DZIOBEK'S CONFIGURATIONS IN RESTRICTED PROBLEMS AND BIFURCATION*

ALAN ALMEIDA SANTOS
Rua A, 111, apto 503 Ibiza, Bairro Grageru, 49027-100 Aracaju, Sergipe, Brazil
(E-mail: alan@ufs.br)

(Received: 19 February 2004; revised: 17 May 2004; accepted: 19 May 2004)

Abstract. We consider some questions on central configurations of five bodies in space. In the first one, we get a general result of symmetry for the restricted problem of $n+1$ bodies in dimension $n-1$. After that, we made the calculation of all c.c. for $n=4$. In our second result, we extend a theorem of symmetry due to [Albouy, A. and Libre, I.: 2002, *Contemporary Math.* **292**, 1–16] on non-convex central configurations with 4 unit masses and an infinite central mass. We obtain similar results in the case of a big, but finite central mass. Finally, we continue the study by [Schmidt, D.S.: 1988, *Contemporary Math.* **81**] of the bifurcations of the configuration with four unit masses located at the vertices of a equilateral tetrahedron and a variable mass at the barycenter. Using Liapunov–Schmidt reduction and a result on bifurcation equations, which appear in [Golubitsley, M., Stewart, L. and Schaeffer, D.: 1988, *Singularties and Groups in Bifurcation Theory*, Vol. II, Springer-Verlag, New York], we show that there exist indeed seven families of central configurations close to a regular tetrahedron parameterized by the value of central mass.

Key words: n body problem, central configurations, bifurcation, symmetry

1. Introduction

The properties and role that central configurations play in the dynamics of the n body problem are explained in countless articles about the subject (see for example Albouy, 2004; Saari, 1980). We intend to show the symmetry of central configurations with n particles in dimension $n-2$, the so called Dziobek configurations and, in some cases, to calculate them. The approach is well known from references (Albouy, 1997; 2004).

Let $q = (q_1, \ldots, q_n) \in (\mathbb{R}^d)^n$ be a configuration of n particles with positive masses $m_1, \ldots, m_n$ in an Euclidean space. Let q_G be the center of mass of the particles.

* This research was partially supported by CNPq, Brazil.

Celestial Mechanics and Dynamical Astronomy **90:** 213–238, 2004.

DEFINITION 1.1. We say that q is a *central configuration* if there exists a constant $\lambda \in \mathbb{R}$ such that

$$\sum_{j \neq i} m_j \|q_i - q_j\|^{2a}(q_i - q_j) = \lambda(q_i - q_G) \quad \text{for any } i \in \{1, \dots, n\} \tag{1.1}$$

The exponent a is taken in the range $(-\infty, -1)$. Newton's problem refers to exponent $a = -3/2$.

Substituting q_G by its expression, the equation (1.1) take the form

$$\sum_{j \neq i} m_j \left(s_{ij}^a - \frac{\lambda}{M} \right)(q_i - q_j) = 0 \tag{1.2}$$

where $M = \sum m_i$ and $s_{ij} = \|q_i - q_j\|^2$.
We define the function $\varphi(s) = -s^a + \frac{\lambda}{M}$ and the variables S_{ij} by

$$S_{ij} = \varphi(s_{ij}).$$

We present as in Albouy (1997) an important estimate originally due to Moeckel: defining the quantities

$$\Sigma_{ij} = (m_i + m_j)S_{ij} + \frac{1}{2} \sum_{k \neq i,j} m_k(S_{ik} + S_{jk})$$

we have that if q is a central configuration then $\Sigma_{ij} \leqslant 0$. The arithmetic mean satisfies

$$\overline{\Sigma} = \frac{2}{n(n-1)} \sum_{i<j} \Sigma_{ij} = \frac{1}{n-1} \sum_{i<j} (m_i + m_j)S_{ij} \leqslant 0 \tag{1.3}$$

DEFINITION 1.2. We call q a *Dziobek configuration* if there exist constants $\lambda, \eta \in \mathbb{R}$ and a non-zero vector $\Delta \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{n} \Delta_j = 0 \qquad \text{and} \qquad \sum_{j=1}^{n} \Delta_j q_j = 0 \tag{1.4}$$

and

$$-s_{ij}^a + \frac{\lambda}{M} = \eta d_i d_j, \quad \text{with} \quad d_i = \Delta_i / m_i \tag{1.5}$$

It's easy to see that Dziobek's configurations are central configurations with dimension at most $n - 2$.

