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SECOND SPECIES PERIODIC ORBITS OF THE ELLIPTIC 3 BODY PROBLEM

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Abstract. We consider the plane restricted elliptic 3 body problem with small mass ratio and small eccentricity and prove the existence of many periodic orbits shadowing chains of collision orbits of the Kepler problem. Such periodic orbits were first studied by Poincaré for the non-restricted 3 body problem. Poincaré called them second species solutions.

Key words: 3-body problem, collision, second species orbits, twist map

1. Introduction

Suppose the Sun of mass 1, Jupiter of mass $\mu \ll 1$ and an Asteroid of negligible mass move in $\mathbb{R}^2$. The Asteroid does not influence the motion of the Sun and Jupiter. Then Jupiter's and Sun's orbits are ellipses with foci at 0 and eccentricity $\epsilon \in (0,1)$. We normalize the variables in such a way that the period of motion equals 2π. Then by Kepler's law the major semiaxis of Jupiter's orbit is $(1+\mu)^{-1/3}$. Jupiter's position $u(t,\mu,\epsilon)$ is a 2π-periodic function of time $t \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ depending analytically on the parameters μ, ϵ. The *elliptic restricted* 3-*body problem* describing the motion of the Asteroid has the Hamiltonian

$$H(p,q,t,\mu,\epsilon) = \frac{|p|^2}{2} - \frac{1}{|q+\mu u(t,\mu,\epsilon)|} - \frac{\mu}{|q-u(t,\mu,\epsilon)|}, \quad p,q \in \mathbb{R}^2. \tag{1.1}$$

For $\mu = 0$ Jupiter disappears and Hamiltonian (1.1) describes the Kepler problem Sun–Asteroid:

$$H_0(p,q) = \frac{|p|^2}{2} - \frac{1}{|q|}. \tag{1.2}$$

It has integrals of energy $H_0 = E$ and angular momentum G. For $E < 0$ and $G \neq 0$ orbits of the Kepler problem are ellipses with major semi axis and eccentricity

$$a = -\frac{1}{2E}, \quad e = \sqrt{1 + 2EG^2}. \tag{1.3}$$

Jupiter's orbit $u(t) = u(t, 0, \epsilon)$ is an ellipse with eccentricity ϵ and major semi axis 1.

For small $\mu > 0$ orbits of the elliptic 3 body problem (1.1) are close to the orbits of the Kepler problem (1.2), because Jupiter's influence is small. However, when the Asteroid comes near Jupiter, Jupiter's gravitational force ceases to be small. Very close to Jupiter, Sun's influence becomes negligible, and the orbit of the Asteroid will be close to a hyperbolic orbit of the Kepler problem Jupiter–Asteroid. After interacting with Jupiter, the Asteroid exits its neighborhood, and then moves again along almost Kepler orbit, till the next almost collision with Jupiter. Such an almost collision orbit of the Asteroid shadows a chain $\sigma = (\gamma_i)_{i=1}^n$ of *collision orbits* $\gamma_i : [t_{i-1}, t_i] \to \mathbb{R}^2 \setminus \{0\}$ of the Kepler problem Sun–Asteroid: $\gamma_i(t_{i-1}) = u(t_{i-1})$, $\gamma(t_i) = u(t_i)$ and $\gamma_i(t) \neq u(t)$ for $t_{i-1} < t < t_i$.

Periodic almost collision orbits were first considered by Poincaré who named them *second species solutions*. Poincaré claimed the existence of such solutions shadowing a 2-chain of collision orbits for the general 3 body problem, but did not provide a complete proof. In fact making the argument rigorous is not easy due to a singular nature of the perturbation. Non-periodic almost collision orbits were used by Alexeev (1970) and others to study capture in the 3 body problem.

For $\epsilon = 0$ Jupiter's orbit is a circle. Then we obtain the *circular 3 body problem* with Hamiltonian H^0. It is autonomous in the rotating coordinate frame, where Sun and Jupiter are fixed, and hence has Jacobi's integral $h = H^0 - G$, where G is the angular momentum with respect to 0. Sometimes Jacobi's constant $-2h$ is used, but we prefer to use the relative Hamiltonian h.

For the circular 3 body problem, second species periodic orbits with given Jacobi's integral h were studied e.g. by Henon (1977), Perko (1981), Bruno (1990), Gomez and Olle (1991). These orbits are periodic in the rotating coordinate frame and quasiperiodic in the fixed coordinate frame. They may exist for Jacobi's integral $h \in (-3/2, \sqrt{2})$, when there are Kepler ellipses with $E - G = h$ crossing Jupiter's orbit which is the unit circle. The corresponding values of angular momentum G lie in $(2 - \sqrt{4h + 6}, -h)$. Marko and Niderman (1995) proved the existence of second species periodic orbits shadowing a symmetric chain of two collision orbits w with mathematical level of rigor.

Bolotin and MacKay (2000) proved the existence of an infinite number of periodic (in the rotating coordinate frame) and chaotic almost collision orbits for the circular 3 body problem for any $h \in (-3/2, \sqrt{2})$ by using variational methods. In the rotating coordinate frame, there exist infinitely many Kepler collision orbits $\{\gamma_k\}_{k\in K}$, with Jacobi's integral $E_k - G_k = h$ and the set of angular momenta $\{G_k\}_{k\in K}$ dense in $(2-\sqrt{4h+6}, -h)$, such that for any finite set $L \subset K$, sufficiently small $\mu > 0$ and any chain $\sigma = (\gamma_{k_i})_{i\in\mathbb{Z}}$ of collision orbits with $k_i \in L$, there exists a unique (up to time translation) almost collision orbit γ of the circular 3 body problem with Jacobi's integral $H^0 - G = h$ which is $O(\mu)$-shadowing (up to time reparametrization) the chain σ. These orbits form a hyperbolic invariant set with Lyapunov exponents of order $|\ln \mu|$. A hyperbolic invariant set was constructed also by Font et al. (2002). If the sequence $(k_i)_{i\in\mathbb{Z}}$ is periodic, then γ is periodic.

Second species periodic orbits of the elliptic 3 body problem were previously studied by Gomez and Olle (1991). The goal of the present paper is to describe second species periodic orbits of the elliptic 3 body problem with small eccentricity $\epsilon \in (0, 1)$. These solutions are periodic with respect to the fixed coordinate frame, and their period is an integer multiple of 2π. Our results are similar to the results of (Bolotin and MacKay 2000) for the circular 3 body problem, but we consider only periodic collision chains. It turns out that the proofs for the elliptic problem are considerably harder. The method can provide also chaotic almost collision shadowing orbits which will be studied in a subsequent paper.

2. Main Results

To give conditions which allow us to shadow a chain of collision orbits of the Kepler problem we need to introduce some notation.

Set $\mu = 0$ and let $u(t) = u(t, 0, \epsilon)$ be Jupiter's orbit. Let Γ be the set of C^1 collision curves $\gamma : [t_0, t_1] \to \mathbb{R}^2 \setminus \{0\}$ such that $\gamma(t_0) = u(t_0)$, $\gamma(t_1) = u(t_1)$ and $\gamma(t) \neq u(t)$ for $t \in (t_0, t_1)$. We identify Γ with an open set in $\Omega \times \mathbb{R}^2$, where $\Omega \subset C^1([0, 1], \mathbb{R}^2)$ is the set of loops at 0, via the reparametrization

$$\gamma \mapsto (\hat{\gamma}, t_0, t_1), \quad \gamma(t) = \hat{\gamma}((t - t_0)/(t_1 - t_0)) + u(t). \tag{2.1}$$

If γ is a collision orbit of the Kepler problem, then it is a critical point of Hamilton's action

$$A(\gamma) = \int_{t_0}^{t_1} \left(\frac{1}{2}|\dot{\gamma}(t)|^2 + \frac{1}{|\gamma(t)|}\right) \mathrm{d}t \tag{2.2}$$

on the set of curves in Γ with fixed end points. We say that γ is non-degenerate if it is a non-degenerate critical point, i.e. time moments

t_0, t_1 are not conjugate along γ. If we allow variations of t_0, t_1, then by the first variation formula,

$$\mathrm{d}A(\gamma) = (p\,\mathrm{d}q - H_0\,\mathrm{d}t)\big|_{t_0}^{t_1} = h(\dot\gamma(t_0), t_0)\,\mathrm{d}t_0 - h(\dot\gamma(t_1), t_1)\,\mathrm{d}t_1, \tag{2.3}$$

where

$$h(p,t) = H_0(p, u(t)) - p\cdot\dot u(t) = \frac{1}{2}|p - \dot u(t)|^2 - \frac{1}{2}|\dot u(t)|^2 - \frac{1}{|u(t)|} \tag{2.4}$$

is the energy relative to Jupiter at the collision.

Fix $m, n \in \mathbb{N}$. Let $\Pi = \Pi_{nm}$ be the set of all $2\pi m$-periodic n-chains $\sigma = (\gamma_i)_{i=1}^n \in \Gamma^n$ of collision curves $\gamma_i : [t_{i-1}, t_i] \to \mathbb{R}^2 \setminus \{0\}$, where $t_0 < t_1 < \cdots < t_n$ and $t_n = t_0 + 2\pi m$. The time moments $t_1 < \cdots < t_n$ are not fixed: they are independent variables in Π. By reparametrization (2.1), we identify Π with an open set in $\Omega^n \times \mathbb{R}^n$.

Define the action functional on Π by $A(\sigma) = \sum_{i=1}^n A(\gamma_i)$. If σ is a critical point of A, then each γ_i is a collision orbit of the Kepler problem and by (2.3),

$$\mathrm{d}A(\sigma) = \sum_{i=1}^n (h(p_i^-, t_i) - h(p_i^+, t_i))\,\mathrm{d}t_i = 0,$$

where $p_i^+ = \dot\gamma_{i-1}(t_i)$ and $p_i^- = \dot\gamma_i(t_i)$. Hence

$$h(p_i^-, t_i) = h(p_i^+, t_i) = h_i. \tag{2.5}$$

Equivalently, $|v_i^+| = |v_i^-|$, where $v_i^\pm = p_i^\pm - \dot u(t_i)$ are relative collision velocities.

We call $\sigma \in \Pi$ a *non-degenerate periodic collision chain* if σ is a non-degenerate critical point of the functional A and satisfies the *changing direction condition:* $v_i^+ \neq \pm v_i^-$ for $i = 1, \ldots, n$.

Remark. Collision chains are break solutions of Kepler's problem (1.2), and they depend on Jupiter only through velocity changes at times t_i. Then solutions undergo elastic reflections from Jupiter.

