GLOBAL REGULARIZATION OF THE RESTRICTED PROBLEM OF THREE BODIES

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(Received: 4 November 2003; revised: 6 January 2004; accepted: 13 January 2004)

Abstract. The global regularizing transformations of the planar, circular restricted problem of three bodies are studied. It is shown that all these transformations can be written in the same general form which is the solution of a first order ordinary differential equation.

Key words: global regularization, restricted three-body problem

1. Introduction

The equations of motion of the planar, circular restricted problem of three bodies are (Szebehely, 1976)

$$\ddot{x} - 2\dot{y} = \Omega_x, \qquad \ddot{y} + 2\dot{x} = \Omega_y, \tag{1}$$

where

$$\Omega = \frac{1}{2}\left[(1-\mu)r_1^2 + \mu r_2^2\right] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2},$$
$$r_1 = [(x-\mu)^2 + y^2]^{1/2}, \qquad r_2 = [(x+1-\mu)^2 + y^2]^{1/2}.$$

The configuration of the system is shown in Figure 1. Equations (1) determine the motion of the third body P_3 of negligible mass, the primaries P_1 and P_2 have masses $1-\mu$ and μ, and coordinates $(\mu, 0)$, $(\mu - 1, 0)$, where μ is the mass parameter ($0 < \mu \leqslant 0.5$).

Equations (1) have one first integral, the Jacobian integral

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C, \tag{2}$$

where C is the Jacobian constant.

Equations (1) are singular for $r_1 = 0$, $r_2 = 0$, that is for collision of the third body with either one of the primaries. Motion near or through collisions can be studied only by using regularized equations of motion. There is an extended literature on regularization. Classical references can be found in Szebehely (1967), and in Hagihara (1975). A recent treatment on regularization is given by Celletti (2002).

Celestial Mechanics and Dynamical Astronomy **90:** 35–42, 2004.

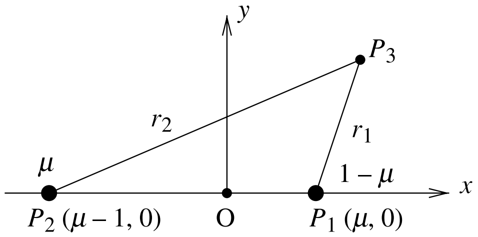

Figure 1. Configuration of the restricted three-body problem.

Regularization of the equations of motion can be achieved by coordinate and time transformations. The time transformation is essential, the idea is to introduce a new time scale in which the motion near singularity slows down. This can be simply done by relating the new and old velocities in a way that we multiply the old velocity by a variable which decreases towards the singularity thus balancing the increasing velocity. Considering for simplicity a one dimensional motion

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = r\frac{\mathrm{d}x}{\mathrm{d}t},$$

where τ is the new time variable, and r $(= |x|)$ is the distance between the bodies. From this

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = r, \tag{3}$$

which is a special case of the Sundman type time transformation (Sundman, 1912). Actually, other functions of r could be on the right hand side. It is good to remember that applying a time transformation the aim is to keep the regularized velocity low, therefore the distances between the interested bodies should be present on the right hand side of the equation of the time transformation.

It is convenient to write regularizing transformations in complex variables. Introducing $z = x + \mathrm{i}y$ $(\mathrm{i} = \sqrt{-1})$, Equations (1) can be written as

$$\ddot{z} + 2\mathrm{i}\dot{z} = \Omega_z, \tag{4}$$

where $\Omega_z = \Omega_x + \mathrm{i}\Omega_y$, and in Ω the distances are $r_1 = |z-\mu|$, $r_2 = |z+1-\mu|$. A coordinate and time transformation can be written in a general form as

$$z = f(w), \qquad \frac{\mathrm{d}t}{\mathrm{d}\tau} = g(w), \tag{5}$$

where f is a complex and g a real function of the complex variable $w = u + \mathrm{i}v$. Assuming

$$g = h(w)\bar{h}(w) = |h|^2, \tag{6}$$

where bar over h means complex conjugate, Equation (4) can be transformed into

$$w'' + \left(\frac{f^{**}}{f^*} - \frac{h^*}{h}\right) w'^{\,2} - \frac{\bar{h}^*}{\bar{h}}|w'|^2 + 2\mathrm{i}|h|^2 w' = \frac{|h|^4}{|f^*|^2}\Omega_w, \tag{7}$$

where prime and star mean derivation according to τ and w, respectively.

This equation can be simplified by eliminating the terms containing w'^2 and $|w'|^2$. The term containing w'^2 can be eliminated by assuming $f^* = h$. The price of this is that the time transformation and the coordinate transformation are no longer independent, but according to Equations (5) and (6)

$$z = f(w), \qquad \frac{\mathrm{d}t}{\mathrm{d}\tau} = |f^*|^2. \tag{8}$$

The term containing $|w'|^2$ can be eliminated by using the Jacobian integral, which in the new variables and already using the assumption $f^* = h$ can be written as

$$|w'|^2 = |f^*|^2(2\Omega - C). \tag{9}$$

After the simplification Equation (7) takes the form

$$w'' + 2\mathrm{i}|f^*|^2 w' = U_w, \tag{10}$$

where

$$U = |f^*|^2(\Omega - \tfrac{1}{2}C). \tag{11}$$

Equation (11) shows the essence of regularization: properly selecting the function f, which determines the time and coordinate transformations through Equations (8), the factor $|f^*|^2$ can simplify out the critical denominators r_1 and r_2 in the potential function Ω.