The following lemma is trivial

LEMMA 1.3. *If q is a configuration of dimension exactly $n - 2$ then there exists, up to a factor, a unique $\Delta \in \mathbb{R}^n$ satisfying* (1.4).

PROPOSITION 1.4. *If q is a central configuration of dimension exactly $n - 2$ then q is a Dziobek's configuration.*

Proof. See Albouy, 2004. □

The next result is a consequence of Moeckel's inequality.

PROPOSITION 1.5. *If q is a Dziobek configuration then $\eta > 0$.*

Proof. We have that

$$0 = \sum \Delta_i \sum d_j = \sum_{i<j} (\Delta_i d_j + \Delta_j d_i) + \sum_{i=1}^{n} \Delta_i d_i = \frac{1}{\eta} \sum_{i<j} (m_i + m_j) S_{ij} + \sum_{i=1}^{n} \frac{\Delta_i^2}{m_i}$$

and by (1.3)

$$\overline{\Sigma} = \frac{-\eta}{n-1} \sum_{i=1}^{n} \frac{\Delta_i^2}{m_i} \leqslant 0.$$ □

The constant η can be normalized in the computation.

PROPOSITION 1.6. *If q is a Dziobek configuration then*

$$(\Delta_i - \Delta_j)(d_i - d_j) \geqslant 0 \tag{1.6}$$

Proof. Substituting $S_{ij} = \eta d_i d_j$ in the expression of Σ_{ij} one has that

$$\Sigma_{ij} = \frac{-\eta}{2} (\Delta_i - \Delta_j)(d_i - d_j) \leqslant 0.$$ □

DEFINITION 1.7. A Dziobek configuration of dimension $n-2$ is said convex if at the least two of the Δ_i's are positive and at the least two of the Δ_i's are negative. If one of the Δ_i's is negative and the others are positive then the configuration is called non-convex.

Roughly speaking, a configuration is convex if none of the particles q_i is strictly inside the convex hull of the others. When the configuration is non-convex with $\Delta_1 < 0$ and $\Delta_j > 0$ $(j > 1)$, q_1 is inside the convex hull of the particles q_j.

PROPOSITION 1.8. *If $q \in \left(\mathbb{R}^d\right)^n$ is a configuration then the equations* (1.4) *are equivalent to the system*

$$\sum_{k \neq i} \Delta_k s_{ik} = \sum_{k \neq j} \Delta_k s_{jk} \quad \text{and} \quad \sum_{k=1}^{n} \Delta_k = 0 \tag{1.7}$$

for any $1 \leqslant i < j \leqslant n$.

Proof. Suppose that (1.4) holds. Deriving the quantity $t_q = \sum \Delta_k \|q_k - q\|^2$ with respect to the vector q we find

$$-2 \sum_{k=1}^{n} \Delta_k (q_k - q) = 0$$

by (1.4). Then, t_q does not depend on q and so $t_{q_i} = t_{q_j}$.

Now, substituting $s_{kj} = \langle q_k - q_j, q_k - q_j \rangle$ in the first of (1.7) we have

$$\sum_{k=1}^{n} \langle \Delta_k (q_k - q_j), q_i - q_j \rangle = 0, \qquad \text{for all } i,j \in \{1, \ldots, n\}.$$

Fixed j, the equation above says that the vector $\sum \Delta_k (q_k - q_j)$ is orthogonal to all generators of the linear space in which it lives. Thus

$$\sum \Delta_k (q_k - q_j) = 0$$

and thereby $\sum \Delta_k q_k = 0$. □

Let's define $t_i = \sum \Delta_k s_{ik}$. By taking $\eta = 1$ and $\lambda = 1$ the search of Dziobek configurations consists in solving the system of $n(n+1)/2$ equations and $n(n+1)/2$ variables

$$t_i = t_j \tag{1.8}$$

$$\sum_{k=1}^{n} \Delta_k = 0 \tag{1.9}$$

$$-s_{ij}^{a} + \frac{1}{M} = \frac{\Delta_i \Delta_j}{m_i m_j}, \qquad 1 \leqslant i < j \leqslant n \tag{1.10}$$

In addition, (1.8) furnishes the following necessary conditions for a configuration be a Dziobek's one:

$$Q_{ijk} = \begin{vmatrix} 1 & 1 & 1 \\ t_i & t_j & t_k \\ \Delta_i & \Delta_j & \Delta_k \end{vmatrix} = 0 \tag{1.11}$$

The sign of each of these determinants is invariant by the transformations $s \mapsto \xi s + \mu$, $\xi > 0$ and $\mu \in \mathbb{R}$.

