CAPTURE IN THE CIRCULAR AND ELLIPTIC RESTRICTED THREE-BODY PROBLEM

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Abstract. In this paper the authors provide a study of the phenomenon of the gravitational capture by using the models of the circular and elliptic restricted three-body problem. In the first part the inadequacy of the circular restricted three-body problem in the study of the phenomenon of the capture in the case of NEAs is shown. In the model of the spatial elliptic restricted three-body problem criteria of the capture are deduced by using the pulsating Hill-regions.

Key words: asteroid, capture, elliptic restricted three-body problem, Hill-region

1. Introduction

Gravitational capture of small bodies by major planets is an interesting phenomenon in planetary systems, having applications to the study of comets, asteroids and moons, and it can be studied by using different models of the celestial mechanics.

Several authors studied this problem, introducing different concepts of capture, like weak capture (Belbruno and Marsden, 1997; Belbruno, 1999), temporary capture (Brunini, 1996), longest capture (Vieira and Winter, 2001), resonant capture (Yu and Tremaine, 2001), etc. Brunini et al. (1996) studied the conditions of the capture in the restricted three-body problem. Murison (1989) pointed out connections between the gravitational capture and chaotic motions. An exciting study has been dedicated recently to the capture of irregular moons – with non-circular orbits – by giant planets (Astakhov et al., 2003). The authors confirmed with three-dimensional Monte Carlo simulations that irregular satellites are captured in a thin spatial region, where orbits are chaotic and that the resulting orbit is either prograde or retrograde depending on the initial energy.

In this paper we give some necessary conditions of the gravitational capture by using the Hill-regions in the spatial elliptic restricted three-body problem. We point out that the Hill-regions are invariant in the circular restricted three-body problem which does not happen in the case of the motion of real bodies, such as Near Earth Asteroids. The variation of the Hill-regions can be modeled in the spatial elliptic

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restricted three-body problem. By using this model a more adequate description of the gravitational capture can be given.

2. Hill-Regions and Capture in the CRTBP

In this part of our study we demonstrate that the circular restricted three-body problem (CRTBP) is inadequate to study the phenomenon of capture in some concrete cases. We illustrate this by using the Hill-regions and the trajectory of the asteroid 2002MN.

On 14 June 2002, an asteroid with a diameter between 50 and 120 m made one of the closest ever-recorded approaches to Earth. Astronomers working on Lincoln Near Earth Asteroid Research (LINEAR) search program first detected the giant rock on 17 June 2002.

A natural question arises: can the Earth capture such an asteroid? The simplest model possible to use in the study of the phenomenon of the capture is the CRTBP. This problem deals with the motion of a massless particle subjected to the gravitational attraction of two massive primaries, with masses m_1 and m_2, $(m_2 < m_1)$ revolving around their common center of mass in circular orbits, under the influence of their mutual gravitational attraction. In the synodic system – using the standard canonical system of units associated with this model: the unit of distance is the distance between the two primaries and the unit of time is chosen such that the period of the motion of m_2 around m_1 is 2π – the equations of motion of the third body are (Szebehely, 1967):

$$\ddot{x} - 2\dot{y} = \frac{\partial \Omega^\circ}{\partial x}, \qquad \ddot{y} + 2\dot{x} = \frac{\partial \Omega^\circ}{\partial y}, \qquad \ddot{z} = \frac{\partial \Omega^\circ}{\partial z}, \tag{1}$$

where Ω° is the pseudo-potential function given by

$$\Omega^\circ = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu), \tag{2}$$

where

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}, \qquad r_2 = \sqrt{(x+\mu-1)^2 + y^2 + z^2}, \tag{3}$$

and

$$\mu = \frac{m_2}{m_1 + m_2}. \tag{4}$$

The Hill-regions, where the motion of the massless particle is possible are bounded by the zero velocity surfaces of equation