2. Regularizing Transformations

Several regularizing transformations are known. The Levi-Civita (1906) parabolic transformation

$$z = w^2, \tag{12}$$

can regularize a singularity in the origin of the coordinate system, and it is used to regularize the planar case of the two-body problem. It can also be applied to the restricted three-body problem, and for this it is enough to translate the origin of the coordinate system to any one of the primaries. Thus

$$z = f(w) = w^2 + \mu, \qquad \frac{\mathrm{d}t}{\mathrm{d}\tau} = |f^*|^2 = 4|w|^2, \tag{13}$$

regularize the equations of motion in P_1. Since now $r_1 = |z - \mu| = |w|^2$, thus $|f^*|^2 = 4r_1$, and this is the reason, why the singularity at P_1 is transformed out.

Similarly,

$$z = f(w) = w^2 - 1 + \mu, \qquad \frac{\mathrm{d}t}{\mathrm{d}\tau} = |f^*|^2 = 4|w|^2, \tag{14}$$

regularize the equations of motion in P_2. Now $r_2 = |z+1-\mu| = |w|^2$, and $|f^*|^2 = 4r_2$. The transformations (13) and (14) eliminate only one of the singularities, therefore they are called local regularizations.

There are transformations, which can remove both singularities simultaneously. These are called global regularizations (though mathematically they are local operations). These transformations can be most conveniently given in a coordinate system, where the primaries are located symmetrically with respect to the origin. So we translate the origin of the coordinate system to the midpoint of the primaries by the transformation

$$q = z + \tfrac{1}{2} - \mu. \tag{15}$$

Then the primaries will be located at $q = \pm 1/2$. The equation of motion (4) changes to

$$\ddot{q} + 2\mathrm{i}\dot{q} = \Omega_q, \tag{16}$$

where in Ω the distances are $r_1 = |q-(1/2)|$, $r_2 = |q+(1/2)|$. The transformations

$$q = f(w), \qquad \frac{\mathrm{d}t}{\mathrm{d}\tau} = |f^*|^2, \tag{17}$$

bring Equation (16) into Equation (10).

The Thiele–Burrau transformation, introduced by Thiele (1895) for $\mu = 0.5$ and generalized by Burrau (1906) for arbitrary μ, is

$$q = \tfrac{1}{2}\cos w. \tag{18}$$

In this case $|f^*|^2 = (1/4)|\sin w|^2$, and since $r_1 = |(1/2)\cos w - (1/2)|$, $r_2 = |(1/2)\cos w + (1/2)|$, thus $|f^*|^2 = r_1 r_2$. Therefore the two critical denominators r_1 and r_2 in the potential function Ω in Equation (11) are regularized together.

The Birkhoff (1915) transformation is

$$q = \frac{1}{4}\left(2w + \frac{1}{2w}\right). \tag{19}$$

It is easy to see that in this case $|f^*|^2 = r_1 r_2/|w|^2$. This function eliminates the singularities at P_1 and P_2, however, it introduces a new singularity $w = 0$ in the transformed plane. However, $w = 0$ corresponds to $q = \infty$ and thus all points in the finite physical plane are regularized by this transformation.

Birkhoff's transformation can be generalized to

$$q = \frac{1}{4}\left(w^2 + \frac{1}{w^2}\right) \tag{20}$$

(Arenstorf, 1963; Deprit and Broucke, 1963). This transformation is called Lemaître transformation due to its relation to the regularization of the general problem of three bodies by Lemaître (1955). In this case $|f^*|^2 = 4r_1 r_2/|w|^2$.

There are also other generalizations. Wintner (1930) generalized the Birkhoff transformation, which in the midpoint coordinate system can be expressed as

$$q = \frac{1}{2}\frac{(w + (1/2))^{2n} + (w - (1/2))^{2n}}{(w + (1/2))^{2n} - (w - (1/2))^{2n}}, \tag{21}$$

where n is any positive integer.

Broucke (1965) generalized the Thiele–Burrau transformation to

$$q = \tfrac{1}{2}\cos nw, \tag{22}$$

and the Birkhoff transformation to

$$q = \frac{1}{4}\left(w^n + \frac{1}{w^n}\right), \tag{23}$$

where n is any nonzero real number.

The Thiele–Burrau, the Birkhoff, and the Lemaître transformations were introduced at different times and by different reasonings. Therefore it is quite extraordinary that all these transformations, and their generalizations as well, can be written in the same general form

$$q = \frac{1}{4}\left[h(w) + \frac{1}{h(w)}\right], \tag{24}$$

where $h(w) = \mathrm{e}^{\mathrm{i}w}$, $2w$, w^2, $\mathrm{e}^{\mathrm{i}nw}$, w^n for the Thiele–Burrau, the Birkhoff, the Lemaître, and Broucke's two transformations, while in the case of the Wintner transformation

$$h = \frac{(w + (1/2))^{n} + (w - (1/2))^{n}}{(w + (1/2))^{n} - (w - (1/2))^{n}}, \tag{25}$$

as one can see.