2. Dziobek Configurations of $n + 1$ Bodies

Let $m_1, \ldots, m_n$ be fixed positive masses and $q(m_0, m_1, \ldots, m_n)$ a Dziobek configuration which varies continuously with the mass m_0. Assume that, as m_0 goes to zero, $q(m_0, \ldots, m_n)$ tends to a well-defined, collision-free configuration $\tilde{q}$ where $\tilde{q}(m_1, \ldots, m_n)$ is $n-1$ dimensional. The limit $\tilde{q}$ is what we call a *Dziobek configuration of* $n+1$ *bodies*.

2.1. THE SYMMETRY

The variables s_{ij} and Δ_i are continuous functions of the mass m_0. Passing to the limit, Equation (1.2) becomes

$$\sum_{j=1}^{n} m_j \tilde{S}_{ij}(\tilde{q}_j - \tilde{q}_i) = 0, \quad i \neq 0.$$

As the configuration $\{\tilde{q}_1, \ldots, \tilde{q}_n\}$ is $n-1$ dimensional the set of $n-1$ vectors $\{q_j - q_i\}$, $i \neq 0$ being fixed, is linearly independent. Then

$$\tilde{S}_{ij} = 0 \quad \Rightarrow \quad \tilde{s}_{ij} = s_0 = \left(\frac{\lambda}{M}\right)^{\frac{1}{a}} \quad \text{for all} \quad 0 < i < j \leqslant n.$$

This means that the n massive particles form a regular $n-1$ dimensional simplex. For $n = 3$ we have an equilateral triangle (Lagrange) and for $n = 4$ a regular tetrahedron (Lehmann–Filhés). The nullity of m_0 implies that of the Δ_i's. Indeed, according to relations (1.10) for all $i, j \neq 0$ the products $\Delta_i \Delta_j$ tend to zero as $m_0 \to 0$. On the other hand, the limit $\widetilde{S}_{0i}$ exists and is finite, which gives

$$\Delta_i \Delta_0 = m_0 m_i S_{0i} \to 0 \quad \text{as} \quad m_0 \to 0.$$

One multiplies $\sum \Delta_i = 0$ by Δ_j and this shows that $\Delta_i \to 0$ as $m_0 \to 0$ for all $0 \leqslant i \leqslant n$.

PROPOSITION 2.1. *If* $\theta_{ij} = S_{ij}/m_0$ *for* $0 < i < j \leqslant n$ *then* $\lim \theta_{ij}$ *exists and is finite.*

Proof. Fix $i \neq 0$ and let $j \neq i, 0$. The limiting configuration $\{\tilde{q}_1, \ldots, \tilde{q}_n\}$ being $n-1$ dimensional and $q(m_0, \ldots, m_n)$ being a continuous function of m_0, $\{q_1, \ldots, q_n\}$ is also $n-1$ dimensional for sufficiently small values of m_0. Now, from equation

$$\sum_{j=0}^{n} m_j S_{ij}(q_j - q_i) = 0$$

we see that the coordinates of the vector $-S_{0i}(q_0 - q_i)$ with respect to the basis $\{q_j - q_i\}_{j \neq 0, i}$ are $m_j m_0^{-1} S_{ij}$. Because the linear independence is preserved at the limit, one sees that the coordinates have a well defined and finite limit. $\square$

Let's call

$$\tilde{\theta}_{ij} = \lim_{m_0 \to 0} \frac{S_{ij}}{m_0}.$$

We now rescale the vector Δ making

$$\Delta = (\sqrt{m_0}\Delta_0, \ldots, \sqrt{m_0}\Delta_n) \tag{2.1}$$

PROPOSITION 2.2. *With the normalization* (2.1), $\lim \Delta_i$ *exists for all i.*

Proof. We substitute the components of Δ in the equations (1.10) to get

$$m_i m_j S_{ij} = m_0 \Delta_i \Delta_j \quad \text{and} \quad m_i S_{0i} = \Delta_0 \Delta_i$$

By the proposition **(2.1)** and the definition of Dziobek configuration of $n+1$ bodies, we have the limits

$$\lim_{m_0 \to 0} \Delta_i \Delta_j = m_i m_j \tilde{\theta}_{ij} \quad \text{and} \quad \lim_{m_0 \to 0} \Delta_0 \Delta_i = m_i \widetilde{S}_{0i}.$$

As above, this implies that Δ_i^2, and thus, by continuity, Δ_i, have a well defined limit as m_0 goes to zero. $\square$

We denote them by

$$\tilde{\Delta}_i = \lim_{m_0 \to 0} \Delta_i$$

REMARK 2.3. The limit $\tilde{\Delta}_0$ cannot be zero for, otherwise, the dimension of the configuration should be n, which contradicts the hypothesis.

