

ROSETTE CENTRAL CONFIGURATIONS, DEGENERATE CENTRAL CONFIGURATIONS AND BIFURCATIONS

J. LEI^{1,2} and M. SANTOPRETE³

¹*Department of Mathematics, University of California, Irvine, U.S.A.*

²*Zhou Pei-Yuan Center for Applied Mathematics, Tsinghua University,
Beijing 100084 China, e-mail: jzlei@mail.tsinghua.edu.cn*

³*Department of Mathematics, University of California, Irvine, Irvine CA 92697 U.S.A.,
e-mail: msantopr@math.uci.edu*

(Received: 17 October 2005; revised: 11 November 2005; accepted: 29 November 2005)

Abstract. In this paper we find a class of new degenerate central configurations and bifurcations in the Newtonian n -body problem. In particular we analyze the Rosette central configurations, namely a coplanar configuration where n particles of mass m_1 lie at the vertices of a regular n -gon, n particles of mass m_2 lie at the vertices of another n -gon concentric with the first, but rotated of an angle π/n , and an additional particle of mass m_0 lies at the center of mass of the system. This system admits two mass parameters $\mu = m_0/m_1$ and $\epsilon = m_2/m_1$. We show that, as μ varies, if $n > 3$, there is a degenerate central configuration and a bifurcation for every $\epsilon > 0$, while if $n = 3$ there is a bifurcation only for some values of ϵ .

Key words: bifurcations, central configurations, degenerate central configurations, n -body problem

1. Introduction

In the planar Newtonian n -body problem the simplest possible motions are such that the whole system of particles rotates as a rigid body about its center of mass. In this case the configuration of the bodies does not change with time. Only some special configurations of point particles are allowed such motions. These configurations are called *central configurations*.

Many questions were raised about the set of central configurations. The main general open problem is the Chazy–Wintner–Smale conjecture: given n positive masses m_1, \dots, m_n interacting by means of the Newtonian potential, the set of equivalence classes of central configurations is finite. Such conjecture was proved for $n = 4$, in the case of equal masses, by Albouy (1995) and (1996) and in the general case by Hampton and Moeckel (2006).

Chazy believed in a stronger statement: namely that any equivalence class of central configuration is nondegenerate. This statement is known to be false: Palmore (1975, 1976) showed the existence of degenerate central configurations in the planar n -body problem with $n \geq 4$. His example consists of $n - 1$ particles lying at the vertices of a regular polygon and one particle at the centroid. Unfortunately only few examples of degenerate central configurations are known. In this paper, we find a new family of degenerate central configurations that arise from some highly symmetrical configurations.

Another interesting problem, that is strictly related to the study of degenerate central configurations, is the study of bifurcations in the n -body problem. The interest in this problem arises because, at a bifurcation, the structure of the phase space changes. Several authors studied bifurcations in the n -body problem (see Sekiguchi, 2004 for a list of references), in particular Sekiguchi analyzed a highly symmetrical configuration of $2n + 1$ -bodies. He considered a Rosette configuration, i.e. a planar configuration where $2n$ particles of mass m lie at the vertices of two concentric regular n -gons, one rotated an angle of π/n from the other and another particle of mass m_0 lies at the center of the two n -gons. He showed that there is a bifurcation in the number of classes of central configurations for any $n \geq 3$.

In this paper we generalize Sekiguchi example and we allow the masses on the two concentric n -gons to be different. This considerably complicates the analysis. Indeed, if one considers two concentric n -gons one with particles of mass m_1 and the other (rotated of an angle π/n from the first) with particles of masses m_2 and a mass m_0 in the center, one has to deal with two mass parameters $\mu = m_0/m_1$ and $\epsilon = m_2/m_1$. In this case we prove that, as μ varies, if $n > 3$, there is a degenerate central configuration and a bifurcation for every $\epsilon > 0$. On the other hand the case $n = 3$ is special and, in this case, as μ is varied, there is a bifurcation for some values of ϵ but not for others.

This paper is organized as follows. In the next section we introduce the equation of the n -body problem. In Section 3 we discuss central configurations. In the following section we introduce the highly symmetrical configurations that are the object of the paper. In Section 5 we present and prove the main results of the paper: the existence, for any $n > 3$, of a bifurcation in the number of classes of central configurations and of a new family of degenerate central configurations. In the last section we analyze the special case where $n = 3$.

2. Equations of Motion

The planar n -body problem concerns the motion of n particles with masses $m_i \in \mathbb{R}^+$ and positions $q_i \in \mathbb{R}^2$, where $i = 1, \dots, n$. The motion is governed by Newton's law of motion

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}. \tag{1}$$

Where $U(q)$ is the Newtonian potential

$$U(q) = \sum_{i < j} \frac{m_i m_j}{|q_i - q_j|}. \tag{2}$$

Let $q = (q_1, \dots, q_n) \in \mathbb{R}^{2n}$ and $M = \text{diag}[m_1, m_2, \dots, m_n]$. Then the equations of motion can be written as

$$\ddot{q} = M^{-1} \frac{\partial U}{\partial q}. \tag{3}$$

In studying this problem it is natural to assume that the center of mass of the system is at the origin, i.e. $m_1 q_1 + \dots + m_n q_n = 0$, and that the configuration avoids the set $\Delta = \{q: q_i = q_j \text{ for some } q_i \neq q_j\}$.