$$2\Omega^\circ = C, \tag{5}$$

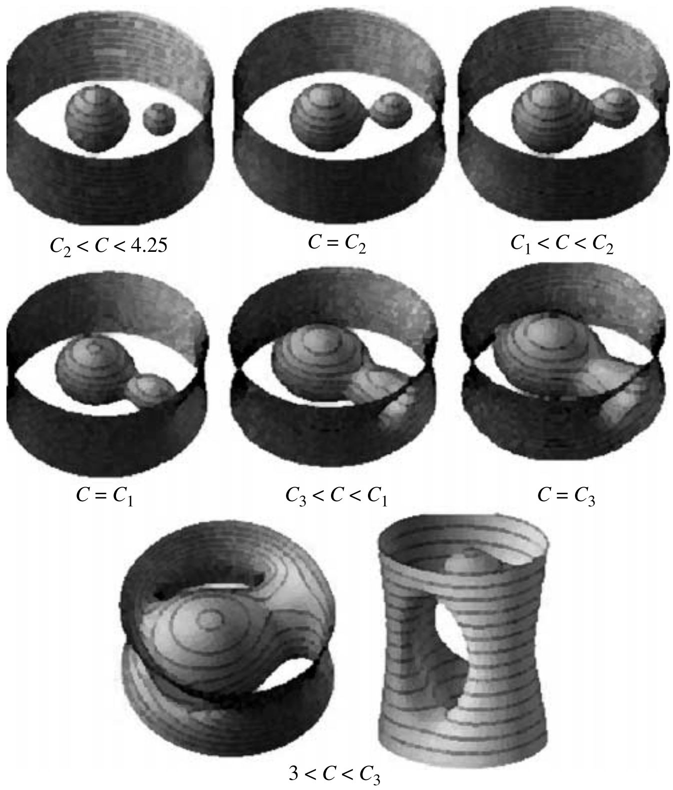

Figure 1. The Hill-zones for different value of C.

with constant C. The shape of the Hill-zones depends on the value of constant C (Figure 1). Each system of primaries is characterized by five critical values

$$C_i = 2\Omega^\circ(L_i), \qquad i = 1, \ldots, 5, \tag{6}$$

where L_i are the Lagrange-points. For these constants we have

$$3 = C_4 = C_5 \leqslant C_3 \leqslant C_1 \leqslant C_2 \leqslant 4.25$$

in generally, and in the case of the Sun–Earth system the critical value for L_2 between the two primaries is

$$C_2 = 3.000893278,$$

and in L_1, the Lagrange-point outside of the Earth the critical value is

$$C_1 = 3.000889276.$$

For $C > C_2$ the zero velocity surfaces delimit three regions where the motion of the small body is possible. Two of these regions are closed around the primaries, the

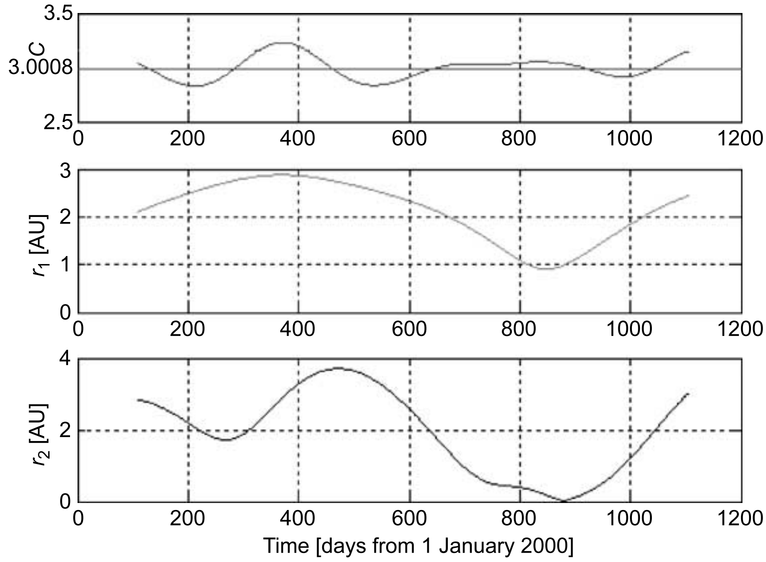

Figure 2. The variation of C in case of the asteroid 2002MN.

third one is the exterior of the cylinder. Between these regions the communication is impossible.