THEOREM 2.1. *For any non-degenerate periodic collision chain $\sigma = (\gamma_i)_{i=1}^n$ of the Kepler problem there exists $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$ there exists a unique $2\pi m$-periodic orbit $\gamma : \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ of the elliptic 3 body problem which is $O(\mu)$-shadowing the chain σ and avoids collisions with Jupiter. More precisely, there exist constants $a, b > 0$ independent of μ such that*

$$|\gamma(t) - u(t, \mu, \epsilon)| \geqslant a\mu, \quad |\gamma(t) - \gamma_i(t)| \leqslant b\mu \quad \textit{for} \quad t_{i-1} \leqslant t \leqslant t_i. \tag{2.6}$$

The proof is given in (Bolotin 2006). Theorem 2.1 holds for systems more general than the elliptic 3 body problem. What is important is the presence of a small Newtonian singularity in the potential. However, we do not discuss such generalizations here.

Formally, the assertion of Theorem 2.1 holds also for 1-chains consisting of one collision orbit $\gamma : [t_0, t_1] \to \mathbb{R}^2 \setminus \{0\}$ with $t_1 - t_0 = 2\pi m$. However, all 1-chains are degenerate, because the action of an elliptic orbit is a function of period only. Note that for the circular 3 body problem, there are many non-degenerate 1-chains, because the notion of non-degeneracy is different (Bolotin and MacKay 2000).

The goal of the present paper is to prove the following:

THEOREM 2.2. *For any $h \in (-3/2, \sqrt{2})$ there is a dense set $\{G_k\}_{k\in L(h)}$ in $(2-\sqrt{4h+6}, -h)$ such that for any $N \geqslant 2$, any sequence $(k_i)_{i=1}^N$ with*[1] *$k_i \in L(h)$ and $k_{i-1} \neq k_i$ for $i = 1, \dots, N$, and any sequence $(l_i)_{i=1}^N$ of positive integers with sufficiently large $\sum_{i=1}^N l_i$, there exists an integer sequence $(m_i)_{i=1}^N$, $|m_j - l_j| \leqslant 1$, such that for sufficiently small $\epsilon > 0$ the Kepler problem has* 2 *non-degenerate periodic collision n-chains $\sigma_\pm$ of the form*

$$\underbrace{\gamma_{11}\cdots\gamma_{1m_1}}_{m_1}\cdots\underbrace{\gamma_{i1}\cdots\gamma_{im_i}}_{m_i}\cdots\underbrace{\gamma_{N1}\cdots\gamma_{Nm_N}}_{m_N}, \qquad n = \sum_{i=1}^N m_i, \tag{2.7}$$

where γ_{ij} is an elliptic collision orbit with angular momentum G_{ij} and energy E_{ij} which are $O(\epsilon, n^{-1})$-close to G_{k_i} and $E_{k_i} = h - G_{k_i}$, respectively.

Each collision orbit γ_{ij} performs several rotations along a Kepler ellipse with major semiaxis and eccentricity given by (1.3), where $E = E_{ij}$ and $G = G_{ij}$. In fact the proof gives more information which we dropped to simplify the formulation. Theorems 2.1 and 2.2 imply:

COROLLARY 2.1. *For small $\mu > 0$, there exist periodic orbits $\gamma_\pm$ of the elliptic* 3 *body problem which are $O(\mu)$-shadowing collision the chains $\sigma_\pm$ in Theorem 2.2. Poincaré's multipliers*[2] *of $\gamma_\pm$ have the form $\alpha_\pm, \alpha_\pm^{-1}, \beta_\pm, \beta_\pm^{-1}$, where*

$$\alpha_\pm = e^{a_\pm\sqrt{\pm\epsilon}+O(\mu,\epsilon)}, \quad \beta_\pm = b_\pm\mu^{-1} + O(\epsilon, \mu), \quad a_\pm, b_\pm \neq 0.$$

The property of the multipliers follows from Theorem 4.3 and Proposition 3.1. Thus γ_- has a pair of multipliers on the unit circle, and the other on the real line. All multipliers of γ_+ are real.

[1]We set $k_0 = k_N$.

[2]Multipliers are eigenvalues of the monodromy map.

A drawback of Theorem 2.1 is that μ_0 depends on the collision chain σ. Hence for fixed μ we are unable to conclude the existence of an infinite number of periodic orbits of the elliptic 3 body problem. In fact they exist, but to show this we need a uniform version of Theorem 2.1 which is proved in (Bolotin 2006). To apply this result, we need also a uniform version of Theorem 2.2. We will prove this version in a subsequent paper. It gives periodic and chaotic collision chains with certain hyperbolicity, so the infinite set of periodic shadowing orbits we will get are hyperbolic. On the other hand, half of the periodic orbits given by Theorems 2.1 and 2.2 are linearly stable in some directions. For applications one has to estimate how small ϵ, μ need to be, but we do not address this problem in the present paper.

Remark. One may hope to find almost collision periodic orbits of the elliptic 3 body problem with small eccentricity ϵ by perturbing the periodic orbits with given Jacobi's integral $h \in (-3/2, \sqrt{2})$ found by Marco and Niderman (1995) and Bolotin and Mackay (2000) for the circular 3 body problem. They are periodic with respect to the rotating coordinate frame with period $T(h)$. If $T(h)/2\pi$ is rational, these orbits are periodic with respect to the fixed frame, and they fill a 2-torus in the extended phase space $\mathbb{R}^4 \times \mathbb{T}$. As shown by Poincaré, for $0 < \epsilon \ll \mu$ such torus generically disintegrates into a pair of periodic solutions. However for fixed $\epsilon > 0$ and small μ such approach does not work. Hence it seems impossible to obtain our results by using similar results for the circular problem.

In the next section we reformulate the definition of a non-degenerate collision chain and show that such chains correspond to non-degenerate periodic orbits of sequences of twist maps of annuli. Then we give sufficient conditions for the existence of such orbits for sequences of almost integrable twist maps. Lastly, we check that these conditions hold for twist maps arising from the Kepler problem with small eccentricity and deduce Theorem 2.2.

3. Twist Maps Reformulation of the Non-degeneracy Condition

In the remaining part of the paper we set $\mu = 0$. The eccentricity $\epsilon > 0$ will be fixed so we do not show it in the notation.

Let $\Sigma \subset \Gamma$ be the set of all *non-degenerate collision orbits* $\gamma : [t_0, t_1] \to \mathbb{R}^2 \setminus \{0\}$ of Kepler's problem. Then Σ is a 2-dimensional manifold and the projection $\pi : \Sigma \to \mathbb{R}^2$, $\pi(\gamma) = (t_0, t_1)$, is a local diffeomorphism. Hence Σ has an open covering $\{\Sigma_k\}_{k \in K}$ such that $\pi_k = \pi|_{\Sigma_k} : \Sigma_k \to U_k \subset \mathbb{R}^2$ is a diffeomorphism. Without loss of generality we assume that the sets U_k are invariant under the translation $(t_0, t_1) \mapsto (t_0 + 2\pi, t_1 + 2\pi)$.

For $(t_0, t_1) \in U_k$ let

$$v_k^-(t_0, t_1) = \dot\gamma(t_0) - \dot u(t_0), \qquad v_k^+(t_0, t_1) = \dot\gamma(t_1) - \dot u(t_1) \tag{3.1}$$

be the relative collision velocities of the collision orbit $\gamma = \pi_k^{-1}(t_0, t_1) \in \Sigma_k$, and let

$$S_k(t_0, t_1) = A(\gamma) \tag{3.2}$$

be the action (2.2). By (2.3),

$$D_1 S_k(t_0, t_1) = h(\dot\gamma(t_0), t_0), \qquad D_2 S_k(t_0, t_1) = -h(\dot\gamma(t_1), t_1), \tag{3.3}$$

where D_i is ith partial derivative.

Let K be the index set for the covering $\{\Sigma_k\}_{k\in K}$. Fix $n, m \in \mathbb{N}$ and for a sequence $\mathbf{k} = (k_i)_{i=1}^n \in K^n$ let

$$X_{\mathbf{k}} = \{\mathbf{t} = (t_i)_{i=1}^n : (t_{i-1}, t_i) \in U_{k_i} \text{ for } i = 1, \dots, n\}, \quad t_n = t_0 + 2\pi m.$$

We identify points of $X_{\mathbf{k}}$ which differ by a translation $(t_i) \mapsto (t_i + 2\pi)$. Define a function $A_{\mathbf{k}}$ on $X_{\mathbf{k}}$ by

$$A_{\mathbf{k}}(\mathbf{t}) = \sum_{i=1}^n S_{k_i}(t_{i-1}, t_i). \tag{3.4}$$

If $\sigma = (\gamma_i)_{i=1}^n$, $\gamma_i : [t_{i-1}, t_i] \to \mathbb{R}^2 \setminus \{0\}$, is a non-degenerate periodic collision chain, then $\gamma_i \in \Sigma_{k_i}$ for some (maybe non-unique) sequence $\mathbf{k}$. Then $\mathbf{t} = (t_i)_{i=1}^n \in X_{\mathbf{k}}$ is a non-degenerate critical point of $A_{\mathbf{k}}$. The relative Hamiltonian (2.5) at the collisions is given by (3.3):

$$h_i = -D_2 S_{k_i}(t_{i-1}, t_i) = D_1 S_{k_{i+1}}(t_i, t_{i+1}). \tag{3.5}$$

Conversely, a non-degenerate critical point $\mathbf{t}$ of $A_{\mathbf{k}}$ satisfying the *changing direction condition*

$$v_{k_i}^+(t_{i-1}, t_i) \neq \pm v_{k_{i+1}}^-(t_i, t_{i+1}), \quad i = 1, \dots, n, \tag{3.6}$$

defines a non-degenerate collision chain $\sigma = (\gamma_i)_{i=1}^n$, $\gamma_i = \pi_{k_i}^{-1}(t_{i-1}, t_i)$.

Remark. The assertion of Theorem 2.1 holds also if $\mathbf{t}$ is a topologically non-degenerate critical point of $A_{\mathbf{k}}$, e.g. if the degree of $\operatorname{grad} A_{\mathbf{k}}$ at $\mathbf{t}$ is non-zero. The only difference is that the shadowing orbit is non-unique in general, and there is no $O(\mu)$-estimate (2.6) for the shadowing distance.

Next we reformulate the non-degeneracy condition in the language of twist maps. We do not assume that the functions S_k are necessarily related to the Kepler problem. In what follows $\{S_k\}_{k\in K}$ are some generic C^2 functions on closures of open sets $U_k\subset\mathbb{R}^2$, invariant under the 2π-translation. We will come back to the Kepler problem later on.

Suppose that $\{S_k\}_{k\in K}$ satisfy the *twist condition*:

$$D_{12}S_k(t_0,t_1)\neq 0 \quad \text{for} \quad (t_0,t_1)\in\overline{U}_k. \tag{3.7}$$

In general the twist condition may not hold in the whole U_k, and we have to cut it in smaller pieces. For simplicity we assume that $I_k(t_0)=\{t_1-t_0:(t_0,t_1)\in U_k\}$ is a non-empty interval of length less than 2π for each t_0. This holds in our application.

