angular momentum, meaning that the constraint $\mathbf{k}\cdot\delta\mathbf{k} = 0$ is active. This simplifies our analysis and means, for our case, that we can disregard the first-order variations in λ. Considering these constant angular momentum variations, and substituting in for our proper value of λ, we find the simpler equation we must evaluate:

$$\delta^2\Lambda = \delta^2\mathcal{E} - \frac{\omega}{k}\left[\delta\mathbf{k}\cdot\delta\mathbf{k} + \mathbf{k}\cdot\delta^2\mathbf{k}\right]. \tag{86}$$

Our next step is to re-write this as a quadratic form in the internal variables, or $\delta^2\Lambda = \delta\mathbf{x}\cdot\Lambda_{\mathbf{xx}}\cdot\delta\mathbf{x}$, subject to the constraint $\mathbf{k}\cdot\mathbf{k_x}\cdot\delta\mathbf{x} = 0$. The term $\Lambda_{\mathbf{xx}}$ can be expressed in matrix notation as follows, while the constraint has already been evaluated in Equation 78.

$$\begin{bmatrix} -\nu\left[\mathcal{U}_{\mathbf{rr}}-\tilde{\boldsymbol{\omega}}\cdot\tilde{\boldsymbol{\omega}}\right] & -\nu^2\frac{\omega}{k}\left[\tilde{\boldsymbol{\omega}}\cdot\tilde{\mathbf{r}}+\widetilde{\boldsymbol{\omega}\times\mathbf{r}}\right]\cdot\tilde{\mathbf{r}} & -\nu\frac{\omega}{k}\left[\tilde{\boldsymbol{\omega}}\cdot\tilde{\mathbf{r}}+\widetilde{\boldsymbol{\omega}\times\mathbf{r}}\right]\cdot\mathcal{J} \\ -\nu^2\frac{\omega}{k}\left[\tilde{\boldsymbol{\omega}}\cdot\tilde{\mathbf{r}}+\widetilde{\boldsymbol{\omega}\times\mathbf{r}}\right]\cdot & & \\ \left[\tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\omega}}-\widetilde{\boldsymbol{\omega}\times\mathbf{r}}\right] & & \\ \hline \nu^2\frac{\omega}{k}\tilde{\mathbf{r}}\cdot\left[\tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\omega}}-\widetilde{\boldsymbol{\omega}\times\mathbf{r}}\right] & \nu\left[\mathbf{U}+\nu\frac{\omega}{k}\tilde{\mathbf{r}}\cdot\tilde{\mathbf{r}}\right] & -\nu\tilde{\mathbf{r}}\cdot\left[\mathbf{U}-\frac{\omega}{k}\mathcal{J}\right] \\ \hline -\nu\frac{\omega}{k}\mathcal{J}\cdot\left[\tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\omega}}-\widetilde{\boldsymbol{\omega}\times\mathbf{r}}\right] & \nu\left[\mathbf{U}-\frac{\omega}{k}\mathcal{J}\right]\cdot\tilde{\mathbf{r}} & \mathcal{J}-\frac{\omega}{k}\mathcal{J}\cdot\mathcal{J} \end{bmatrix}. \tag{87}$$

We note that this is a symmetric matrix and that this 9×9 matrix is easily evaluated once a relative equilibrium is found. Again, the dimension of the matrix is reduced to 8×8 by removing an element of the angular velocity variation, $\delta\boldsymbol{\omega}$, which can always be found due again to the non-singularity of $\mathcal{J}$. The eigenvalues of this reduced matrix are computed and, if all are positive, we know that the relative equilibrium is energetically stable as the energy is at a local minimum and the system cannot shed additional energy without also reducing its angular momentum. Such stability conditions have been found to be extremely important for the rotational dynamics of satellites, and are often used to ensure their non-linear stability. Should any of the eigenvalues of the reduced matrix be negative, then the state can vary along the direction of the corresponding eigenvector and further reduce the energy of the system. This also implies that the system will move away from the current equilibrium point.