3. Central Configurations

DEFINITION 1. A configuration $q \in \mathbb{R}^2 \setminus \Delta$ is called a central configuration if there is some constant λ such that

$$M^{-1} \frac{\partial U}{\partial q} = \lambda q.$$

Central configurations, as it was shown by Smale (see, Smale, 1970; Abraham and Marsden, 1978), can be viewed as rest points of a certain gradient flow. Introduce a metric in \mathbb{R}^{2n} such that $\langle q, q \rangle = q^T M q$ and let

$$S = \{q: \langle q, q \rangle = 1, m_1 q_1 + \dots + m_n q_n = 0\}$$

denote the unit sphere S^{2n-3} with respect to this metric in the subspace where the center of mass is at the origin. The scalar product $I = \langle q, q \rangle$ is called moment of inertia. Let $S^* = S \setminus \Delta$. The vector field $X = M^{-1} \frac{\partial U}{\partial q} + \lambda q$ where $\lambda = U(q)$ is the gradient of U_S , the restriction of U to the unit sphere S with respect to the metric $\langle \cdot, \cdot \rangle$. This is because X is tangent to S , it has rest points at exactly the central configurations with $\langle q, q \rangle = 1$ and $\langle X(q), v \rangle = DU(q)v$ for every $q \in S$ and $v \in T_q S$. Furthermore the rest

points of X are exactly the central configurations in S . Note that, since the Newtonian potential is an homogeneous function, any central configuration is homothetic to one in S . Therefore the problem of finding central configurations is essentially that of finding rest points of the gradient flow of U_S or, equivalently, finding the critical points of U_S .

The gradient flow preserves some sets of configurations with symmetry. In this paper we study one of such sets of configurations with symmetry.

We denote by C_n the set of central configuration of the n -body problem. We say that two relative equilibria in S^* are equivalent (and belong to the same equivalence class) if one is obtained from the other by a rotation and an homothety. The set \tilde{C}_n is the set of equivalence classes of central configurations.

Clearly I and Δ are invariant under the action of S^1 . Thus, we can conclude that S^* is diffeomorphic to the $(2n-3)$ -dimensional sphere S^{2n-3} (it is actually an ellipsoid E^{2n-3}) with all the points Δ removed, that is

$$S^* = E^{2n-3} \setminus (E^{2n-3} \cap \Delta) \approx S^{2n-3} \setminus (S^{2n-3} \cap \Delta).$$

Since U_S is invariant under the action of S^1 it defines a map $\tilde{U}_S: S^*/S^1 \rightarrow \mathbb{R}$. If we let $\pi: S^* \rightarrow S^*/S^1$ denote the canonical projection, $\tilde{\Delta} = \pi(E^{2n-3} \cap \Delta)$, and recalling that $E^{2n-3}/S^1 \approx S^{2n-3}/S^1 \approx \mathbb{C}P^{n-2}$, complex projective space, we are led to the investigation of the critical points of $\tilde{U}_S: \mathbb{C}P^{n-2} \setminus \tilde{\Delta} \rightarrow \mathbb{R}$.

Consequently one can show that the set of equivalence classes of central configurations is given by the set of critical points of the map $\tilde{U}_S: \mathbb{C}P^{n-2} \setminus \tilde{\Delta} \rightarrow \mathbb{R}$. More precisely we have the following result of Smale (see, Smale, 1970, 1971; Abraham and Marsden, 1978).

PROPOSITION 1. *For any $n \geq 2$ and any choices of the masses in the planar n -body the set of equivalence classes of central configurations is diffeomorphic to the set of critical points of the map $\tilde{U}_S: \mathbb{C}P^{n-2} \setminus \tilde{\Delta} \rightarrow \mathbb{R}$.*

Let q be a critical point of \tilde{U}_S . A critical point of \tilde{U}_S is degenerate provided that the hessian $D^2\tilde{U}_S(q)$ has a nontrivial nullspace. We have the following definition

DEFINITION 2. An equivalence class of central configurations is degenerate (nondegenerate) provided that the corresponding critical point q of \tilde{U}_S is degenerate (nondegenerate).

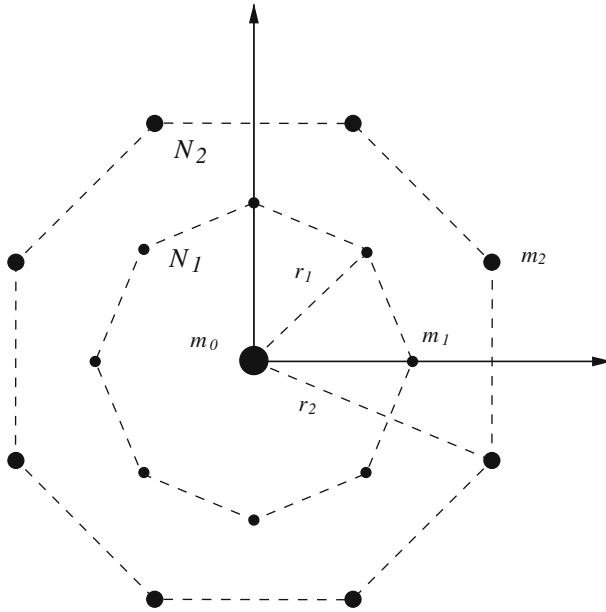


Figure 1. Rosette configuration for $n=6$.

4. Symmetrical Configurations

Consider the set Σ of all the configuration in \mathbb{R}^2 consisting of two concentric regular n -gons, one rotated of an angle π/n from the other, with a mass in their common center of symmetry (see Figure 1).