Let $\widetilde{U}_k$ be the quotient of U_k under the translation $(t_0,t_1)\mapsto(t_0+2\pi,t_1+2\pi)$, and let $\mathbb{A}=\mathbb{T}\times\mathbb{R}$ be the annulus. The map g_k: $\widetilde{U}_k\to\mathbb{A}$ given by

$$g_k(t_0,t_1)=(t_0,h_0),\qquad h_0=D_1S_k(t_0,t_1)$$

is a diffeomorphism of $\widetilde{U}_k$ onto the annulus $V_k=g_k(\widetilde{U}_k)$. Define the map f_k: $V_k\to\mathbb{A}$ by $f_k(t_0,h_0)=(t_1,h_1)$, where

$$(t_0,t_1)=g_k^{-1}(t_0,h_0),\qquad h_1=-D_2S_k(t_0,t_1). \tag{3.8}$$

The symplectic map f_k is called a *twist map* with the generating function S_k. Since the closure of $\widetilde{U}_k$ is compact, the twist condition is uniform in U_k. Hence f_k can be extended to a twist map of a neighborhood of $\overline{V}_k$ in $\mathbb{A}$.

Let $\mathbf{k}=(k_i)_{i=1}^n$. A critical point $\mathbf{t}=(t_i)_{i=1}^n\in X_{\mathbf{k}}$ of $A_{\mathbf{k}}$ defines a sequence $\mathbf{x}=(x_i)_{i=0}^{n-1}$, $x_i=(t_i,h_i)\in V_{k_{i+1}}$, where h_i is given by (3.5). Then $x_i=f_{k_i}(x_{i-1})$ for $i=1,\dots,n$. Hence $\mathbf{x}$ is a periodic orbit of a sequence $f_{k_1},\dots,f_{k_n}$ of twist maps, and x_0 is a fixed point of the composition $f_{\mathbf{k}}=f_{k_n}\circ\cdots\circ f_{k_1}$. The orbit $\mathbf{x}$ defines $\mathbf{t}$ modulo 2π-translations.

The following result is well known for periodic orbits of twist maps (MacKay and Meiss 1983; Kozlov and Treschev 1991), and of course holds also for their random compositions.

LEMMA 3.1. *A sequence* $\mathbf{t}=(t_i)_{i=1}^n\in X_{\mathbf{k}}$ *is a non-degenerate critical point of* $A_{\mathbf{k}}$ *iff the corresponding fixed point* x_0 *of* $f_{\mathbf{k}}$ *is non-degenerate:*

$$\det(f_{\mathbf{k}}'(x_0)-I)\neq 0. \tag{3.9}$$

Moreover

$$\det(f_{\mathbf{k}}'(x_0)-I)=-\det(A_{\mathbf{k}}''(\mathbf{t}))\prod_{i=1}^{n}(-D_{12}S_{k_i}(t_{i-1},t_i))^{-1}. \tag{3.10}$$

Condition (3.9) often holds even when each map f_k is completely integrable and hence has no non-degenerate periodic orbits, (see Section 4). In fact a composition of completely integrable maps is generically chaotic.

Equation (3.10) gives a relation between the Morse index $m(\mathbf{t})$ of the critical point $\mathbf{t}$ and stability of the periodic orbit $\mathbf{x}$. Suppose for simplicity that the twist of all maps f_{k_i} has the same sign, for example $D_{12}S_{k_i} < 0$ in $\overline{U}_{k_i}$. Then

$$\text{sign } \det(f'_{\mathbf{k}}(x_0) - I) = -\text{sign } \det A''_{\mathbf{k}}(\mathbf{t}).$$

Hence if $m(\mathbf{t})$ is even, the fixed point x_0 is hyperbolic, and if $m(\mathbf{t})$ is odd, it is elliptic.

If we find a non-degenerate fixed point of $f_{\mathbf{k}}$, we find a non-degenerate periodic collision chain σ of the Kepler problem and hence, by Theorem 2.1, a periodic shadowing orbit γ of the elliptic 3 body problem with small $\mu > 0$. We only need to check that the changing direction condition (3.6) is satisfied.

The proof of Theorem 2.1 in (Bolotin 2006) implies:

PROPOSITION 3.1. *Let λ, λ^{-1} be the multipliers of a non-degenerate fixed point of $f_{\mathbf{k}}$. If the corresponding collision chain σ satisfies the changing direction condition, then for small $\mu > 0$ the multipliers $\alpha, \alpha^{-1}, \beta, \beta^{-1}$ of the shadowing orbit γ in Theorem* 2.1 *satisfy $\alpha = \lambda + O(\mu)$, $\beta = b\mu^{-1} + O(\mu)$.*

In the next section, for almost autonomous generating functions S_k, we give sufficient conditions for the existence of non-degenerate critical points of $A_{\mathbf{k}}$, or, equivalently, non-degenerate fixed points of $f_{\mathbf{k}}$. The proofs are contained in section 5. In Sections 6 and 7 we compute the functions S_k for the elliptic 3 body problem with small eccentricity ϵ and show that these sufficient conditions are satisfied.

4. Almost Autonomous Twist Maps

If the generating function S_k satisfies an additional non-degeneracy condition

$$D_{22}S_k(t_0, t_1) \neq 0 \quad \text{for} \quad (t_0, t_1) \in \overline{U}_k, \tag{4.1}$$

we can represent f_k by another generating function F_k: $f_k(t_0, h_0) = (t_1, h_1)$, where

$$h_0 = h_1 + D_1 F_k(t_0, h_1), \qquad t_1 = t_0 + D_2 F_k(t_0, h_1). \tag{4.2}$$

Set $J_k(t_0) = -D_2 S_k(t_0, t_0 + I_k(t_0))$. Then $h_1 \in J_k(t_0) \mapsto F_k(t_0, h_1)$ is the Legendre transform of the function $s \in I_k(t_0) \mapsto -S_k(t_0, t_0 + s)$. Thus

$$F_k(t_0, h_1) = h_1(t_1 - t_0) + S_k(t_0, t_1), \quad h_1 = -D_1 S_k(t_0, t_1).$$

Suppose that the twist maps f_k are *almost autonomous,* i.e. the generating functions S_k have the form

$$S_k(t_0, t_1) = \Psi_k(s) + \epsilon\psi_k(t_0, s) + O(\epsilon^2), \quad s = t_1 - t_0, \tag{4.3}$$

where ϵ is a small parameter. For example, for the elliptic 3 body problem with small eccentricity, S_k has the form (4.3).

If f_k is almost autonomous, we can assume that $U_k = \{(t_0, t_1) : t_1 - t_0 \in I_k\}$ with I_k a fixed interval. To satisfy the twist conditions (3.7) and (4.1), suppose that $\Psi_k''(s) \neq 0$ for $s \in \bar{I}_k$.[3] Let $J_k = -\Psi_k'(I_k)$ and let $h \in J_k \mapsto \Phi_k(h)$ be the Legendre transform of the function $s \in I_k \mapsto -\Psi_k(s)$. Denote $\rho_k(h) = \Phi_k'(h) = -1/\Psi_k'(s)$. Then $I_k = \rho_k(J_k)$. The generating function F_k in (4.2) has the form

$$F_k(t_0, h_1) = \Phi_k(h_1) + \epsilon\phi_k(t_0, h_1) + O(\epsilon^2),$$

where

$$\phi_k(t, h) = \psi_k(t, \rho_k(h)), \quad (t, h) \in \mathbb{T} \times J_k. \tag{4.4}$$

The twist map f_k: $(t_0, h_0) \mapsto (t_1, h_1)$ is given by (4.2):

$$\begin{aligned} h_0 &= h_1 + \epsilon D_1\phi_k(t_0, h_1) + O(\epsilon^2), \\ t_1 &= t_0 + \rho_k(h_1) + \epsilon D_2\phi_k(t_0, h_1) + O(\epsilon^2). \end{aligned} \tag{4.5}$$

It is defined on an annulus V_k which is $O(\epsilon)$-close to $\mathbb{T} \times J_k$. For $\epsilon = 0$, f_k is an integrable twist map $(t, h) \mapsto (t + \rho_k(h), h)$.[4]

To prove the existence of complicated dynamics for each map f_k is a transcendental problem related to exponentially small splitting of separatrices, and no finite Taylor expansion of F_k in ϵ provides enough information for that. Getting non-degenerate periodic orbits is easier. Poincaré proved (Arnold et al. 1989) that if

$$\phi_k(t, h) = \sum_{n \in \mathbb{Z}} a_{kn}(h) e^{int} \tag{4.6}$$

has many Fourier harmonics, then f_k has many periodic orbits for small $\epsilon \neq 0$. If there is a resonance at $h \in J_k$, i.e. $a_{kn}(h) \neq 0$ and $\rho_k(h) = 2\pi m/n$ for some $m \in \mathbb{Z}$, then f_k has at least two n-periodic orbits $O(\epsilon)$-close to $\mathbb{T} \times \{h\}$.

[3]If the twist condition does not hold in the whole I_k, we cut it in smaller intervals.
[4]In the application to the 3 body problem, h is Jacobi's integral.

However, we will see that in the elliptic 3 body problem,

$$\phi_k(t,h) = a_k(h)e^{it} + \bar{a}_k(h)e^{-it}. \tag{4.7}$$

Thus we may obtain only fixed points of f_k corresponding to $\rho_k(h) = 2\pi m$. They do not give any non-degenerate collision chains of the Kepler problem, because all 1-chains are degenerate.

To get more Fourier harmonics in F_k of higher order in ϵ, we need to perform several steps of the perturbation scheme (Arnold et al. 1989) which is difficult analytically. Hence it is hard to get non-degenerate collision chains by using orbits of just one map f_k.

This difficulty disappears if we take compositions of random sequences of different maps f_k. Suppose for simplicity (this holds in our application) that ϕ_k is a Fourier polynomial and there are no resonances[5] in J_k, i.e. if $a_{kn}(h) \neq 0$ for some $h \in J_k$ and $n \in \mathbb{Z}$, then $n\rho_k(h) \notin 2\pi\mathbb{Z}$. Then f_k has an approximate first integral

$$H_k(t,h) = h + \epsilon D_1\chi_k(t,h), \qquad H_k \circ f_k = H_k + O(\epsilon^2),$$

where χ_k is the solution of the homological equation[6]

$$\chi_k(t+\rho_k(h),h) - \chi_k(t,h) = \phi_k(t,h), \quad (t,h) \in V_k = \mathbb{T} \times J_k. \tag{4.8}$$

Hence

$$\chi_k(t,h) = \sum_{n\neq 0} b_{kn}(h)e^{int}, \quad b_{kn}(h) = \frac{a_{kn}(h)}{e^{in\rho_k(h)} - 1}. \tag{4.9}$$

Then for small ϵ, f_k has a family of approximate invariant curves[7]

$$\Gamma_k(\epsilon,h) = \{(t, h - \epsilon D_1\chi_k(t,h)) : t \in \mathbb{T}\}, \quad h \in J_k.$$

Suppose that $J_k \cap J_l \neq \emptyset$ for a pair $k,l \in K$, and the integrals H_k, H_l are independent in $V_k \cap V_l$. Then there is an interval $J \subset J_k \cap J_l$ such that

$$t \mapsto \chi_{kl}(t,h) = \chi_k(t,h) - \chi_l(t,h) \not\equiv 0, \quad h \in J. \tag{4.10}$$

Equivalently, there is $n \neq 0$ such that $b_{kn}(h) \neq b_{ln}(h)$. Then for small $\epsilon > 0$ the curves $\Gamma_k(\epsilon,h)$ and $\Gamma_l(\epsilon,h)$ intersect but do not coincide.