Indeed, the above condition trivially yields the conditions for the energetic stability of a rigid body rotating in free space (i.e., with no forces present, or $\mathcal{U}\equiv0$). This situation is modeled by setting $\nu\equiv0$, meaning that the non-spherical body has all the mass and there is no gravitational attraction from the sphere. Then, the variations in the position and velocity of the spherical particle are immaterial and the energetic stability condition $\delta^2\Lambda$ reduces to the condition

$$\delta\boldsymbol{\omega}\cdot\left[\mathcal{I}-\frac{\omega}{k}\mathcal{I}\cdot\mathcal{I}\right]\cdot\delta\boldsymbol{\omega}>0 \tag{88}$$

subject to the constraint

$$\boldsymbol{\omega}\cdot\mathcal{I}\cdot\mathcal{I}\cdot\delta\boldsymbol{\omega}=0. \tag{89}$$

For a body rotating in free space the relative equilibrium condition is just that the angular velocity lies along a principal axis of the body. If the inertia dyad is expressed in a principal axis frame $\mathcal{I}=\mathrm{diag}[I_1, I_2, I_3]$ with the

rotation occurring about the third axis, we have $\boldsymbol{\omega} = [0, 0, \omega]$, $\omega/k = 1/I_3$, and the angular momentum constraint reduces to $\delta\omega_3 = 0$ with no constraint on $\delta\omega_1$ and $\delta\omega_2$. The matrix can be trivially reduced to the 2×2 matrix

$$
\begin{bmatrix}
\frac{I_1}{I_3}(I_3 - I_1) & 0 \\
0 & \frac{I_2}{I_3}(I_3 - I_2)
\end{bmatrix}. \tag{90}
$$

The positive definite condition is that the two diagonal terms be positive, or that $I_3 > I_1$ and $I_3 > I_2$. This, of course, is the well-known condition that the body rotate about its maximum moment of inertia. Such a rotating body is spectrally stable when it rotates about either its maximum or minimum axis, however. Thus, with this simple example, we see that our condition recovers this standard result.

5. Study of relative equilibria for Toutatis

Now we consider the solution of the equilibrium conditions for a general gravity field and inertia tensor. For a general gravity field we cannot hope to find analytical solutions to these equations, although it may be possible to develop approximate solutions under the assumption that $r \gg 1$ by expanding the gravity field to the lowest order. We do not pursue such an approach here, but instead describe how we can solve these equations under general conditions when the two bodies are in close proximity to each other. To motivate our general approach we will use a realistic asteroid shape model for Toutatis and its associated constant density gravity field (Werner and Scheeres, 1997). We use this asteroid due to its clearly non-symmetric shape (Scheeres et al., 1998).

We have investigated solutions to both the sufficient and the necessary and sufficient conditions. We have found, for reasons not fully understood, that the sufficient conditions are better suited to solving for relative equilibria which spin about the intermediate and minimum axes of the locked inertia tensor $\mathcal{J}$, while the necessary and sufficient condition is better suited to finding relative equilibria that spin about the maximum inertia axis of $\mathcal{J}$. We hypothesize that the local structure of the equilibrium point controls this aspect of solution. In the following we concentrate on providing a detailed discussion and presentation of our solutions of the necessary and sufficient conditions, as systems rotating about their maximum moment of inertia are of the most physical interest and are the only ones which exhibit energetic stability. We do provide some example solutions using the sufficiency conditions in Table I.

TABLE I

Details of select relative equilibria about Toutatis.

ν	r	Relative position		Spin pole		Spectral Stability # Real Eigenvalues	Energetic Stability # Negative Eigenvalues
—	km	Latitude deg	Longitude deg	Latitude deg	Longitude deg		
0.5	3	−1.61	0.70	88.62	8.64	No 2	No 1
0.5	4	−0.99	0.75	89.09	9.21	Yes 0	Yes 0
0.2	2	−0.99	−96.89	89.03	89.44	No 4	No 2
0.2	3.5	1.29	−94.32	88.74	89.65	Yes 0	No 2
0.2	5	1.10	−93.15	88.92	89.74	No 2	No 1
0.5	3	6.52	93.27	0.41	3.87	No 4	No 2
0.5	3	3.68	−92.22	0.16	−2.65	No 4	No 2
0.5	3	−2.44	0.46	−0.19	90.40	No 4	No 2
0.5	3	−1.21	177.72	0.47	−91.97	No 4	No 2