Let m_0 , m_1 and m_2 be the masses in the center of mass, on the n -gon N_1 and on the n -gon N_2 respectively. Then it follows from the symmetry of the configuration that the gradient of \tilde{U}_S is tangent to $\tilde{\Sigma}$ (where $\tilde{\Sigma} = \pi(E^{2n-3} \cap \Sigma)$). Thus to find equivalence classes of central configurations in $\tilde{\Sigma}$ it is sufficient to study the critical points of $\tilde{U}_S|_{\tilde{\Sigma}}$. Since $\tilde{\Sigma}$ is one dimensional, only one parameter is needed to describe such symmetric configuration. This is a great simplification. Figure 1 shows two parameters (r_1, r_2) which can be used to describe such a configuration.

The potential in these coordinates is

$$U(q) = (nm_1)^2 \bar{U}(r_1, r_2)$$

where

$$\bar{U}(r_1, r_2) = \frac{\mu}{n} \left(\frac{1}{r_1} + \frac{\epsilon}{r_2} \right) + k_n \left(\frac{1}{r_1} + \frac{\epsilon^2}{r_2} \right) + \frac{1}{n} \sum_{k=1}^n \frac{\epsilon}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos \phi_k}}$$

with, $\mu = m_0/m_1$, $\epsilon = m_2/m_1$, $\phi_k = (2k - 1)\pi/n$ and

$$k_n = \frac{1}{4n} \sum_{k=1}^{n-1} \csc \frac{\pi}{n} k$$

is the potential of a regular n -gon of unit size and unit masses. The last term in \bar{U} is the moment of inertia

$$I(q) = \langle q, q \rangle = \bar{I}(r_1, r_2) = m_1 n (r_1^2 + \epsilon r_2^2).$$

The central configurations are the solutions of the equation $\nabla \tilde{U}_S|_{\tilde{\Sigma}} = 0$ or $\nabla U = \frac{\lambda}{2} \nabla I$ (with $\lambda = U$), that in this case can be written as

$$\begin{aligned} \frac{\partial \bar{U}}{\partial r_1} &= m_1 n \lambda r_1 \\ \frac{\partial \bar{U}}{\partial r_2} &= m_1 n \lambda \epsilon r_2. \end{aligned} \tag{4}$$

Solving the equations above for λ one gets

$$\frac{1}{r_2^3} F(x) = 0 \tag{5}$$

where

$$F(x, \epsilon, \mu) = \frac{\mu}{n} (1 - x^3) + k_n (\epsilon - x^3) + \frac{x^3}{n} \sum_{k=1}^n \frac{(1 - \epsilon) - \frac{1}{x} (1 - \epsilon x^2) \cos \phi_k}{(1 + x^2 - 2x \cos \phi_k)^{3/2}} \tag{6}$$

and $x = r_2/r_1$. The equation for the central configuration above depends only on one parameter and is invariant under the transformation $(x, \epsilon, \mu) \rightarrow (\frac{1}{x}, \frac{1}{\epsilon}, \mu\epsilon)$. Thus it suffices to study the central configurations with $0 < \epsilon \leq 1$.

The case $\epsilon = 1$ was studied in detail by Sekiguchi (2004) that proved the following

THEOREM 1. *If $n = 2$ the number of central configurations is one for any value of μ . If $n \geq 3$ the number of central configurations is three for $\mu < \mu_c(n)$ and one for $\mu \geq \mu_c(n)$, where*

$$\mu_c(n) = \frac{1}{12} \sum_{k=1}^n \frac{\cos \phi_k}{\sin^3(\phi_k/2)} - n k_n$$

It is therefore sufficient to analyze the problem with $0 < \epsilon < 1$.

PROPOSITION 2. *For every $\mu > 0$ and $\epsilon > 0$ there is at least one Rosette central configuration.*

Proof. Since $\lim_{x \rightarrow 0} F(x) = (\frac{\mu}{n} + k_n) > 0$ and $\lim_{x \rightarrow \infty} F(x) = -\infty < 0$, by the intermediate value theorem, the equation $F(x) = 0$ has at least one solution. □

When $n = 2$ it can be shown that, for every value of ϵ , there is only one class of central configurations and no bifurcation occur, or more precisely we have the following

PROPOSITION 3. *If $n = 2$ for every $\mu > 0$ and $\epsilon > 0$ there is only one Rosette central configurations.*

Proof. In this case

$$F(x, \epsilon, \mu) = \frac{\mu}{2}(1 - x^3) + \frac{1}{8}(\epsilon - x^3) + \frac{x^3(1 - \epsilon)}{(1 + x^2)^{3/2}} \tag{7}$$

$\lim_{x \rightarrow 0} F(x) = \frac{\mu}{2} + \frac{\epsilon}{8}$ and $\lim_{x \rightarrow \infty} F(x) = -\infty$ so $F(x) = 0$ has at least one solution. We need only to prove the statement for $0 < \epsilon \leq 1$. If $\epsilon = 1$ $F(x)$ is a monotonically decreasing and the statement follows. If $0 < \epsilon < 1$ consider

$$F'(x) = 3x^2 \left(- \left(\frac{\mu}{2} + \frac{1}{8} \right) + \frac{(1 - \epsilon)}{(1 + x^2)^{5/2}} \right). \tag{8}$$