[5]If there are resonances in J_k, we divide it in non-resonance subintervals.

[6]For simplicity of notation we assume that the time average of $\phi_k(t,h)$ is zero. This holds in our application. In general, we need to subtract the average.

[7]There are also exact KAM invariant curves corresponding to Diophantine $\rho_k(h)/(2\pi)$.

Any sequence $\mathbf{k}\in\{k,l\}^n$ has the form

$$\mathbf{k}=\underbrace{k,\ldots,k}_{m_1},\underbrace{l,\ldots,l}_{m_2},\underbrace{k,\ldots,k}_{m_3},\ldots,\underbrace{l,\ldots,l}_{m_N},\qquad \sum_{j=1}^{N} m_j=n. \tag{4.11}$$

We assumed that N is even, because the corresponding periodic orbits do not change if we make a cyclic permutation of $\mathbf{k}$. Hence for odd N we may replace N by $N-1$ and m_1 by m_1+m_N.

The next result is a generalization of Poincaré's theorem on periodic orbits of perturbed Hamiltonian systems.

THEOREM 4.1. *For any even $N\geqslant 2$ there exists an infinite set $\mathcal{M}_N\subset\mathbb{N}^N$ of integer sequences $\mathbf{m}=(m_i)_{i=1}^N$ such that for any $\mathbf{m}\in\mathcal{M}_N$ there exists $h_{\mathbf{m}}\in J$ such that:*

- *There exist $\epsilon_0>0$, $c>0$ such that for any $\epsilon\in(0,\epsilon_0)$ the composition*

  $$f_{\mathbf{k}}=f_l^{m_N}\circ f_k^{m_{N-1}}\circ\cdots\circ f_l^{m_2}\circ f_k^{m_1}$$

  has a pair of topologically nondegenerate fixed points $x_\pm$ in $\mathbb{T}\times(h_{\mathbf{m}}-c\epsilon,h_{\mathbf{m}}+c\epsilon)$.
- *The multipliers of $x_\pm$ are*

  $$\lambda_\pm,\lambda_\pm^{-1},\quad \lambda_\pm=e^{c_\pm\sqrt{\pm\epsilon}+O(\epsilon)},\quad c_\pm\geqslant 0. \tag{4.12}$$

- *The set $\{h_{\mathbf{m}}\in J:\mathbf{m}\in\mathcal{M}_N\}$ is dense in J.*

Hence the fixed point x_- has multipliers on the unit circle, and x_+ on the real line. Generically x_- is elliptic and x_+ hyperbolic.

We will see that the set $\mathcal{M}_N$ is relatively dense in $\mathbb{N}^N$ in the following sense: there exist constants $a,b>0$ independent of N such that for any sequence $(l_j)_{j=1}^N$ with $\sum l_j\geqslant b$ there exists $\mathbf{m}\in\mathcal{M}_N$ with $\max_j|m_j-l_j|\leqslant a$.

Note that any $\mathbf{m}\in\mathcal{M}_N$ defines a sequence $j\mathbf{m}$ in $\mathcal{M}_{jN}$ for any integer j by repeating $\mathbf{m}$ several times. Such a sequence gives the same periodic orbit, so we do not count it twice. Thus for non-prime N we delete from $\mathcal{M}_N$ all sequences of the form $j\mathbf{m}$, where j is a divisor of N.

Conditions of Theorem 4.1 holds also if both maps f_k, f_l are completely integrable, provided their invariant curves are different. In a subsequent paper we will show that the skew product dynamical system on $\{k,l\}^{\mathbb{Z}}\times J$ obtained by composing the maps f_k, f_l in random order has a chaotic compact hyperbolic invariant set containing an infinite number of hyperbolic periodic orbits. There exist also diffusion orbits. This can be shown using the arguments similar to these of Moeckel (2002) and Marco and Sauzin (2004).

We will prove Theorem 4.1 and its generalization in Section 5. For the elliptic 3 body problem we can obtain a more explicit statement. Indeed, then the functions ϕ_k have the form (4.7), and hence by (4.9),

$$\chi_k(t,h) = b_k(h)e^{it} + \bar{b}_k(h)e^{-it}, \quad b_k(h) = \frac{a_k(h)}{e^{i\rho_k(h)} - 1}. \tag{4.13}$$

We will compute the coefficients b_k for the elliptic 3 body problem in Section 7.

Till the end of this section we assume for simplicity that χ_k has the form (4.13). If we take $k, l \in K$ such that $h \in J_k \cap J_l$ and $b_k(h) \neq b_l(h)$, then the function $t \mapsto \chi_{kl}(t,h)$ has two non-degenerate critical points, and for small $\epsilon > 0$, the curves $\Gamma_k(\epsilon, h)$ and $\Gamma_l(\epsilon, h)$ have two points of *transverse* intersection. In this case we can get non-degenerate periodic orbits.

For a sequence $\mathbf{k}$ of the form (4.11), let

$$\begin{aligned} \Omega_{\mathbf{k}}(h) &= (m_1 + m_3 + \cdots + m_{N-1})\rho_k(h) + (m_2 + m_4 + \cdots + m_N)\rho_l(h), \\ c_{\mathbf{k}}(h) &= 1 - \mathrm{e}^{im_1\rho_k} + \mathrm{e}^{i(m_1\rho_k + m_2\rho_l)} - \mathrm{e}^{i((m_1+m_3)\rho_k + m_2\rho_l)} \\ &\quad + \cdots - \mathrm{e}^{i((m_1+m_3+\cdots+m_{N-1})\rho_k + (m_2+\cdots+m_{N-2})\rho_l)}. \end{aligned} \tag{4.14}$$

THEOREM 4.2. *Suppose for some $h \in J$ we have*

$$\Omega_{\mathbf{k}}(h) = 0, \quad \Omega'_{\mathbf{k}}(h) \neq 0, \quad c_{\mathbf{k}}(h) \neq 0. \tag{4.15}$$

Then there exist $\epsilon_0 > 0$, $c > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ the composition $f_{\mathbf{k}}$ has a pair of non-degenerate fixed points $x_\pm$ in $\mathbb{T} \times (h - c\epsilon, h + c\epsilon)$. The multipliers of $x_\pm$ have the form (4.12).

Let us show that the set of $h \in J$ such that (4.15) holds for some sequence $\mathbf{m}$ is dense in J. Suppose for example that $N = 2$. Then

$$\mathbf{k} = \underbrace{k, \ldots, k}_{m_1}, \underbrace{l, \ldots, l}_{m_2} \tag{4.16}$$

and $c_{\mathbf{k}}(h) = 1 - e^{im_1\rho_k(h)}$. Hence condition (4.15) gives

$$m_1\rho_k(h) + m_2\rho_l(h) \in 2\pi\mathbb{Z}, \tag{4.17}$$

$$m_1\rho_k(h) \notin 2\pi\mathbb{Z}, \quad m_2\rho_l(h) \notin 2\pi\mathbb{Z}. \tag{4.18}$$

Suppose for simplicity that $\rho'_k(h)$ and $\rho'_l(h)$ have the same sign. Then for large $n = m_1 + m_2$ the distance between the neighbors of the solution set of (4.17) is approximately

$$d = 2\pi(m_1|\rho'_k(h)| + m_2|\rho'_l(h)|)^{-1} + O(n^{-2}).$$

On the other hand, the length of the components of the set (4.18) is approximately

$$d' = 2\pi(\min\{m_1|\rho_k'(h)|, m_2|\rho_l'(h)|\})^{-1} + O(n^{-2}).$$

If m_1 and m_2 are both non-zero and n is large, then $d' > d$. Then there are many[8] $h \in J$ such that (4.17) and (4.18) holds. Hence for $N = 2$ the set of $h \in J$ satisfying the conditions of Theorem 4.2 is dense in J.

A similar argument works for any even N. For example, if $N = 4$, then

$$\mathbf{k} = \underbrace{k, \dots, k}_{m_1}, \underbrace{l, \dots, l}_{m_2}, \underbrace{k, \dots, k}_{m_3}, \underbrace{l, \dots, l}_{m_4},$$

where $m_1 \neq m_3$ or $m_2 \neq m_4$ (or else we get the case $N = 2$). Condition (4.15) gives

$$\Omega_{\mathbf{k}}(h) = (m_1 + m_3)\rho_k(h) + (m_2 + m_4)\rho_l(h) \in 2\pi\mathbb{Z},$$
$$c_{\mathbf{k}}(h) = 1 - e^{im_1\rho_k} + e^{i(m_1\rho_k + m_2\rho_l)} - e^{i((m_1+m_3)\rho_k + m_2\rho_l)} \neq 0.$$

If the sum $c_{\mathbf{k}}(h)$ of four unit complex numbers is zero, they form a parallelogram in $\mathbb{C} = \mathbb{R}^2$. Then there are two possibilities:

- $e^{im_1\rho_k} = e^{im_3\rho_k} = 1$.
- $e^{i(m_1\rho_k + m_2\rho_l)} = e^{i(m_3\rho_k + m_2\rho_l)} = -1$.

In the first case $m_1\rho_k \in 2\pi\mathbb{Z}$ and $m_3\rho_k \in 2\pi\mathbb{Z}$. In the second case $m_1\rho_k + m_2\rho_l \in \pi + 2\pi\mathbb{Z}$ and $m_3\rho_k + m_2\rho_l \in \pi + 2\pi\mathbb{Z}$. If n is sufficiently large and all m_1, m_2, m_3, m_4 are non-zero, a density argument as above gives many $h \in J$ such that (4.15) holds. Hence for $N = 4$ the set of $h \in J$ satisfying the condition of Theorem 4.2 is dense in J.

We can proceed in a similar way getting periodic orbits corresponding to any even N.

We will see that for the elliptic 3 body problem, for almost any pair k, l the collision chains of the Kepler problem corresponding to the fixed points in Theorem 4.2 satisfy the changing direction condition (see Lemma 6.3). Thus for an alternating sequence $(k_i)_{i=1}^N$, Theorem 2.2 follows from Theorem 4.2. To prove the complete version of Theorem 2.2, we need a more general result.

[8]We use a crude argument to show the existence of solutions. One can easily get more information.

Let $K(h)=\{k\in K: h\in J_k\}$. For definiteness we fix a set of $k\in K(h)$ such that all $\rho_k'(h)$ have the same sign.[9] For example, let

$$L(h)=\{k\in K(h): \rho_k'<0 \text{ in } J_k\}. \tag{4.19}$$

Let

$$D(h)=\{(k,l)\in L(h)\times L(h): b_k(h)\neq b_l(h)\}.$$

Let Δ be a closed interval and Λ a finite set such that $\Lambda\subset D(h)$ for all $h\in\Delta$.