In the following we first present results for relative equilibria at a range of distances and mass parameters ν for Toutatis. All these results assume rotation about the maximum moment of inertia of the system, and the use of the necessary and sufficient conditions from Proposition 12 to compute the relative equilibrium. We note that these results are qualitatively similar to those found for an ellipsoid (Scheeres, 2004), albeit with clear deviations in the relative equilibrium positions due to the asymmetry of the mass distribution. Figure 2 shows a view down the maximum moment of inertia of Toutatis, plotting the location of the relative equilibria for values of radius ranging from the surface to 3 km in the y direction and 5 km in the x direction. For each distance the values of the mass parameter range from 0 to 1. In this plot, Toutatis is lined up along its principal axes of inertia, while the locations of the relative equilibria are clearly offset from the principal axes. Thus, we directly see the effect of the mass distribution asymmetry. The plotted locations of the relative equilibria are actually extended swaths, as shown in Figure 3, as they are computed for a range of radius and ν values. Specifically, we choose a
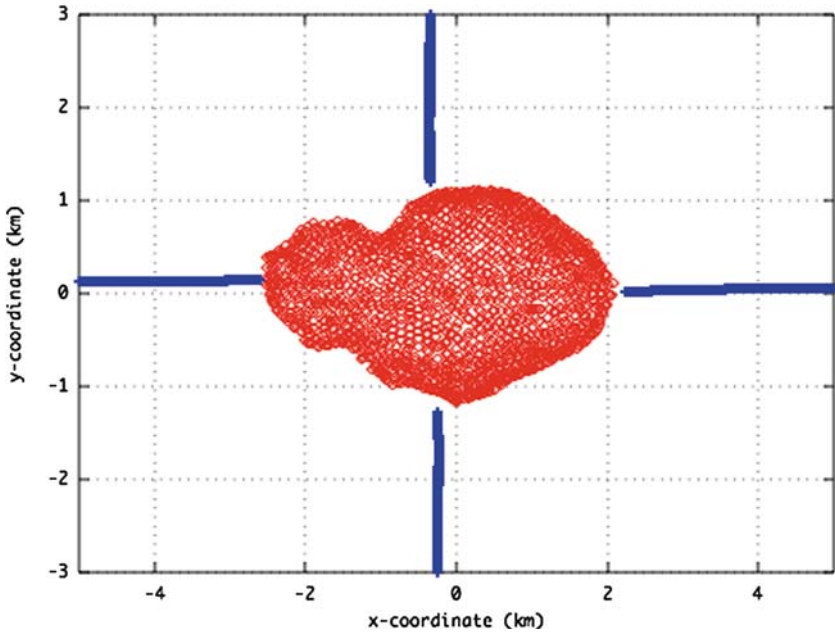
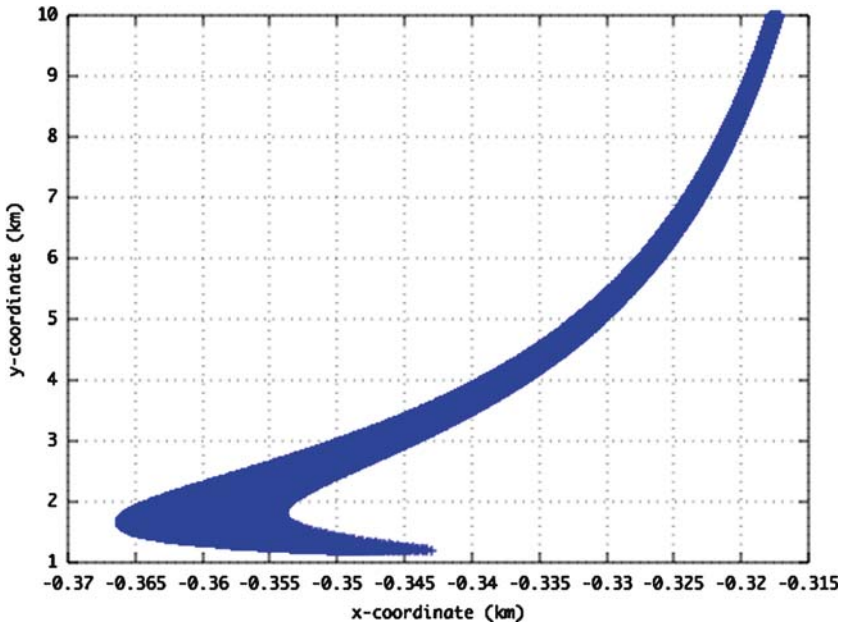*Figure 2.* Locations of relative equilibria about maximum moment of inertia.