Clearly one solution of $F'(x) = 0$ is $x = 0$. The other solutions can be found studying the equation $\eta(x) = \frac{\mu}{2} + \frac{1}{8}$, where

$$\eta(x) = \frac{(1 - \epsilon)}{(1 + x^2)^{5/2}}$$

is a monotonically decreasing function and $\eta(0) = (1 - \epsilon)$. The equation $\eta(x) = \frac{\mu}{2} + \frac{1}{8}$ has no solutions if $\mu \geq \frac{7}{4}$ or $\mu < \frac{7}{4}$ and $\epsilon \in (\frac{7}{8} - \frac{\mu}{2}, 1)$. It has one solution x^* if $\mu < \frac{7}{4}$ and $\epsilon \in (0, \frac{7}{8} - \frac{\mu}{2}]$. Consequently if $\mu \geq \frac{7}{4}$ or $\mu < \frac{7}{4}$ and $\epsilon \in (\frac{7}{8} - \frac{\mu}{2}, 1)$ $F'(x)$ is always negative, $F(x)$ monotonically decreasing and $F(x) = 0$ has only one solution. On the other hand, if $\mu < \frac{7}{4}$ and $\epsilon \in (0, \frac{7}{8} - \frac{\mu}{2}]$, $F'(x)$ is positive for $x \in (0, x^*)$ and negative for $x \in (x^*, \infty)$. Thus $F(x)$ is increasing for $x \in (0, x^*)$, decreasing for $x \in (x^*, \infty)$ and $F(x) = 0$ has one solution since $F(0) > 0$. □

5. Bifurcations and Degenerate Central Configurations for $n > 3$

In this section we consider the Rosette central configurations for $n > 3$. The main result is the existence of a bifurcations for every value of ϵ as the

parameter μ increases. The case $n = 3$ is studied in the next section. More precisely we prove the following

THEOREM 2. *For any $n > 3$ and $\epsilon > 0$ there is at least one value μ_0 corresponding to a bifurcation in the number of equivalence classes of Rosette central configurations as the parameter $\mu > 0$ increases.*

An important consequence of the existence of a bifurcation is the existence of a degenerate equivalence class of Rosette central configurations

COROLLARY 1. *For any $n > 3$ and $\epsilon > 0$ there is at least one value μ_0 of μ for which there is a degenerate equivalence class of Rosette central configuration.*

Proof. The proof is by contradiction. Consider the potential $\tilde{U}_S(q; \mu)$ for the configuration under discussion in this paper, where we put into evidence the dependence on the mass μ . Let q_1^0, \dots, q_l^0 be the critical points of \tilde{U}_S for $\mu = \mu_0$, where μ_0 is the bifurcation value. Assume that the class of central configurations is nondegenerate for every q_l^0 . This means that $D^2\tilde{U}_S(q_l^0; \mu)$ has bounded inverse. But then by the implicit function theorem, there exist a neighborhood B of μ_0 and unique functions $\{q_l(\mu)\}_{l=1}^n$ defined in B , such that $q_l(\mu_0) = q_l^0$ and $D\tilde{U}_S(q_l(\mu); \mu) = 0$. This contradicts the assumption that μ_0 is a bifurcation value. \square

The proof of Theorem 2 requires several preparations. The reminder of this section is devoted to such preparations and to the proof of Theorem 2.

First of all observe that the central configurations, when $\epsilon \neq 1$ can also be viewed as the solutions of $h(x, \epsilon) = \mu$ where

$$h(x, \epsilon) = -n k_n \frac{\epsilon - x^3}{1 - x^3} - \frac{x^2}{(1 - x^3)} \sum_{k=1}^n \frac{x(1 - \epsilon) - (1 - \epsilon x^2) \cos \phi_k}{(1 + x^2 - 2x \cos \phi_k)^{3/2}}$$

Hereinafter, we say x to be a Rosette central configuration if x is solution of the equation $\mu = h(x, \epsilon)$. Let $u_k = \cos \phi_k$ then

$$h(x, \epsilon) = h_0(x) + (1 - \epsilon)h_1(x)$$

where

$$h_0(x) = -n k_n + \frac{x^2(1 - x^2)}{(1 - x^3)} \sum_{k=1}^n \frac{u_k}{(1 + x^2 - 2x u_k)^{3/2}}$$

and

$$h_1(x) = \frac{nk_n}{(1-x^3)} - \frac{x^3}{1-x^3} \sum_{k=1}^n \frac{(1-xu_k)}{(1+x^2-2xu_k)^{3/2}}.$$

5.1. THE CASE $x > 1, \epsilon \in (0, 1)$

We now study the number of central configurations for $x > 1$ and $\epsilon \in (0, 1)$. It is easy to show that, for any $\mu > 0$, there is at least one Rosette central configuration with $x > 1$. This follows from the limits

$$\lim_{x \rightarrow 1^+} h(x, \epsilon) = \infty, \quad \lim_{x \rightarrow \infty} h(x, \epsilon) = -n k_n < 0$$

and an application of the Intermediate Value Theorem. The first limit is

$$\lim_{x \rightarrow 1^+} h(x, \epsilon) = \infty \times \text{sgn}((\epsilon - 1)A_n)$$

where

$$A_n = nk_n - \frac{1}{4} \sum_{k=1}^n \csc\left(\frac{\phi_k}{2}\right) \tag{9}$$

and $\text{sgn}((\epsilon - 1)A_n) = 1$ since $\epsilon - 1 < 0$ and $A_n < 0$ by the following Lemma