Any sequence $\mathbf{k}\in K^n$ has the form

$$\mathbf{k}=\underbrace{k_1,\ldots,k_1}_{m_1},\ldots,\underbrace{k_i,\ldots,k_i}_{m_i},\ldots,\underbrace{k_N,\ldots,k_N}_{m_N},\quad \sum_{j=1}^{N} m_j=n. \tag{4.20}$$

Since we study periodic orbits, set $k_0=k_N$. Without loss of generality, $k_{i-1}\neq k_i$ for $i=1,\ldots,N$.

THEOREM 4.3. *There exist constants $b,c,d>0$ depending on Δ and Λ such that for any $h\in\Delta$, any sequence $(k_i)_{i=1}^N$ such that $(k_{i-1},k_i)\in\Lambda$ for $i=1,\ldots,N$ and any sequence $(l_j)_{j=1}^N$ of positive integers with $\sum_{j=1}^N l_j\geqslant b$, there exist an integer sequence $(m_i)_{i=1}^N$, $|l_i-m_i|\leqslant 1$, and $\tilde h\in(h-cn^{-1},h+cn^{-1})$, $n=\sum m_i$, such that for small $\epsilon_0>0$ and all $\epsilon\in(0,\epsilon_0)$ the map $f_{\mathbf{k}}=f_{k_N}^{m_N}\circ\cdots\circ f_{k_1}^{m_1}$ has 2 non-degenerate fixed points $x_\pm$ in $\mathbb{T}\times(\tilde h-d\epsilon,\tilde h+d\epsilon)$. The multipliers of $x_\pm$ have the form* (4.12).

In Section 7 we will deduce Theorem 2.2 from Theorem 4.3. Theorems 4.1–4.3 are proved in the next section. First we consider arbitrary almost autonomous generating functions S_k and prove Theorem 4.1 and its generalization, Theorem 5.1. Then we will assume that ϕ_k has the form (4.7) and prove Theorems 4.2 and 4.3.

5. Periodic Orbits of Almost Autonomous Maps

By Lemma 3.1, to prove the existence of non-degenerate periodic orbits of sequences of twist maps f_k, we need to find $n,m\in\mathbb{N}$ and a sequence $\mathbf{k}=(k_i)_{i=1}^n$ such that the function $A_{\mathbf{k}}$ in (3.4) has non-degenerate critical points.

[9]If we take indices k with different signs of ρ_k', the description of admissible sequences will need a minor modification. Since we do not aim for higher generality, we make this assumption.

If S_k has the form (4.3), then

$$A_{\mathbf{k}}(\mathbf{t}) = \Psi_{\mathbf{k}}(\mathbf{s}) + \epsilon \psi_{\mathbf{k}}(\mathbf{t}) + O(\epsilon^2),$$

where $\mathbf{s} = (s_j)_{j=1}^n$ with $s_j = t_j - t_{j-1} \in I_{k_j}$ and

$$\Psi_{\mathbf{k}}(\mathbf{s}) = \sum_{j=1}^n \Psi_{k_j}(s_j), \qquad \psi_{\mathbf{k}}(\mathbf{t}) = \sum_{j=1}^n \psi_{k_j}(t_{j-1}, s_j).$$

Since $t_n = t_0 + 2\pi m$,

$$\sum_{j=1}^n s_j = 2\pi m. \tag{5.1}$$

If $\mathbf{s}$ is a critical point of $\Psi_{\mathbf{k}}$ subject to the constraint (5.1), then there exists $h \in \cap_{j=1}^n J_{k_j}$ such that $-\Psi'_{k_j}(s_j) = h$ for $j = 1, \dots, n$. Hence $s_j = \rho_{k_j}(h)$ and

$$\Omega_{\mathbf{k}}(h) = \sum_{j=1}^n \rho_{k_j}(h) = 2\pi m. \tag{5.2}$$

We assume for simplicity[10] that all k_i belong to the set $L(h)$ in (4.19). Then $\Psi''_{k_i}(s_i) > 0$. Hence $\mathbf{s}$ is a non-degenerate *minimum* point of $\Psi_{\mathbf{k}}$ subject to the constraint (5.1). Then for $\epsilon = 0$ the function $A_{\mathbf{k}}$ has a one-dimensional non-degenerate minimal critical manifold

$$Z = \left\{ \mathbf{t} = (t_j)_{j=1}^n : t_j = t + \sum_{i=1}^j \rho_{k_i}(h), \ \ t \in \mathbb{T} \right\} \subset X_{\mathbf{k}}.$$

Define the Poincaré function on Z by

$$\phi_{\mathbf{k}}(t, h) = \psi_{\mathbf{k}}|_Z = \sum_{j=1}^n \psi_{k_j}(t_{j-1}, \rho_{k_j}(h)) = \sum_{j=1}^n \phi_{k_j}(t_{j-1}, h). \tag{5.3}$$

It is 2π-periodic in time and has zero average. By the Lyapunov–Schmidt reduction, if t is a non-degenerate critical point of Poincaré's function $t \mapsto \phi_{\mathbf{k}}(t, h)$, then $A_{\mathbf{k}}$ has a non-degenerate critical point $\mathbf{t} \in X_{\mathbf{k}}$ in a $O(\epsilon)$-neighborhood of Z for small $\epsilon > 0$. If all $k_i \in L(h)$, then Z is a minimal critical manifold. Hence for small $\epsilon > 0$ the Morse index $m(\mathbf{t})$ equals the Morse index $m(t)$ of the corresponding critical point of Poincaré's function. Then for small $\epsilon > 0$, $f_{\mathbf{k}}$ will have a non-degenerate fixed point x_0 near (t, h).

[10]If we drop this assumption, we have to assume that $\Omega'_{\mathbf{k}}(h) \neq 0$.

By Lemma 3.1, x_0 is elliptic if $m(t)=1$, and hyperbolic if $m(t)=0$. Since $\det A''_{\mathbf{k}}(\mathbf{t})=O(\epsilon)$, the multipliers of x_0 have the form $e^{\pm O(\sqrt{\epsilon})}$.

This works also without the non-degeneracy assumption. If Poincaré's function $t\mapsto\phi_{\mathbf{k}}(t,h)$ is non-constant, then for small $\epsilon>0$ the function $A_{\mathbf{k}}$ will have at least two topologically non-degenerate critical points in $O(\epsilon)$-neighborhood of Z, one minimum and the other mountain pass. The corresponding fixed points of $f_{\mathbf{k}}$ may be degenerate, but one of them has multipliers on the unit circle, and the other on the real line.

Let us write the sequence $\mathbf{k}$ in the form (4.20). Then

$$\Omega_{\mathbf{k}}(h)=\sum_{j=1}^{N} m_j\rho_{k_j}(h)=2\pi m, \tag{5.4}$$

$$\phi_{\mathbf{k}}(t,h)=\sum_{j=1}^{N}\phi_{k_j m_j}(\tau_{j-1},h), \tag{5.5}$$

where

$$\tau_j=t+\theta_j, \quad \theta_j=\sum_{i=1}^{j} m_i\rho_{k_i}(h), \tag{5.6}$$

$$\phi_{km}(t,h)=\sum_{j=0}^{m-1}\phi_k(t+j\rho_k(h),h). \tag{5.7}$$

By the homological equation (4.8), the sum telescopes:

$$\phi_{km}(t,h)=\chi_k(t+m\rho_k(h),h)-\chi_k(t,h). \tag{5.8}$$

Hence

$$\phi_{\mathbf{k}}(t,h)=\sum_{j=1}^{N}(\chi_{k_j}(\tau_j,h)-\chi_{k_j}(\tau_{j-1},h))=\sum_{j=1}^{N}\chi_{k_{j-1}k_j}(\tau_{j-1},h), \tag{5.9}$$

where $\chi_{k_{j-1}k_j}$ is the function (4.10) and we set $\tau_N=\tau_0+2\pi m$.

We will find t and sequences $(k_i)_{i=1}^N$, $(m_i)_{i=1}^N$ such that

$$\chi_{k_{j-1}k_j}(\tau_{j-1},h)>0 \quad \text{for} \quad j=1,\dots,N. \tag{5.10}$$

Then Poincaré's function $t\mapsto\phi_{\mathbf{k}}(t,h)$ is non-constant.

Let $\Lambda\subset L(h)\times L(h)$ be a finite set of pairs (k,l) such that $t\mapsto\chi_{kl}(t,h)\not\equiv 0$. There exist $\delta,r>0$ such that for each $(k,l)\in\Lambda$ there is an interval

$$P_{kl}=[\theta_{kl}-3r,\ \theta_{kl}+3r]\subset\mathbb{T} \tag{5.11}$$

such that $\chi_{kl}(t,\tilde{h}) > 0$ for all $t \in P_{kl}$ and $\tilde{h} \in \Delta = [h-\delta, h+\delta]$. Let $I_{kl} = [\theta_{kl} - r, \theta_{kl} + r]$ and take $a > \pi/(2r)$. The set $\{m\rho_k(h)\}_{m=-a}^{a}$ has two points with distance mod 2π less than $\pi/a < 2r$ in $\mathbb{T}$. Thus for any $\tau \in \mathbb{T}$ and $l \in \mathbb{N}$ there exists an integer m such that $|m-l| \leqslant a$ and $\tau + m\rho_k(h) \in I_{kl}$.

Take any sequence $(k_i)_{i=1}^N$ such that $(k_{i-1}, k_i) \in \Lambda$ for $i = 1, \ldots, N$ and any sequence $(l_i)_{i=1}^N$ of positive integers. Let $t \in I_{k_0k_1}$. We will choose the sequence $(m_i)_{i=1}^N$ as follows. Take any $m_1 \in \mathbb{N}$ such that $|m_1 - l_1| \leqslant a$ and $\tau_1 = t + m_1\rho_{k_1}(h) \in I_{k_1k_2}$. Then take any $m_2 \in \mathbb{N}$ such that $|m_2 - l_2| \leqslant a$ and $\tau_2 = \tau_1 + m_2\rho_{k_2}(h) \in I_{k_2k_3}$. We proceed indefinitely defining m_j for all $j = 1, \ldots, N$. Then $\tau_j \in I_{k_jk_{j+1}}$, $\tau_j - \tau_{j-1} = m_j\rho_{k_j}(h)$ and $|m_j - l_j| \leqslant a$ for all j. In particular (5.10) holds.