Figure 3. Detail of location of relative equilibria along the $+y$ axis.

value of ν and then generate relative equilibria for radius values ranging
from just off the Toutatis surface to some maximum limit, then increment
the value of ν and repeat. At a given value of radius, the relative equilibria
tend to sweep from one side of the region to the other as the mass fraction
is changed. In Figure 3 it is clear that the lateral location of these relative
equilbria shift as a function of distance from the center of mass and as a
function of mass ratio.

In Figures 4 and 5 we show the results of our spectral stability com-
putations for relative equilibria along the $+x$ and $-y$ axes of Toutatis. In
these figures, each point corresponds to a relative equilbrium. In Figure 4
the shaded regions are spectrally stable, and in this case are also all ener-
getically stable, while the clear regions are hyperbolically unstable and are
energetically unstable as well. In Figure 5 the shaded region is spectrally
stable. The region below the stable area has a set of four roots correspond-
ing to hyperbolic spirals, while the region above the stability area has a
set of two roots corresponding to 1-dimensional hyperbolic manifolds. In
both cases the remaining characteristic exponents correspond to oscillatory
motions. We note that all the relative equilibria represented in Figure 5 are
energetically unstable.

We also present a few detailed results, chosen from the many relative
equilibria we have computed. In Table I we present a number of detailed
parameters related to the computed relative equilibria. In the table we also
provide the longitude and latitude of the relative position and of the angu-
lar velocity vector, information on its spectral stability and information on
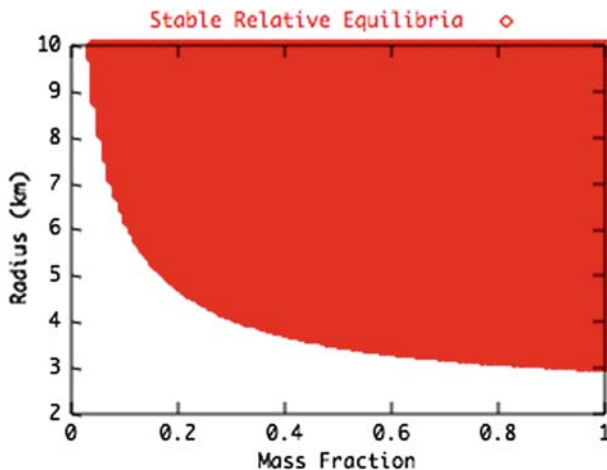its energetic stability. For spectral stability, if not stable we provide the

Figure 4. Spectral stability of relative equilibria along the $+x$ axis.
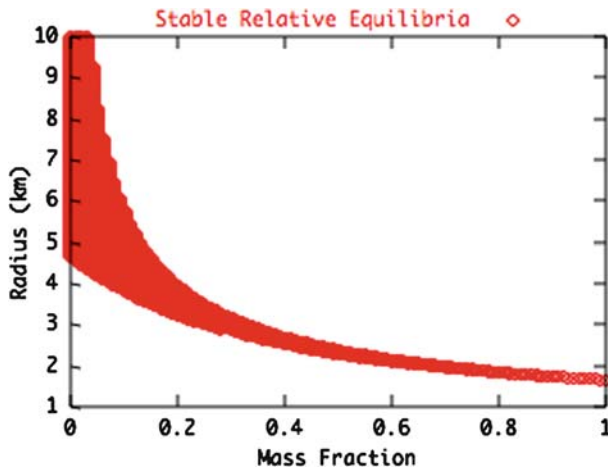
Figure 5. Spectral stability of relative equilibria along the $-y$ axis.

number of eigenvalues with real terms. For energetic stability, if not stable we provide the number of negative eigenvalues. In the first five entries we chose one relative equilibria from each stability and instability region from Figures 4 and 5. In the last four entries we use the sufficiency conditions from Proposition 11 to compute relative equilibria that are not about the maximum moment of inertia of the $\mathcal{J}$ dyad. From the table we, again, clearly see that the system does not rotate about a principal moment of inertia nor does the sphere lie along a principal axis. This situation should actually be the generic situation found for all relative equilbria, even those between a spacecraft and planet, due to the non-symmetric distribution of mass in general. However, for those cases we find that the distance between the spacecraft and planet are so large that these small deviations from principal axis may be difficult to detect.