LEMMA 1. For all $n \geq 2$,

$$A_n < 0.$$

Proof. Clearly

$$A_n = \frac{1}{4} \left(\sum_{k=1}^{n-1} \csc \frac{k \pi}{n} - \sum_{k=1}^n \csc \left(\frac{k \pi}{n} - \frac{\pi}{2n} \right) \right) \tag{10}$$

therefore when n is even one has

$$\begin{aligned} & \sum_{k=1}^{n-1} \csc \frac{k \pi}{n} - \sum_{k=1}^n \csc \left(\frac{k \pi}{n} - \frac{\pi}{2n} \right) \\ &= \sum_{k=1}^{n/2} \left(\csc \frac{k \pi}{n} - \csc \left(\frac{k \pi}{n} - \frac{\pi}{2n} \right) \right) - \csc \left(\frac{\pi}{2} + \frac{\pi}{2n} \right) \\ & \quad + \sum_{k=\frac{n}{2}+1}^{n-1} \left(\csc \frac{k \pi}{n} - \csc \left(\frac{k \pi}{n} + \frac{\pi}{2n} \right) \right) < 0 \end{aligned} \tag{11}$$

while when n is odd

$$\begin{aligned}
 & \sum_{k=1}^{n-1} \csc \frac{k \pi}{n} - \sum_{k=1}^n \csc \left(\frac{k \pi}{n} - \frac{\pi}{2n} \right) \\
 &= \sum_{k=1}^{(n-1)/2} \left(\csc \frac{k \pi}{n} - \csc \left(\frac{k \pi}{n} - \frac{\pi}{2n} \right) \right) - \csc \frac{\pi}{2} \\
 & \quad + \sum_{k=(n+1)/2}^{n-1} \left(\csc \frac{k \pi}{n} - \csc \left(\frac{k \pi}{n} + \frac{\pi}{2n} \right) \right) < 0.
 \end{aligned} \tag{12}$$

□

We now want to show that when μ is large enough, for every $\epsilon \in (0, 1)$, there is exactly one Rosette central configuration with $x > 1$, i.e., we prove the following

PROPOSITION 4. *For every $\epsilon \in (0, 1)$ there exists a $\hat{\mu}$ such that for every $\mu > \hat{\mu}$ there is one and only one Rosette central configuration*

Proof. Observe that one can write

$$h(x, \epsilon) = -\frac{(1 - \epsilon)A_n}{3(x - 1)} + O((x - 1)^0). \tag{13}$$

Therefore there exist $\hat{\mu}_0 > 0$ and $\delta > 0$ such that for any $\mu > \hat{\mu}_0$ the equation $h(x, \epsilon) = \mu$ has a unique solution in $(1, 1 + \delta)$. Moreover the function $h(x, \epsilon)$ has a maximum value $\hat{\mu}_1$ in $[1 + \delta, \infty)$, since $\lim_{x \rightarrow \infty} h(x, \epsilon) = -nk_n$. Let

$$\hat{\mu} = \max(\hat{\mu}_0, \hat{\mu}_1)$$

then, if $\mu > \hat{\mu}$, the equation $h(x, \epsilon) = \mu$ has a unique solution. □

5.2. THE CASE $0 < x < 1, \epsilon \in (0, 1)$

We now study the number of central configurations for $x < 1$ and $\epsilon \in (0, 1)$. In particular we show that

PROPOSITION 5. *For any $n > 3$ and $\epsilon \in (0, 1)$,*

1. *there is a $\mu_n^* > 0$ such that for every $0 < \mu < \mu_n^*$ there are at least two Rosette central configurations, with $x \in (0, 1)$*
2. *there is a $\check{\mu} \geq \mu_n^*$ such that for every $\mu > \check{\mu}$ there are no Rosette central configurations with $x \in (0, 1)$.*

Proof. (a) Observe that, if $\epsilon \in (0, 1)$,

$$h(0, \epsilon) = -n k_n \epsilon < 0, \quad \lim_{x \rightarrow 1^-} h(x, \epsilon) = -\infty \times \operatorname{sgn}((\epsilon - 1)A_n) = -\infty,$$

where the limit follows from Lemma 1. If there exists $x_n^* \in (0, 1)$ such that $h_1(x_n^*) = 0$ and $h_0(x_n^*) > 0$, then by the Intermediate Value Theorem, $\mu = h(x, \epsilon)$ has at least two solutions for every $0 < \mu < \mu_n^* = h_0(x_n^*)$. To complete the proof it is necessary to show the existence of x_n^* . The existence of x_n^* will be proved in Lemma 3.

(b) Since $h(0, \epsilon) < 0$ and $\lim_{x \rightarrow 1^-} h(x, \epsilon) = -\infty < 0$ the function $h(x, \epsilon)$ has a maximum value in $[0, 1]$. Let $\check{\mu}$ be such maximum. Then the equation $\mu = h(x, \epsilon)$ has no solutions for $x \in (0, 1)$ if $\mu > \check{\mu}$. \square

To complete the proof of the proposition above, and prove Lemma 3 we need the following technical result.

LEMMA 2. *Let*

$$k_n^- = \frac{1}{4\pi} \ln \left(\frac{1 + \cos \frac{\pi}{n}}{1 - \cos \frac{\pi}{n}} \right) + \frac{1}{4n \sin \frac{\pi}{n}} \tag{14}$$

then for all $n \geq 3$

$$k_n > k_n^-.$$

Moreover k_n^- is monotonically increasing with n .