We also need to satisfy (5.4): $\Omega_{\mathbf{k}}(h) = \tau_N - t \in 2\pi\mathbb{Z}$. Since $t, \tau_N \in I_{k_0k_1}$ mod 2π, there exists $m \in \mathbb{Z}$ such that $|\Omega_{\mathbf{k}}(h) - 2\pi m| < 2r$. There exists $\rho > 0$ such that $\rho_k' < -\rho$ in J_k for any $(k,l) \in \Lambda$. Then $\Omega_{\mathbf{k}}'(h) < -n\rho < 0$ for all $h \in \Delta$. Hence for large n we can find $\tilde{h} \in \Delta$ such that $|h - \tilde{h}| < 2r/(n\rho) < \delta$ and $\Omega_{\mathbf{k}}(\tilde{h}) = 2\pi m$. If we replace h by $\tilde{h}$, then τ_i will be replaced by $\tilde{\tau}_i = t + \tilde{\theta}_i$, where $\tilde{\theta}_i$ is given by (5.6). Then $\tilde{\tau}_N = t + 2\pi m$, while all $(\tau_i)_{i=1}^{N-1}$ change by less than $2r$. Hence new $\tilde{\tau}_j$ lie in $P_{k_jk_{j+1}}$, and thus (5.10) holds for all j.

We proved:

THEOREM 5.1. *There exist $a, b, c, d > 0$ such that for any $h \in \Delta$, any sequence $(k_i)_{i=1}^N$ such that $(k_{i-1}, k_i) \in \Lambda$ for $i = 1, \ldots, N$ and any sequence $(l_i)_{i=1}^N$ of positive integers with $\sum_{i=1}^N l_i \geqslant b$, there exist an integer sequence $(m_i)_{i=1}^N$, $|l_i - m_i| \leqslant a$, and $\tilde{h} \in (h - cn^{-1}, h + cn^{-1})$, $n = \sum_{i=1}^N m_i$, such that for small $\epsilon_0 > 0$ and all $\epsilon \in (0, \epsilon_0)$ the map $f_{k_N}^{m_N} \circ \cdots \circ f_{k_1}^{m_1}$ has a pair of topologically non-degenerate fixed points $x_\pm$ in $\mathbb{T} \times (\tilde{h} - d\epsilon, \tilde{h} + d\epsilon)$. The multipliers of $x_\pm$ have the form* (4.12). □

If $(k_i)_{i=1}^N$ is an alternating sequence as in (4.11), then we obtain Theorem 4.1. If the functions ϕ_k have the form (4.7), then

$$\phi_{\mathbf{k}}(t,h) = a_{\mathbf{k}}(h)e^{it} + \bar{a}_{\mathbf{k}}(h)e^{-it}. \tag{5.12}$$

Hence if Poincaré's function $t \mapsto \phi_{\mathbf{k}}(t,h)$ is non-constant, it has only non-degenerate critical points. Moreover $\{t \in \mathbb{T} : \chi_{kl}(t,h) > 0\}$ is a half the circle. Then Theorem 4.3 follows immediately. □

To prove Theorem 4.2, we may use a more direct argument. Computation of Poincaré's function simplifies for an alternating sequence (4.11). By (5.6),

$$\phi_{\mathbf{k}}(t,h)=\sum_{j=0}^{N-1}(-1)^j\chi_{kl}(t+\theta_j,h).$$

If ϕ_k has the form (4.7), then by (4.13), $\phi_{\mathbf{k}}$ has the form (5.12) with

$$a_{\mathbf{k}}(t,h)=(b_k(h)-b_l(h))c_{\mathbf{k}}(h), \tag{5.13}$$

where

$$c_{\mathbf{k}}(h)=\sum_{j=0}^{N-1}(-1)^j e^{i\theta_j}$$

is given by (4.15). If $c_{\mathbf{k}}(h)\neq 0$, then Poincaré's function $t\mapsto\phi_{\mathbf{k}}(t,h)$ has two non-degenerate critical points. Theorem 4.2 is proved. □

In the remaining part of the paper we compute the generating functions S_k for the elliptic 3 body problem with small eccentricity ϵ of Jupiter's orbit and check that the conditions of Theorems 4.2 and 4.3 are satisfied. Then the assertion of Theorem 2.2 will follow.

First we compute the autonomous generating functions Ψ_k. Then we will apply a perturbation argument and find the non-autonomous perturbation ψ_k.

6. Collision Orbits for the Circular Jupiter's Orbit

In this section we set $\epsilon=0$. Then Jupiter moves counterclockwise with angular velocity one along a unit circle, and its position[11] is $u(t,0,0)=e^{it}$. We need to find intervals I_k such that for $s=t_1-t_0\in I_k$ there exists a non-degenerate collision orbit $\sigma\colon[t_0,t_1]\to\mathbb{R}^2$ of the Kepler problem such that $\sigma(t_0)=e^{it_0}$ and $\sigma(t_1)=e^{it_1}$. Since the circular problem is time-invariant, we may set $t_0=0$ and $t_1=s$.

Let a, e be the major semiaxis and eccentricity of the elliptic orbit of the Asteroid. Let $\pm\theta\in(-\pi,0)\cup(0,\pi)$ be the polar angles corresponding to intersections of Asteroid's and Jupiter's orbits, measured counterclockwise from the perihelion of Asteroid's orbit. Then the polar radius at the collision is

$$r=\frac{a(1-e^2)}{1+e\cos\theta}=1.$$

[11]We write vectors in $\mathbb{R}^2$ as complex numbers.

Suppose that $-\theta$ corresponds to collision at $t=0$, and θ to collision at $t=s$. Then $s=2\theta+2\pi n$ for some $n\in\mathbb{N}$. Let $\eta\in(-\pi,0)\cup(0,\pi)$ be the eccentric anomaly corresponding to θ. We choose η mod 2π so that it has the same sign as θ. Then for $r=1$,

$$\cos\theta=a(\cos\eta-e),\qquad \sin\theta=a\sqrt{1-e^2}\sin\eta.$$

By Kepler's time equation (Arnold et al. 1989), $\omega s=2(\eta-e\sin\eta)+2\pi m$, $m\in\mathbb{Z}$, where $\omega=\pm a^{-3/2}$ is the mean angular velocity of the Asteroid. We obtain a system of three equations (compare with e.g. (Henon 1977; Bruno 1981, Marko and Niderman 1995))

$$\omega(\theta+\pi n)=\eta-e\sin\eta+\pi m, \tag{6.1}$$

$$\cos\theta=a(\cos\eta-e), \tag{6.2}$$

$$1+e\cos\theta=a(1-e^2) \tag{6.3}$$

with four variables $a=\omega^{-2/3}$, $e\in(0,1)$, $\theta\in(-\pi,0)\cup(0,\pi)$, $\eta\in(-\pi,0)\cup(0,\pi)$ and integer parameters $n\in\mathbb{N}$, $m\in\mathbb{Z}$.

Using more work (including computer simulation) one can obtain a complete description of the solution set, (see e.g. (Henon 1977; Bruno 1981). However, for simplicity we consider only solutions with large n corresponding to many rotations of collision orbits along the Kepler ellipse. Then no technique is needed except the implicit function theorem. The computation below is similar to that of Marco and Niderman (1995) for collision orbits with fixed Jacobi's constant. However, we need more information on time intervals since the perturbed system is nonautonomous.

We will find values of θ for which system (6.1)–(6.3) has a non-degenerate[12] solution $a(\theta)$, $e(\theta)$, $\eta(\theta)$. Equation (6.3) implies

$$e=e(a,\theta)=(2a)^{-1}(-\cos\theta\pm\sqrt{\cos^2\theta+4a(a-1)}). \tag{6.4}$$

For $a>1$ the solution $e\in(0,1)$ is non-degenerate for θ in the intervals $L_\pm=L_\pm(a)$, where $L_-=(-\pi,0)$ and $L_+=(0,\pi)$. We have to take plus in (6.4) to ensure that $e>0$. The range of the function $\theta\in L_\pm\mapsto e_+(a,\theta)$ is $(1-a^{-1},1)$.

For $1/2<a<1$ we need to assume that $\cos^2\theta>4a(1-a)$. Let $\vartheta(a)=\arccos(-\sqrt{4a(1-a)})$. A non-degenerate solution $e\in(0,1)$ of (6.3) exists for θ in $L_-(a)=(-\pi,-\vartheta(a))$ or $L_+(a)=(\vartheta(a),\pi)$, and we may take both plus and minus in (6.4). Let $e_\pm(a,\theta)$ be the corresponding functions. The range of the function $\theta\in L_\pm(a)\mapsto e_+(a,\theta)$ is $(\sqrt{a^{-1}-1},1)$, and the range of the function $\theta\in L_\pm(a)\mapsto e_-(a,\theta)$ is $(a^{-1}-1,\sqrt{a^{-1}-1})$.

[12]This means that the Jacobian with respect to a,e,η has rank 3.

For $a=1$ we need to take θ in $L_-(1)=(-\pi,-\pi/2)$ or $L_+(1)=(\pi/2,\pi)$, and $\theta\in L_\pm(1)\mapsto e_+(1,\theta)=-\cos\theta$ has the range $(0,1)$.

To take care of all cases simultaneously, we introduce the following notation. Set $\alpha=\pm$ in $L_\pm(a)$. For $1/2<a<1$ set $\beta=\pm$ in $e_\pm(a,\theta)$, and for $a\geqslant 1$ set $\beta=+$. Equation (6.2) gives

$$\eta=\eta_{\alpha\beta}(a,\theta)=\alpha\ \arccos(a^{-1}\cos\theta+e_\beta(a,\theta)),\quad \theta\in L_\alpha(a), \tag{6.5}$$

where α and β take values $+$ or $-$ as explained above. Substituting (6.4) and (6.5) in (6.1), we get

$$\omega=\nu+(\pi n)^{-1}(-\omega\theta+\eta_{\alpha\beta}(\omega^{-2/3},\theta)-e_\beta(\omega^{-2/3},\theta)\sin\eta_{\alpha\beta}(\omega^{-2/3},\theta)), \tag{6.6}$$

where $\nu=m/n$ and $\theta\in L_\alpha(a)$.