Using the asteroid 2002MN data from Digital Ephemeris 403 (DE403) at the date 2000. January 01.5, and transforming these data in the rotating coordinate system, we found for that moment

$$C > C_2.$$

This means that the asteroid cannot approach the Earth. The Hill-zones for this value are closed and the asteroid at this moment was outside of the external Hill-cylinder. Using then a fourth order Wisdom–Holman (Wisdom and Holman, 1991, 1992) n-body integrator we found that the value of C decrease with time (Figure 2), and the asteroid become closer to the Earth – the corresponding Hill-regions opening in time.

This phenomenon shows us that the model of the CRTBP is not adequate to study the eventually capture of the Near Earth Asteroids. It is necessary to have a such a model, which permits the change of the type of Hill-regions in time.

Such a model, which gives us the possibility to study the real motion of the Near Earth Asteroids is the spatial elliptic restricted three-body problem.

3. The Hill-Regions in the ERTBP

In the elliptic restricted three-body problem (ERTBP) the two massive primaries, with masses m_1 and m_2 revolve on elliptical orbits under their mutual gravitational attraction and the motion of a third, massless body is studied. The orbit of m_2 around m_1, in an inertial system is

$$r = \frac{a(1-e^2)}{1+e\cos f}, \tag{7}$$

where r is the mutual distance, a and e are the semimajor axis and the eccentricity of the elliptical orbit, and f is the true anomaly.

There are several systems of reference that can be used to describe the elliptic restricted three-body problem. In our study a non-uniformly rotating and pulsating coordinate system is used. In this system of reference the origin is in the center of mass of the two massive primaries (e.g., Sun and Earth), and the $\tilde{\xi}$ axis is directed towards m_2. The $\tilde{\xi}\tilde{\eta}$ coordinate-plane rotates with variable angular velocity, in such a way, that the two massive primaries are always on the $\tilde{\xi}$ axis, and the period of the rotation is 2π. Besides the rotation, the system also pulsates, to keep the primaries in fixed positions ($\tilde{\xi}_1 = -\mu, \tilde{\eta}_1 = \tilde{\zeta}_1 = 0, \tilde{\xi}_2 = 1-\mu, \tilde{\eta}_2 = \tilde{\zeta}_2 = 0$). In this system the equations of motion of the third massless particle are

$$\tilde{\xi}'' - 2\tilde{\eta}' = \frac{\partial\omega}{\partial\tilde{\xi}}, \qquad \tilde{\eta}'' + 2\tilde{\xi}' = \frac{\partial\omega}{\partial\tilde{\eta}}, \qquad \tilde{\zeta}'' = \frac{\partial\omega}{\partial\tilde{\zeta}}, \tag{8}$$

where the derivatives are taken with respect to the true anomaly f, and

$$\omega = (1+e\cos f)^{-1}\Omega,$$

with

$$\Omega = \frac{1}{2}(\tilde{\xi}^2 + \tilde{\eta}^2 - e\tilde{\zeta}^2\cos f) + \frac{1-\mu}{\sqrt{(\tilde{\xi}+\mu)^2 + \tilde{\eta}^2 + \tilde{\zeta}^2}} + \\ + \frac{\mu}{\sqrt{(\tilde{\xi}-1+\mu)^2 + \tilde{\eta}^2 + \tilde{\zeta}^2}} + \frac{1}{2}\mu(1-\mu). \tag{9}$$

Performing the same operations, which in the RTBP leads to the Jacobi-integral, in the case of the spatial ERTBP we obtain an invariant relation of the form:

$$\left(\frac{\mathrm{d}\tilde{\xi}}{\mathrm{d}f}\right)^2 + \left(\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}f}\right)^2 + \left(\frac{\mathrm{d}\tilde{\zeta}}{\mathrm{d}f}\right)^2 = 2\omega - e\int_0^f \frac{\tilde{\zeta}^2\sin h}{1+e\cos h}\,\mathrm{d}h - \\ - 2e\int_0^f \frac{\Omega\sin h}{(1+e\cos h)^2}\,\mathrm{d}h - C. \tag{10}$$

This is the generalization of Szebehely's invariant relation (Szebehely, 1967, pp. 595) for the spatial ERTBP. Assuming that the eccentricity $e > 0$ is small (as it is in the case of the Sun–Earth system, $e = 0.017$), the sum of the two integrals in Equation (10) is smaller than the term 2ω (In the case of the Sun–Earth system see