Proof. The sum:

$$\sum_{k=1}^{n-1} \operatorname{csc} \frac{\pi k}{n}$$

can be estimated using the trapezoidal rule. Since $g(u) = \operatorname{csc} \frac{\pi u}{n}$ is convex on $[1, n - 1]$ the trapezoidal rule gives an upper bound for the integral over $[1, n - 1]$:

$$\int_1^{n-1} g(x) \, dx < \frac{1}{2}g(1) + g(2) + \dots + g(n - 2) + \frac{1}{2}g(n - 1).$$

This gives the formula for k_n^- . Moreover k_n^- is monotonically increasing since the derivative of the function, obtained replacing $\frac{\pi}{n}$ in k_n^- with the continuous variable u , is negative. \square

We can finally prove the following

LEMMA 3. *For any $n > 3$, there exists a $x_n^* \in (\frac{54}{100}, 1)$ such that $h_1(x_n^*) = 0$ and $h_0(x_n^*) > 0$.*

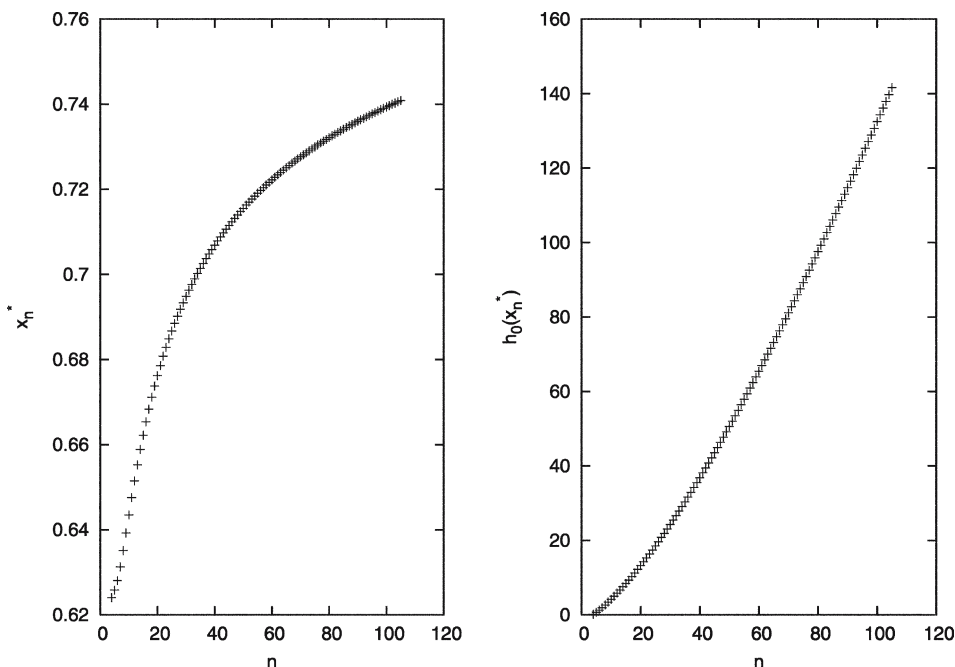


Figure 2. The root x_n^* such that $h_1(x_n^*)=0$ and corresponding $h_0(x_n^*)$ for $4 \leq n \leq 106$.

Proof. Verifying these conclusions numerically for small n is trivial by common mathematical software (for example, Mathematica¹ or Matlab²). Numerical results for $4 \leq n \leq 106$ are given in Figure 2 (The solutions x_n^* are found numerically through the function FindRoot provided by Mathematica). The proof for $n \geq 107$ is given below.

The proof will be completed by showing firstly that $h_1(x)=0$ has solution $x_n^* \in (\frac{54}{100}, 1)$ and secondly that $h_0(x) > 0$ for any $x \in (\frac{54}{100}, 1)$ such that $h_1(x)=0$.

1. We first show that for any $n \geq 107$, the equation $h_1(x)=0$ has at least one solution $x_n^* \in (\frac{54}{100}, 1)$. To this end, it is sufficient to show that $h_1(\frac{54}{100}) > 0$ and $h_1(1) < 0$. Equivalently, let

$$\tilde{h}_1(x) = (1-x^3)h_1(x) = nk_n - x^3 \sum_{k=1}^n \frac{1-xu_k}{(1+x^2-2xu_k)^{3/2}}$$

we will show that $\tilde{h}_1(\frac{54}{100}) > 0$ and $\tilde{h}_1(1) < 0$.

¹<http://www.wolfram.com/>

²<http://www.mathworks.com/>

When $u \in [-1, 1]$ and $x \in [0, 1]$, we have

$$\frac{1 - xu}{(1 + x^2 - 2xu)^{3/2}} < \frac{1}{(1 - x)^2}.$$

and therewith

$$\tilde{h}_1(x) > n \left(k_n - \frac{x^3}{(1 - x)^2} \right)$$

Thus, when $n \geq 107$

$$\tilde{h}_1 \left(\frac{54}{100} \right) > n \left(k_n - \frac{\left(\frac{54}{100}\right)^3}{\left(1 - \frac{54}{100}\right)^2} \right) > n \left(k_n^- - \frac{75}{100} \right) \geq n \left(k_{107}^- - \frac{75}{100} \right) > 0$$

where $k_{107}^- = 0.7514096544$ was computed using Lemma 5.2 and $k_n^- > k_{107}^-$ since k_n^- is monotonically increasing with n .

A simple computation shows that

$$\tilde{h}_1(1) = A_n = \frac{1}{4} \left(\sum_{k=1}^{n-1} \csc \frac{k\pi}{n} - \sum_{k=1}^n \csc \left(\frac{k\pi}{n} - \frac{\pi}{2n} \right) \right), \tag{15}$$

and thus, by Lemma 1, $\tilde{h}_1(1) < 0$ for any n . Hence, we conclude that for any $n \geq 3$, there exist $x_n^* \in \left(\frac{54}{100}, 1\right)$, such that $h_1(x_n^*) = 0$.

2. We now show that for any $n \geq 107$ and $x_n^* \in \left(\frac{54}{100}, 1\right)$ such that $h_1(x_n^*) = 0, h_0(x_n^*) > 0$.