For large n on the right-hand side of Equation (6.6) is a contraction with respect to ω, and hence it can be solved for $\omega=\nu+O(n^{-1})$. More precisely, fix small $\delta>0$ and take sufficiently large $N>0$. Let K be the set of triples $k=(\nu,\alpha,\beta)$, where $\nu=m/n$ is a rational number with relatively prime $m\in\mathbb{Z}$, $n>N$ such that $\delta<|\nu|<1-\delta$ or $1+\delta<|\nu|<2^{3/2}-\delta$, $\alpha=\pm$, and $\beta=+$ if $|\nu|<1$, and $\beta=\pm$ if $|\nu|>1$. We delete δ-neighborhoods of the endpoints of the intervals $L_\alpha(\nu^{-2/3})$ and set

$$L_k=\{\theta:[\theta-\delta,\theta+\delta]\subset L_\alpha(\nu^{-2/3})\}.$$

If $N=N(\delta)$ is sufficiently large, then for any $k=(\nu,\alpha,\beta)\in K$ and $\theta\in L_k$ Equation (6.6) has a non-degenerate analytic solution

$$\omega=\omega_k(\theta)=\nu+(\pi n)^{-1}R_k(\theta), \tag{6.7}$$

$$R_k(\theta)=-\theta+\eta_{\alpha\beta}(\nu^{-2/3},\theta)-e_{\alpha\beta}(\nu^{-2/3},\theta)\sin\eta_\beta(\nu^{-2/3},\theta)+O(n^{-1}). \tag{6.8}$$

Then

$$a=\omega_k^{-2/3}(\theta)=\nu^{-2/3}+\tfrac{2}{3}n^{-1}\nu^{-5/3}R_k(\theta)+O(n^{-2}).$$

Substituting in (6.4), we get

$$e=e_k(\theta)=\tfrac{1}{2}\nu^{2/3}\left(-\cos\theta+\beta\sqrt{\cos^2\theta+4\nu^{-2/3}(\nu^{-2/3}-1)}\right)+O(n^{-1}). \tag{6.9}$$

For $\delta\to 0$ and $N\to\infty$ and the range $e_k(L_k)$ approaches one of the intervals $(1-\nu^{2/3},1)$ or $(\sqrt{\nu^{2/3}-1},1)$ or $(\nu^{2/3}-1,\sqrt{\nu^{2/3}-1})$, depending on $|\nu|$ and α.

Thus for $k\in K$ and $\theta\in L_k$ we obtain a countable set of families of non-degenerate solutions $a_k(\theta)$, $e_k(\theta)$, $\eta_k(\theta)$ of system (6.1)–(6.3). Let $I_k=2L_k+2\pi n$. If $\theta\in L_k$, then $s=2\theta+2\pi n\in I_k$. We obtain:

LEMMA 6.1. *For any $k\in K$ and $s\in I_k$ there exists a unique non-degenerate elliptic collision orbit $\sigma\colon [0,s]\to\mathbb{R}^2\setminus\{0\}$ of the Kepler problem such that $\sigma(0)=1$, $\sigma(s)=e^{is}$. The action function $A(\sigma)=\Psi_k(s)$ satisfies the twist condition $\Psi_k''\neq 0$ in $\bar I_k$. Moreover*

$$\operatorname{sign}\Psi_k''=\iota_k=\alpha\beta\operatorname{sign}\nu,\quad k=(\nu,\alpha,\beta). \tag{6.10}$$

The first statement is already proved. Next we prove (6.10). By (3.3), $\Psi_k'(s)=-h$, where h is the relative Hamiltonian at the collision. In the circular case, h is Jacobi's integral $h=E-G$, where E is the energy of the Asteroid, and G is the angular momentum. By (1.3),

$$E=-\tfrac{1}{2}\omega^{2/3},\qquad G=\omega^{-1/3}(1-e^2)^{1/2}, \tag{6.11}$$

where ω and e are given by (6.7) and (6.9). Then

$$\Psi_k'(s)=-h=\tfrac{1}{2}\omega^{2/3}+G. \tag{6.12}$$

Since $\omega=\omega_k(\theta)$ is constant up to $O(n^{-1})$, for large n we have

$$\Psi_k''(s)=-\omega^{-1/3}(1-e^2)^{-1/2}eD_1e(\theta,a)+O(n^{-1}).$$

Differentiating (6.3) we get

$$D_1e(\theta,a)=\frac{e\sin\theta}{\cos\theta+2ae}.$$

Since $\sin\theta\neq 0$ for $\theta\in\bar L_k$, we have $\Psi_k''(s)\neq 0$ in $\bar I_k$ for large n.

By (6.4), $\cos\theta+2ae=\beta\sqrt{\cos^2\theta+4a(a-1)}$. Hence for large n the sign of $\Psi_k''(s)$ equals the sign of $\beta\nu\sin\theta$. Lemma 6.1 is proved. □

For large n we can obtain an asymptotic formula for the inverse function $s=\rho_k(h)$ of $h=-\Psi_k'(s)$. First we compute its domain $J_k=-\Psi_k'(I_k)$.

By (6.11) and (6.12), for $a>1$ the interval $e\in(1-a^{-1},1)$ corresponds to $0<|h+(2a)^{-1}|<(2-a^{-1})^{1/2}$. Hence, for given $k=(\nu,\alpha,\beta)\in K$ with $|\nu|<1$ and $\beta=+$, the domain of $\rho_k(h)$ is an interval J_k such that

$$J_k\subset(-\nu^{2/3}/2,\,-\nu^{2/3}/2+(2-\nu^{2/3})^{1/2}),\quad \nu>0,$$
$$J_k\subset(-\nu^{2/3}/2-(2-\nu^{2/3})^{1/2},\,-\nu^{2/3}/2),\quad \nu<0.$$

For $\delta\to 0$ and $n\to\infty$, J_k approaches the corresponding interval.

For $1/2<a<1$, the interval $e\in(\sqrt{a^{-1}-1},1)$ corresponds to $0<|h+(2a)^{-1}|<\sqrt{2a-1}$, and the interval $e\in(a^{-1}-1,\sqrt{a^{-1}-1})$ corresponds to $\sqrt{2a-1} < |h+(2a)^{-1}| < \sqrt{2-a^{-1}}$. Hence for $k=(\nu,\alpha,\beta)\in K$ with $|\nu|>1$, the domain of $\rho_k(h)$ is an interval J_k such that

$$\begin{aligned}
&J_k \subset (-\nu^{2/3}/2, -\nu^{2/3}/2+\sqrt{2\nu^{-2/3}-1}), && \nu<0,\ \beta>0,\\
&J_k \subset (-\nu^{2/3}/2-\sqrt{2\nu^{-2/3}-1}, -\nu^{2/3}/2), && \nu>0,\ \beta>0,\\
&J_k \subset (-\nu^{2/3}/2+\sqrt{2\nu^{-2/3}-1}, -\nu^{2/3}/2+\sqrt{2-\nu^{2/3}}), && \nu<0,\ \beta<0,\\
&J_k \subset (-\nu^{2/3}/2-\sqrt{2-\nu^{2/3}}, -\nu^{2/3}/2-\sqrt{2\nu^{-2/3}-1}), && \nu>0,\ \beta<0
\end{aligned}$$

and it approaches the corresponding interval for $\delta\to 0$ and $n\to\infty$.

A formula for $\rho_k(h)$ is obtained by solving (6.1) for $\cos\theta$:

$$\cos\theta = e^{-1}(a(1-e^2)-1) = (G^2-1)(1-\nu^{2/3}G^2)^{-1/2} + O(n^{-1}),$$

where

$$G = G_k(h) = -\nu^{2/3} - h + O(n^{-1}). \tag{6.13}$$

Hence

$$s = \rho_k(h) = 2\pi n + 2\alpha \arccos\big((G^2-1)(1-\nu^{2/3}G^2)^{-1/2}\big) + O(n^{-1}). \tag{6.14}$$

We obtain:

LEMMA 6.2. *For any $k\in K$ and $h\in J_k$ there exists a non-degenerate elliptic collision orbit $\sigma=\sigma_k(h)\colon [0,\rho_k(h)]\to\mathbb{R}^2\backslash\{0\}$ of the Kepler problem with energy $E_k(h)$, angular momentum $G_k(h)$, and Jacobi's integral $E_k(h)-G_k(h)=h$. Moreover:*

- *$\rho_k'(h)\neq 0$ in J_k and* sign $\rho_k'(h) = -\iota_k$.
- *For any closed interval $\Delta\subset(-3/2,\sqrt{2})$ and sufficiently small $\delta>0$, the intervals $(J_k)_{k\in K}$ cover Δ, and any $h\in\Delta$ is contained in an infinite number of intervals $\{J_k\}_{k\in K(h)}$.*
- *The set $\{G_k(h)\}_{k\in K(h)}$ of angular momenta is dense in $(2-\sqrt{4h+6}, -h)$, except for a $O(\delta)$-neighborhood of the boundary.*

The last statement follows from the description of intervals J_k and (6.13). □

In fact we missed one condition in the definition of a collision orbit. We need to verify that the collision orbit $\sigma\colon[0,s]\to\mathbb{R}^2$, $s=\rho_k(h)$, has no early collisions: $\sigma(t)\neq e^{it}$ for $t\in(0,s)$. If there is a collision at $t=\tau$, then $e^{i\tau}=1$ or $e^{i\tau}=e^{is}$. Suppose for example that $e^{i\tau}=e^{is}$. Then $s-\tau=2\pi p$ and $\omega(s-\tau)=2\pi q$ for some $p\in\mathbb{N}$ and $q\in\mathbb{Z}$. This means that Jupiter and Asteroid make

p full rotations in the time interval $[\tau, s]$ colliding twice at the same point. Then $\omega = q/p$ with $|q| < |m|$, $p < n$. By (6.7), we obtain the equation

$$R_k(\theta) = \pi(nq - mp)/p, \qquad \theta = s/2 - \pi n.$$

We will see that the analytic function $R_k(\theta)$ is non-constant in a neighborhood of $\bar{L}_k$. Hence there is a finite (or empty) set C_k of $s \in I_k$ which give collision orbits admitting early collisions. We delete these values of s thus replacing I_k by a finite union $\tilde{I}_k = I_k \setminus C_k$ of smaller intervals.

It is convenient to express $R_k(\theta)$ as a function of $h \in J_k$. By (6.8),

$$\begin{aligned} R_k(\theta) &= R_k(\rho_k(h)/2 - \pi/2) = \alpha r_k(h) + O(n^{-1}), \\ r_k(h) &= \arccos \frac{1 - \nu^{2/3}}{\sqrt{1 - \nu^{2/3} G^2}} - \arccos \frac{G^2 - 1}{\sqrt{1 - \nu^{2/3} G^2}} \\ &\quad - \nu^{1/3} \sqrt{2 - \nu^{2/3} - G^2}, \end{aligned} \tag{6.15}$$

where G is given by (6.13). Evidently, $r_k(h)$ is a non-constant analytic function of h. Hence if N is sufficiently large, for any $k \in K$ there is at most a finite set D_k of $h \in J_k$ such that the corresponding collision orbit σ has early collisions. We delete these values of h replacing J_k by a finite union $\tilde{J}_k = J_k \setminus D_k$ of smaller intervals. Then $\tilde{I}_k = \rho_k(\tilde{J}_k)$.

The deleted values of $h \in D_k$ correspond to shorter collision orbits $\tilde{\sigma}$: $[0, \tau] \to \mathbb{R}^2$ with $\nu = m/n$ replaced by $\tilde{\nu} = (m - q)/(n - p)$. Thus $k = (\nu, \alpha, \beta)$ is replaced by $\tilde{k} = (\tilde{\nu}, \alpha, \beta)$ and I_k is replaced by $I_{\tilde{k}} = I_k - 2\pi p$. To insure non-degeneracy of $\tilde{\sigma}$ we have to assume that the collision is not too early, i.e. $p < n - N$.