Finally, in Figures 6 and 7 we show a dynamical simulation of the relative trajectory between Toutatis and a sphere, the system having a mass ratio of 0.5. The simulation is initialized in the close vicinity of the relative equilibrium given in the first line of Table I, located at a radius of 3 km. Two runs are made, one just inside of this radius and the other just outside of this radius. The initial position just inside the radius departs the relative equilibrium on a hyperbolic manifold and impacts the Toutatis surface. The starting position just outside the radius departs on a hyperbolic manifold but is instead trapped within a bounded region defined by the energy and angular momentum integrals, similar to the zero-velocity regions in the restricted 3-body problem. Applying the analysis from (Scheeres, 2002b) we note that this system has a negative total energy and thus is Hill stable,
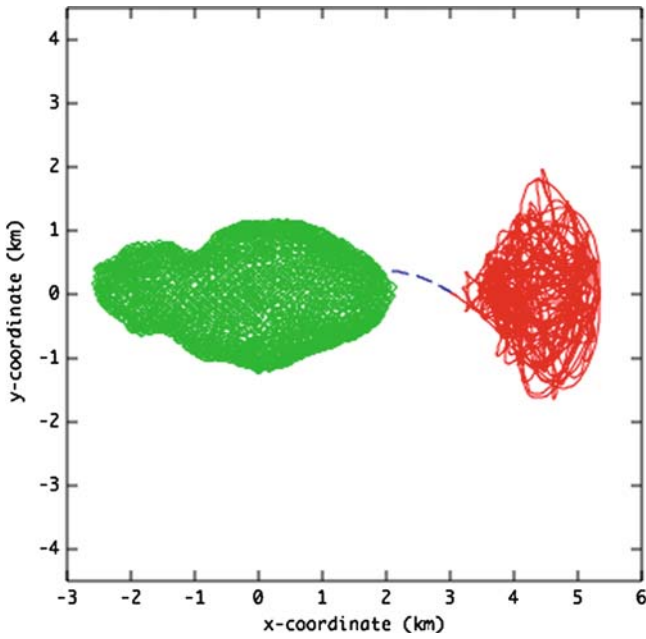
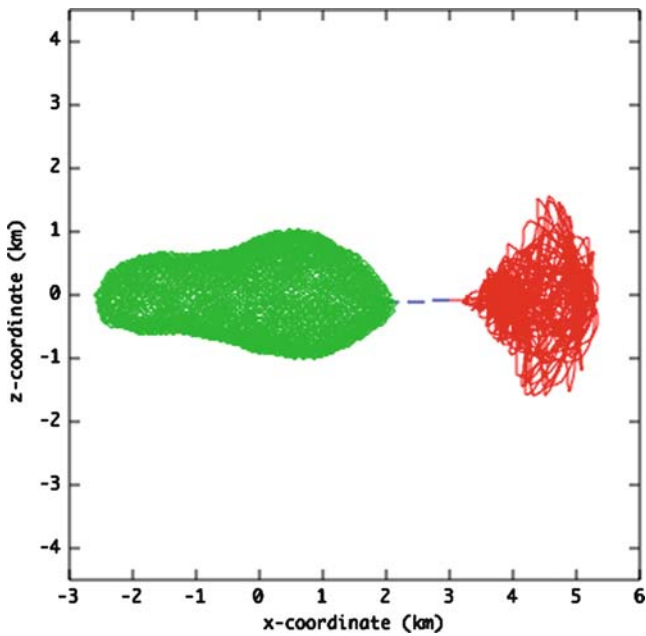*Figure 6.* Unstable, Hill stable trajectory in x–y plane, $\nu = 0.5, r = 3$.