Let

$$h_2(x) = \frac{1 - x^3}{x^2(1 - x^5)} (h_0(x) + (1 - x^3) h_1(x))$$

then

$$h_0(x_n^*) = \frac{x_n^{*3}}{R_1(x_n^*)} h_2(x_n^*)$$

where

$$R_1(x) = \frac{x(1 + x + x^2)}{1 + x + x^2 + x^3 + x^4}$$

Thus, it is sufficient to prove that $h_2(x) > 0$ for any $x \in \left(\frac{54}{100}, 1\right)$. To this end, introduce the notations

$$R_2(x) = 0.15 R_1(x) + 0.85$$

$$g(x, u) = \frac{u - R_1(x)}{(1 + x^2 - 2xu)^{3/2}}$$

then

$$h_2(x) = \sum_{k=1}^n g(x, u_k).$$

It is easy to have

$$0 < R_1(x) < R_2(x) < 1, \quad \forall x \in (0, 1).$$

Thus, grouping the subscripts k in the summation as follows

$$J_1 = \{k \mid 1 \leq k \leq n, u_k < R_1(x)\}$$

$$J_2 = \{k \mid 1 \leq k \leq n, u_k \geq R_2(x)\}$$

we have

$$h_2(x) \geq \sum_{k \in J_1} g(x, u_k) + \sum_{k \in J_2} g(x, u_k).$$

Now, the function $g(x, u)$ of $u \in [0, 1]$ (with given $x \in (\frac{54}{100}, 1)$) has minimum at

$$u = u_-(x) = \frac{3x R_1(x) - 1 - x^2}{x}$$

and is increasing when $R_2(x) < u < 1$. Thus, when $k \in J_1$, we have

$$0 > g(x, u_k) > g(x, u_-(x))$$

and when $k \in J_2$,

$$g(x, u_k) \geq g(x, R_2(x)) > 0.$$

The number of elements in J_1 and J_2 are respectively

$$N(J_1) = \left\lfloor n \left(1 - \frac{\arccos R_1(x)}{\pi} \right) \right\rfloor < \frac{n}{\pi} (\pi - \arccos R_1(x))$$

$$N(J_2) = 2 \left\lfloor \frac{n}{2\pi} \arccos R_2(x) + \frac{1}{2} \right\rfloor \geq \frac{n}{\pi} \arccos R_2(x) - 1.$$

Therefore, we have

$$\begin{aligned} h_2(x) &> \sum_{k \in J_1} g(x, u_-(x)) + \sum_{k \in J_2} g(x, R_2(x)) \\ &= N(J_1) g(x, u_-(x)) + N(J_2) g(x, R_2(x)) \\ &> \frac{n}{\pi} ((\pi - \arccos R_1(x)) g(x, u_-(x)) \\ &\quad + \arccos R_2(x) g(x, R_2(x))) - g(x, R_2(x)) \\ &:= h_3(x; n) \end{aligned}$$

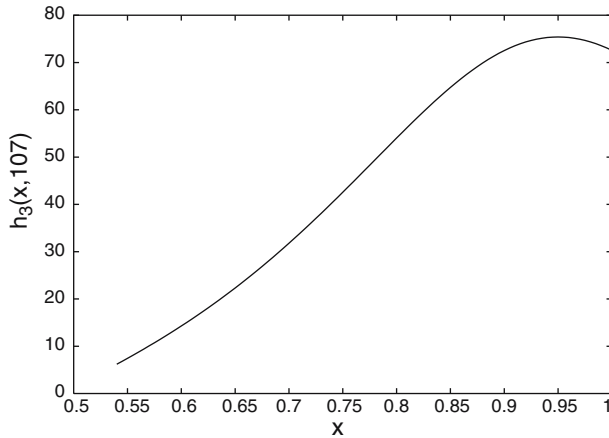


Figure 3. The function $h_3(x; n)$ with $n = 107$ and $x \in (0.54, 1)$.

Now, we only need to verify $h_3(x; n) > 0$ for any $n \geq 107$ and $x \in (\frac{54}{100}, 1)$. It is evident that $h_3(x, n)$ is increasing with respect to n , and thus $h_3(x; 107) > 0$, which is shown in Figure 3, is enough to complete the proof.

The Lemma has been proved. □

5.3. PROOF OF THEOREM 2

With all the preparations above we are now well on our way to proving Theorem 2.

On one hand, using Proposition 4 and 5, we have, for every $\epsilon \in (0, 1)$, that if $\mu > \max(\hat{\mu}, \check{\mu})$ the equation $\mu = h(x, \epsilon)$ has a unique solution.

On the other hand, by Proposition 5, we have, for every $\epsilon \in (0, 1)$, that if $\mu < \mu_n^*$ the equation $\mu = h(x, \epsilon)$ has at least two solutions for $x \in (0, 1)$ and at least one solution for $x > 1$. Moreover if $\epsilon \neq 1$, $x = 1$ is not a solution of $\mu = h(x, \epsilon)$. Thus the number of Rosette central configurations changes as the parameter μ increases. The fact that this result holds for every $\epsilon > 0$ follows from Theorem 1 and the invariance under the transformation $(x, \epsilon, \mu) \rightarrow (\frac{1}{x}, \frac{1}{\epsilon}, \mu\epsilon)$.

This concludes the proof of Theorem 2.

6. The case $n = 3$

The case $n = 3$ is special, indeed for $n = 3$ the proof of Lemma 3 fails. This is because $x_3^* = 0.617364 > \frac{54}{100}$ but $h(x_3^*) = -0.188154 < 0$.