We know that for small $\delta > 0$ any $h \in \Delta$ is contained in an infinite number of intervals $\{J_k\}_{k \in K(h)}$. For an infinite number of $k \in K(h)$ the orbit $\sigma = \sigma_k(h)$ has no early collisions, and so $h \in \tilde{J}_k$. Indeed, for fixed h, $r_k(h)$ is a non-trivial analytic function of ν. This is evident from (6.13) and (6.15), for example by looking at one of the singular points. Now Lemma 6.2 is completely proved with the intervals I_k and J_k replaced by $\tilde{I}_k$ and $\tilde{J}_k$. We will drop tildes for simplicity.

To find non-degenerate collision chains, we need to take care of the changing direction condition (3.6). Let us compute the relative collision velocities $v_k^{\pm} = v_k^{\pm}(0, s)$ of the collision orbit σ in Lemma 6.2. Represent the velocity of σ in the moving basis:[13]

$$\dot{\sigma}(t) = v_1(t) e^{it} + v_2(t) i e^{it}. \tag{6.16}$$

[13]We write vectors in $\mathbb{R}^2$ as complex numbers.

The tangent component $v_2 = G_k(h)$ at the collisions $t=0$ or $t=s$ equals the angular momentum (6.13), while the radial component v_1 can be obtained from the energy integral at the collision:

$$E = \tfrac{1}{2}(v_1^2 + v_2^2) - 1, \quad v_1 = \pm\sqrt{2+2E-G^2}.$$

By (6.12) and (6.13), the radial velocity at the collision $t=0$ is

$$v_1(0) = V_k(h) = \pm\sqrt{2-\nu^{2/3}-(h+\nu^{2/3}/2)^2} + O(n^{-1}). \tag{6.17}$$

For $k=(\nu,\alpha,\beta)$ with $\alpha\nu>0$, we take plus, and for $\alpha\nu<0$ we take minus.

The relative velocity v_k^- at the collision $t=0$ has components $v_2(0)-1$ and $v_1(0)$ in the moving basis. Hence

$$v_k^- = V_k(h) + (G_k(h)-1)i,$$
$$v_k^+ = (-V_k(h) + (G_k(h)-1)i)e^{is}.$$

The absolute value is $|v_k^\pm| = \sqrt{2h+3}$.

Take some $k=(\nu_k,\alpha_k,\beta_k)$ and $l=(\nu_l,\alpha_l,\beta_l)$ in K such that $h\in J_k\cap J_l$. Then Lemma 6.2 gives a pair of collision orbits σ_k,σ_l with $E_k - G_k = E_l - G_l = h$. They can be linked in a collision chain $\sigma_k, \tilde\sigma_l$, where σ_l is replaced by an appropriate time translation $\tilde\sigma_l(t) = e^{is}\sigma_l(t-s)$. The changing direction condition is not satisfied at $t=s$ if $v_k^+ = e^{is}v_l^-$ or $v_k^+ = -e^{is}v_l^-$. In the first case $G_k = G_l$ and $V_k = -V_l$. In the second case $G_k + G_l = 2$ and $V_k = V_l$. By (6.13) and (6.17) the changing direction condition holds automatically for large N if $|\nu_k| \neq |\nu_l|$ and $\nu_k^{2/3} + \nu_l^{2/3} \neq -4h-2$.

Even if there is an equality, there is still the radial velocity to take into account. For example, suppose that $\nu_k^{2/3} + \nu_l^{2/3} = -4h-2$. Then if $\alpha_k\nu_k$ and $\alpha_l\nu_l$ are of opposite sign, we have $V_k \neq V_l$, and hence the changing direction condition holds. We obtain:

LEMMA 6.3. *The changing direction condition holds for the link $\sigma_k, \tilde\sigma_l$ of a collision chain except maybe in one of the following cases:*

- $\alpha_k\nu_k = -\alpha_l\nu_l$,
- $\nu_k^{2/3} + \nu_l^{2/3} = -4h-2$ *and* $\alpha_k \operatorname{sign} \nu_k = -\alpha_l \operatorname{sign} \nu_l$.

For example, the case $\nu_k = \nu_l$ and $\alpha_k = -\alpha_l$ corresponds to continuing after the collision along the same Kepler ellipse. We have to avoid such trivial collision chains. If $k=l$ then the changing direction condition always holds.

7. Collision Orbits for the Almost Circular Problem

Suppose that Jupiter's orbit has small eccentricity ϵ. Then Jupiter's position is

$$u(t) = u(t, 0, \epsilon) = e^{it} + \epsilon\xi(t) + O(\epsilon^2).$$

By Lemma 6.1, for any $k \in K$ and $t_0 \in \mathbb{R}$, $s = t_1 - t_0 \in I_k$, there exists a non-degenerate collision orbit $\sigma\colon [t_0, t_1] \to \mathbb{R}^2$ of the Kepler problem with $\sigma(t_0) = e^{it_0}$, $\sigma(t_1) = e^{it_1}$. Let $h = -\Psi_k'(s)$ be its Jacobi's integral. By the implicit function theorem, we obtain:

LEMMA 7.1. *For small $\epsilon > 0$ there exists a non-degenerate collision orbit γ: $[t_0, t_1] \to \mathbb{R}^2 \setminus \{0\}$ joining $u(t_0)$ and $u(t_1)$, and $\gamma(t) = \sigma(t) + \epsilon\zeta(t) + O(\epsilon^2)$. The action $A(\gamma) = S_k(t_0, t_1)$ has the form (4.3), where*

$$\psi_k(t_0, s) = 2G_k(h)(\sin(t+s) - \sin t) + V_k(h)(\cos(t+s) + \cos t). \qquad (7.1)$$

If there are no early collisions for σ, then for small ϵ there will be no early collisions for γ.

To prove (7.1), let us compute $A(\gamma)$ by using the first variation formula (2.3):

$$A(\gamma) = A(\sigma) + \epsilon \mathrm{d}A(\sigma)(\zeta) + O(\epsilon^2) = A(\sigma) + \epsilon\dot{\sigma}(t) \cdot \zeta(t)\big|_{t_0}^{t_1} + O(\epsilon^2).$$

Since $\xi(t_0) = \zeta(t_0)$ and $\xi(t_1) = \zeta(t_1)$, we obtain

$$\psi_k(t_0, s) = \dot{\sigma}(t_1) \cdot \xi(t_1) - \dot{\sigma}(t_0) \cdot \xi(t_0). \qquad (7.2)$$

As in (6.16), we represent $\xi(t)$ in the moving basis:

$$\xi(t) = \xi_1(t)e^{it} + \xi_2(t)ie^{it}. \qquad (7.3)$$

To compute $\xi_1(t)$ and $\xi_2(t)$, write the equation of Jupiter's orbit $u(t) = re^{i\theta}$ in polar coordinates:

$$r = 1 - \epsilon \cos \eta, \qquad r \sin \theta = \sqrt{1 - \epsilon^2} \sin \eta, \qquad t = \eta - \epsilon \sin \eta,$$

where η is the eccentric anomaly. We used that the major axis equals 1 and the period equals 2π. Solving for r, θ we get

$$r = 1 - \epsilon \cos t + O(\epsilon^2), \qquad \theta = t + 2\epsilon \sin t + O(\epsilon^2).$$

Hence

$$\xi_1(t) = -\cos t, \qquad \xi_2(t) = 2 \sin t. \qquad (7.4)$$

Using (6.16) and (7.3)–(7.4), we compute the dot products in (7.2) and obtain

$$\begin{aligned}\psi_k(t_0, s) &= V_k(h)(\xi_1(t_1) + \xi_1(t_0)) + G_k(h)(\xi_2(t_1) - \xi_2(t_0)) \\ &= 2G_k(h)(\sin(t+s) - \sin t) + V_k(h)(\cos(t+s) + \cos t),\end{aligned}$$

where $G_k(h)$ and $V_k(h)$ are given by (6.13) and (6.17). Now Lemma 7.1 is proved. □

LEMMA 7.2. *For the elliptic* 3 *body problem with small eccentricity, the functions* $\{\chi_k\}_{k\in K}$ *in* (4.9) *have the form*

$$\chi_k(t, h) = B_k(h) \sin t, \quad (t, h) \in \mathbb{T} \times J_k. \tag{7.5}$$

For any $h \in \Delta$ *and* $k \in K(h)$, *the set* $\{l \in K(h) : B_l(h) = B_k(h)\}$ *is finite.*

Proof. Let $\phi_k(t, h) = \psi_k(t, \rho_k(h))$. Rewriting (7.1) in a complex form, we get (4.7), where

$$a_k(h) = c_k + \bar{c}_k e^{i\rho_k}, \quad c_k = V_k/2 + iG_k.$$

A computation shows that the coefficients in (4.13) are

$$b_k = -iB_k/2, \quad B_k(h) = \frac{V_k \sin \rho_k}{1 - \cos \rho_k} + 2G_k.$$

Hence $\chi_k(t, h)$ has the form (7.5).

Using (6.13), (6.14) and (6.17), it is easy to see that for fixed h, $B_k(h)$ is a non-trivial analytic function of ν. Hence for any $k \in K(h)$, there exist at most a finite number of $l \in K(h)$ such that $B_k(h) = B_l(h)$. □

Proof of Theorem 2.2. Take a closed interval $\Delta \subset (-3/2, \sqrt{2})$ and sufficiently small $\delta > 0$. Then for any $h \in \Delta$, $L(h)$ in (4.19) is an infinite set. We modify the definition of $L(h)$ a little.

If there is a pair $k, l \in L(h)$ which does not satisfy the condition of Lemma 6.3, we delete one of k or l from $L(h)$. We do the same also if $B_k(h) = B_l(h)$. Let $\tilde{L}(h)$ be the remaining set. If $k, l \in \tilde{L}(h)$, the changing direction condition holds for the collision orbits σ_k, σ_l of the circular problem. Then for sufficiently small $\epsilon > 0$ it will hold also for collision orbits γ_k, γ_l of the elliptic problem which are perturbations of time translations of σ_k, σ_l.

By Lemma 7.2, $\tilde{D}(h) = \{(k, l) \in \tilde{L}(h) \times \tilde{L}(h) : k \neq l\}$ is an infinite set. We replace $D(h)$ by $\tilde{D}(h) \subset D(h)$ in Theorem 4.3 and take any finite set

$\Lambda \subset \tilde{D}(h)$. For each of the fixed points of $f_{\mathbf{k}}$ in Theorem 4.3, we get non-degenerate collision chains of the form (2.6). The changing direction condition will hold automatically.

Taking all possible sets $\Lambda \subset \tilde{D}(h)$, we obtain the assertion of Theorem 2.2 for $h \in \Delta$, with the set of angular momenta $\{G_k\}_{k \in \tilde{L}(h)}$ dense in $(2 - \sqrt{4h+6}, -h)$, except for an $O(\delta)$-neighborhood of the boundary. Taking $\delta \to 0$ and $\Delta \to (-3/2, \sqrt{2})$ we complete the proof of Theorem 2.2.

Note that we can get much more collision chains than is required for the proof of Theorem 2.2.

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