It is an intriguing question, why all these transformations have the same general form, and although Equation (24) is mentioned in Szebehely's (1967) book in connection with the Thiele–Burrau, Birkhoff and Lemaître transformations, however there is no answer in the literature for this question.

3. Differential Equation of the Regularizing Transformations

Now we show that this interesting problem has a simple solution (Érdi, 1995). First we note that in order to eliminate the singularities at $r_1 = 0$, $r_2 = 0$, it should be that $|f^*|^2 = \gamma(w)(r_1 r_2)^n$, where the unknown function $\gamma(w)$ must be regular at the singularities, and n is a nonzero integer. Considering n, if $n > 1$, then it follows from Equation (11) that $U = 0$ at the singularities, and from Equations (9) and (10) that also $w' = 0$, $w'' = 0$ at the singularities, that is a steady-state solution is obtained, where the third body P_3 rests at the singularities, which can not be the case. So necessarily it should be that $n = 1$.

Thus $|f^*|^2 = \gamma(w) r_1 r_2$, and assuming that $\gamma = |\alpha(w)|^2$, where $\alpha(w)$ is unknown, and considering that $r_1 = |q-(1/2)| = |f(w)-(1/2)|$, $r_2 = |q+(1/2)| = |f(w) + (1/2)|$, the following first order differential equation can be obtained for the function $f(w)$, which determines the regularizing transformations (17)

$$\frac{\mathrm{d}f}{\mathrm{d}w} = \alpha\sqrt{f^2 - \frac{1}{4}}. \tag{26}$$

This equation can be integrated, giving

$$\ln(2f + \sqrt{4f^2 - 1}) = \int \alpha\,\mathrm{d}w = \beta(w),$$

where $\beta(w)$ is unknown, and the constant of integration may be set zero. Solving this equation for f we obtain

$$f = \frac{1}{4}\left(\mathrm{e}^{\beta} + \frac{1}{\mathrm{e}^{\beta}}\right). \tag{27}$$

Taking the unknown function $\beta(w)$ as $\beta = \ln h(w)$, where now $h(w)$ is unknown, Equation (27) leads to Equation (24)

$$q = f(w) = \frac{1}{4}\left[h(w) + \frac{1}{h(w)}\right]. \tag{28}$$

Thus this equation represents the unique form of the global regularizing transformations. It can be seen that

$$\gamma = \left|\frac{h^*}{h}\right|^2, \tag{29}$$

thus for a regularizing function $f(w)$ given by Equation (28), the critical multiplier will be

$$|f^*|^2 = \left|\frac{h^*}{h}\right|^2 r_1 r_2, \tag{30}$$

and here $h(w)$ can be freely selected.

This gives a possibility to define global regularizing transformations, different from the already known. The simplest case is when $\gamma = 1$. This can be achieved for $\beta = w$, or $\beta = \mathrm{i}w$. Actually, $\beta = \mathrm{i}w$ gives the Thiele–Burrau transformation, while $\beta = w$ results in $f = (1/2)\cosh w$. Both transformations, written explicitly in the coordinates, offer similar formalism of the equations of motion in elliptic coordinates. Another example for $\gamma = 1$ is obtained for $h = \mathrm{e}^{\mathrm{i}w}/\mathrm{i}$, which results in $f = (1/2)\sin w$. All these transformations can be generalized to $f = (1/2)\cos nw$

(Broucke, 1965), $f = (1/2)\cosh nw$, $f = (1/2)\sin nw$, with $\gamma = n^2$, where n is any nonzero real number.

Considering Equation (28) it is natural to look for the simplest form of the function $h(w)$, and thus one may take a linear function $h(w) = aw + b$, with a, b arbitrary constants. Among these transformations the only one which leaves the primaries at their places is obtained for $a = 2$, $b = 0$, and this is the Birkhoff transformation.

It seems that Equation (26) is not written in the literature. Interestingly, there are similar kinds of equation in more general problems. In a four-body problem, where three primaries of arbitrary masses move in circular orbits around their center of mass in the Lagrangian equilateral configuration, and a fourth body of negligible mass moves in their gravitational field, Giacaglia (1967) studied the following type of equation in connection with global regularization

$$\left|\frac{\mathrm{d}f}{\mathrm{d}w}\right|^2 = |\alpha(w)|^2 |f - f_1||f - f_2||f - f_3|.$$

For the global regularization of the magnetic-binary problem Mavraganis (1988) solved the equation

$$\left|\frac{\mathrm{d}f}{\mathrm{d}w}\right|^2 = r_1^3 r_2^3.$$

Other examples in more general cases can be found in Hagihara (1975).

Acknowledgements

The support of the Hungarian National Research Fund, grant number OTKA T043739 is acknowledged.

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