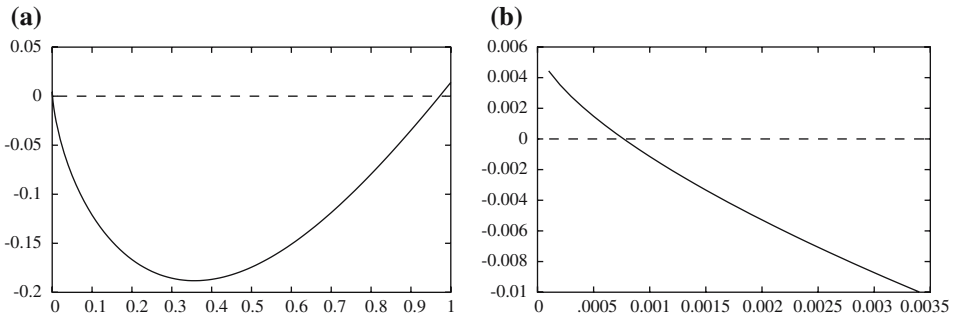


Figure 4. (a) The maximum of the function $h(x, \epsilon)$ as a function of x for $0 < \epsilon < 1$. (b) Magnification of (a) near $\epsilon = 0$.

In this case we study numerically the maximum $h_{\max}(\epsilon)$ of the function $h(x, \epsilon)$ (as a function of x) on the interval $(0, 1)$. Figure 4a depicts $h_{\max}(\epsilon)$ for $\epsilon \in (0, 1)$. Figure 4b shows a magnification of Figure 4(a) near $\epsilon = 0$ making apparent that, near $\epsilon = 0$, $h_{\max}(\epsilon) > 0$. From Figure 4a,b it is apparent that $h_{\max}(\epsilon)$ is always negative except when ϵ is close to 0 or to 1. More precisely we find that $h_{\max}(\epsilon) > 0$ for $\epsilon \in (0, \epsilon_1)$ and $\epsilon \in (\epsilon_2, 1)$ while $h_{\max}(\epsilon) < 0$ for $\epsilon \in (\epsilon_1, \epsilon_2)$, where $\epsilon_1 = 0.00076760883$ and $\epsilon_2 = 0.97198893434$. On the other hand it can be proved that $h(x, \epsilon)$ is a monotone decreasing function with respect to x for $x \in (1, \infty)$ and $\epsilon \in (0, 1)$. In fact, when $n = 3$, we have

$$h(x, \epsilon) = h_0(x) + (1 - \epsilon)h_1(x)$$

where

$$h_0(x) = -\frac{\sqrt{3}}{3} + \frac{x^2(1+x)}{1+x+x^2} \left(\frac{1}{(1-x+x^2)^{3/2}} - \frac{1}{(1+x)^3} \right)$$

$$h_1(x) = \frac{\sqrt{3}}{3(1-x^3)} - \frac{x^3}{1-x^3} \left(\frac{1}{(1+x)^2} + \frac{2-x}{(1-x+x^2)^{3/2}} \right)$$

When $x > 1$, we have $h'_0(x) < 0$ and $h'_0(x) + h'_1(x) < 0$. From which it is easy to conclude that $h'_x(x, \epsilon) < 0$ for any $x > 1$ and $\epsilon \in (0, 1)$. Detailed computations will be omitted.

Consequently for every $\epsilon \in (\epsilon_1, \epsilon_2)$ there is one and only one Rosette central configuration for every value of $\mu > 0$. On the other hand, our numerical study shows that, if $\epsilon \in (0, \epsilon_1)$ or $\epsilon \in (\epsilon_2, 1)$ there is a μ^* such that if $0 < \mu < \mu^*$ there are three Rosette central configurations and if $\mu > \mu^*$ there is only one.

In conclusion, we have the following.

PROPOSITION 6. *For $n = 3$ and $\epsilon \in (\epsilon_1, \epsilon_2)$, there is exactly one Rosette configuration for any $\mu > 0$. For $n = 3$ and $\epsilon \in (0, \epsilon_1)$ or $\epsilon \in (\epsilon_2, 1)$, there exists a value $\mu_0(\epsilon) > 0$, such that when $\mu > \mu_0(\epsilon)$, $\mu = \mu_0(\epsilon)$, and $\mu < \mu_0(\epsilon)$, there are exactly one, two and three Rosette configurations, respectively.*

Acknowledgements

MS wishes to thank Giampaolo Cicogna and Donald Saari for their comments and suggestions regarding this work. This work was supported by NNSFC (National Natural Science Foundation of China) with a grant No. 10301006.

References

- Abraham, R. and Marsden, J. E.: 1978, *Foundation of Mechanics*, 2nd edn, Benjamin, New York.
- Albouy, A.: 1995, 'Symétrie des configurations centrales de quatre corps', *C.R. Acad. Sci. Paris* **320**, 217–220.
- Albouy, A.: 1996, 'The symmetric central configurations of four equal masses', *Contemp. Math.* **198**, 131–135.
- Hampton, M. and Moeckel, R.: 2006, 'Finiteness of relative equilibria of the four-body problem', *Inv. Math.*, to appear.
- Palmore, J. I.: 1975, 'Classifying relative equilibria II', *Bull. Amer. Math Soc.* **81**, 489–491.
- Palmore, J. I.: 1976, 'Measure of degenerate equilibria I', *Ann. Math.* **104**, 421–429.
- Sekiguchi, M.: 2004, 'Bifurcations of central configurations in the $2N + 1$ body problem', *Cel. Mech. & Dyn. Astr.* **90**, 355–360.
- Smale, S.: 1970, 'Topology and mechanics II', *Inv. Math.* **11**, 45–64.
- Smale, S.: 1971, 'Problems on the nature of relative equilibria in celestial mechanics', *Lect. Notes Math.* **197**, 194–198.