Figure 7. Unstable, Hill stable trajectory in x–z plane, $\nu = 0.5, r = 3$.

meaning that the two bodies cannot escape from each other. Applying the stability against impact criterion from the same reference we also find that for this second case the system is stable against impact, meaning that the system is trapped in this state ad-infinitum. For a real system, we would expect energy dissipation to occur, forcing the bounded region to shrink and eventually forcing the system to fall into a new relative equilibrium (Scheeres, 2002a). By definition, this new relative equilibrium would be energetically (and spectrally), stable, as it would be sought out by energy dissipation. It is interesting to note that the angular momentum magnitude of this new relative equilibria would be equal to that of the original, meaning that the angular momentum magnitude must be double-valued in the radius in this range of initial conditions.

6. Conclusions

We derive a set of conditions for a relative equilibrium that are amenable to the computation of relative equilibrium for systems without any spatial symmetry. We find and discuss three different versions of these conditions, corresponding to necessary, sufficient, and necessary and sufficient conditions for a relative equilibrium. Given a solution for the relative equilibrium of such a system we also derive explicit conditions for spectral and energetic stability. The algorithm is applied to a model of the asteroid Toutatis.

Acknowledgements

Appendix A: Dyad Notation

In this paper we make extensive use of dyad notation, as it fits well with our use of vectors and arbitrary coordinate frames for our final results. In this Appendix we state a number of identities involving dyads that we apply, usually without comment, in the paper. In the following we assume Cartesian vectors and tensors throughout.

We may always specify a vector **a** in an arbitrary coordinate frame, so long as we keep track of its defining unit vectors. Then $\mathbf{a} = \sum_{i=1}^{3} a_i \hat{\mathbf{e}}_i = a_i \hat{\mathbf{e}}_i$ using the Einstein summation convention. The simplest dyad can be constructed by the term-by-term multiplication of two vectors **a** and **b** and is simply stated as

$$\mathbf{ab} = a_i b_j \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j, \tag{A.1}$$

where it is clear that the order of multiplication is important. It is important to note that the unit vectors for **a** and **b** need not be specified in the same coordinate frame. This operation is similar to the outer-product of two column vectors in matrix analysis. Given this basic form, we can generalize the concept of a dyad beyond the product of two vectors and define a general dyad as

$$\mathbf{A} = a_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j. \tag{A.2}$$

A dyad can have similar properties to a matrix, such as symmetry ($a_{ij} = a_{ji}$), skew-symmetry ($a_{ij} = -a_{ji}$), positive definiteness, etc. Similarly, operations that can be performed on matrices can also be performed on dyads, such as transpose, inversion, eigenvalue and eigenvector analysis, etc. Perhaps their most convenient property is that multiplications between vectors, dyads, and higher-order tensors can be made notationally without worrying about the coordinate frames these are occurring in, at least not until the actual computation is to be made.

A particularly useful application of the dyad is to the representation of the cross-product operator in vector mechanics. Given a cross-product, $\mathbf{c} = \mathbf{a} \times \mathbf{b} = \tilde{\mathbf{a}} \cdot \mathbf{b} = \mathbf{a} \cdot \tilde{\mathbf{b}}$. The term $\tilde{\mathbf{a}}$ is the cross-product dyad and is skew-symmetric. In general, given a vector $\mathbf{a} = a_i \hat{\mathbf{e}}_i$ we have:

$$\tilde{\mathbf{a}} = a_1 \left(\hat{\mathbf{e}}_3 \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_3 \right) + a_2 \left(\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_3 - \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_1 \right) + a_3 \left(\hat{\mathbf{e}}_2 \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2 \right). \tag{A.3}$$

A further generalization of this operator as a third-rank tensor also exists, but is not needed for our current analysis. From the basic properties of this operator, we find the following useful identities:

$$\tilde{\mathbf{a}} \cdot \tilde{\mathbf{b}} = \mathbf{ba} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{U}, \tag{A.4}$$

$$\widetilde{\mathbf{a} \times \mathbf{b}} = \mathbf{ba} - \mathbf{ab}, \tag{A.5}$$

$$\mathbf{a} \cdot \tilde{\mathbf{b}} \cdot \tilde{\mathbf{b}} \cdot \mathbf{a} = \mathbf{b} \cdot \tilde{\mathbf{a}} \cdot \tilde{\mathbf{a}} \cdot \mathbf{b}, \tag{A.6}$$

where **U** is the identity dyad, i.e., $\mathbf{a} \cdot \mathbf{U} = \mathbf{U} \cdot \mathbf{a} = \mathbf{a}$.

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