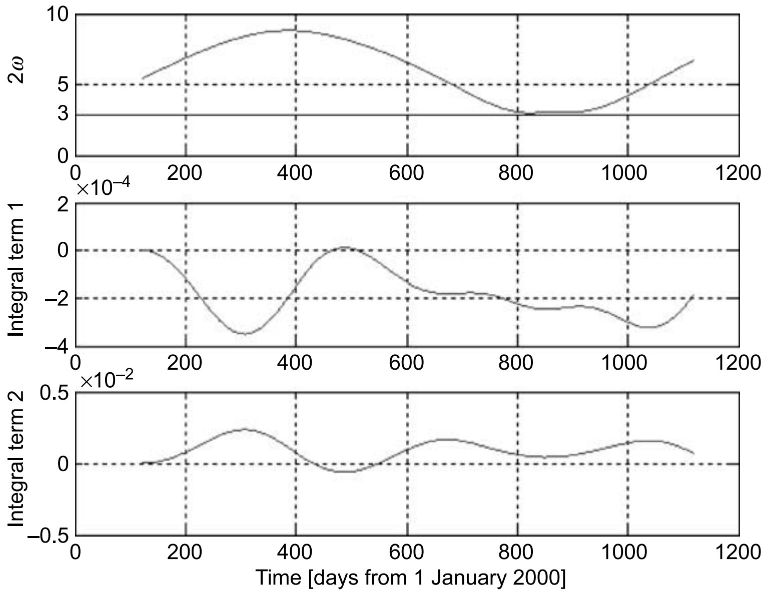

Figure 3. The integral terms in the Sun–Earth system.

Figure 3). Neglecting these small terms, we have the approximate equation of the surfaces of zero velocity

$$2\Omega - C\,(1 + e\cos f) = 0, \tag{11}$$

or

$$2\Omega - C^{*} = 0, \tag{12}$$

for each fixed value of f. Geometrically it means that at every time – or at every value of the true anomaly f – a different set of surfaces of zero velocity are to be constructed. The shape and dimension of these zero velocity surfaces vary in time. This variation is governed by

$$C^{*} = C(1 + e\cos f) \tag{13}$$

and therefore we might speak about pulsating surfaces of zero velocity.

4. Necessary Conditions of the Capture

To give necessary conditions of the capture, we approximate the zero velocity surfaces (12) with the equations

$$2\Omega^{\circ}(\tilde{\xi}, \tilde{\eta}, \tilde{\zeta}) = C^{*}. \tag{14}$$

This approximation is possible when e is small, and the third body moves near to the plane of the primaries. In this case the term $e\tilde{\zeta}^{2}\cos f$ in (9) may be neglected, and we can write $\Omega = \Omega^{\circ}$. Equations (13) and (14) show that the zero velocity surfaces pulsate, and so an do the Hill-zones delimited by them.

Suppose that, for $f = f_0$ the position and velocity of the third body is given, and

$$C^* = C_0^* = C\,(1 + e\cos f_0)\,. \tag{15}$$

Then, from (13) and (15) we have

$$C^*(f) = C_0^* \frac{1 + e\cos f}{1 + e\cos f_0}. \tag{16}$$

As we have seen in Section 2, as time as

$$C_2 < C^*,$$

the third body can not near to m_2 if initially is not inside of the Hill-region surrounding m_2. By using this property we are able to give a necessary condition to the close approach of one of the primaries by the massless body.

If the massless body in the moment corresponding to f_0 was not in the Hill-zone surrounding m_2, and satisfied the condition

$$C_2 < C_0^* \frac{1 - e}{1 + e\cos f_0} \tag{17}$$

then it never enter in this zone, and it can not be captured by m_2.

An other condition can be formulated in the next form:

If the massless body in the moment corresponding to f_0 is in the exterior of the cylinder, and satisfies the condition

$$C_1 < C_0^* \frac{1 - e}{1 + e\cos f_0} \tag{18}$$

then it never enter in the Hill-zone around m_2.

The advantage of this conditions consist in fact that is not necessary the integration of the equations of motion of the third-body, a simple evaluation of the expression (17) or (18) is only necessary.

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