Again, the continuity hypothesis says that the $\tilde{s}_{ij}$ and $\widetilde{\Delta}_i$ satisfy the equations (1.8). Recording that $\tilde{s}_{ij} = \tilde{s}_0$ for all $i, j \neq 0$ the necessary conditions (1.11) for a Dziobek configuration take the form

$$Q_{ijk} = \begin{vmatrix} 1 & 1 & 1 \\ \tilde{s}_{0i} & \tilde{s}_{0j} & \tilde{s}_{0k} \\ m_i \widetilde{S}_{0i} & m_j \widetilde{S}_{0j} & m_k \widetilde{S}_{0k} \end{vmatrix} = 0$$

where we put $\widetilde{\Delta}_0 \widetilde{\Delta}_i = m_i \widetilde{S}_{0i}$.

THEOREM 2.4 (symmetry). *All Dziobek configurations in the* $n+1$ *body problem with equal masses possess a symmetry.*

Proof. We know that

$$S_{ij} = \varphi(s_{ij})$$

where φ has the properties $\varphi' > 0$ and $\varphi'' < 0$. Let $0 < i < j < k \leqslant n$. If $\widetilde{\Delta}_i < \widetilde{\Delta}_j < \widetilde{\Delta}_k$ then $\widetilde{S}_{0i} < \widetilde{S}_{0j} < \widetilde{S}_{0k}$. The convexity of φ implies that

$$Q_{ijk} = \begin{vmatrix} 1 & 1 & 1 \\ \tilde{s}_{0i} & \tilde{s}_{0j} & \tilde{s}_{0k} \\ \varphi(\tilde{s}_{0i}) & \varphi(\tilde{s}_{0j}) & \varphi(\tilde{s}_{0k}) \end{vmatrix} < 0$$

what is impossible for a Dziobek configuration. Thus, we must have

$$\widetilde{\Delta}_i = \widetilde{\Delta}_j \quad \text{or} \quad \widetilde{\Delta}_j = \widetilde{\Delta}_k \quad \text{or} \quad \widetilde{\Delta}_i = \widetilde{\Delta}_i. \qquad \square$$

REMARK 2.5. For $n = 3$ the configuration has a symmetry axis. For $n = 4$ the configuration can admit a symmetry axis $(\widetilde{\Delta}_1 = \widetilde{\Delta}_2 = \widetilde{\Delta}_3)$ or a symmetry plane $\widetilde{\Delta}_1 = \widetilde{\Delta}_2$ and $\widetilde{\Delta}_3 = \widetilde{\Delta}_4$). For $n > 4$ we note the variables $\widetilde{\Delta}_i$ $(i \neq 0)$ assume no more than two different values in the case of equal masses.

2.2. THE DZIOBEK CONFIGURATIONS OF $4+1$ BODY WITH EQUAL MASSES

We follow the strategy presented in [Albouy and Llibre, 2002]. We are going to search Dziobek configurations with the following symmetries

$$\Delta_1 = \Delta_2 \quad \text{and} \quad \Delta_3 = \Delta_4 \quad \text{two planes of symmetry} \tag{2.2}$$

$$\Delta_1 = \Delta_2 = \Delta_3 \quad \text{one axis of symmetry} \tag{2.3}$$

For this end, we take $\lambda = M$ and $a = -3/2$ (Newton's case). Calculation consists in determining the coordinates s_{0i} with $i \in \{1, 2, 3, 4\}$.

SYMMETRY (2.2). In this case we have $s_{01} = s_{02}$ and $s_{03} = s_{04}$. Put $\Delta_1 = \delta(1 + r)$ and $\Delta_3 = \delta(-1 + r)$ so that by (1.4) $\Delta_0 = -4\delta r$.

The equation (1.8) become the system

$$\begin{cases} (s_{01} - s_{03})r = -\frac{1}{2} \\ -(1 + 3r)s_{01} + (1 - r)s_{03} = \frac{1-3r}{2} \end{cases} \tag{2.4}$$

while $S_{0i} = s_{0i}^{-3/2} + 1$ gives us

$$\begin{cases} s_{01}^{-3/2} = 1 + \rho(1 + r) \\ s_{03}^{-3/2} = 1 + \rho(-1 + r) \end{cases} \tag{2.5}$$

with the notation $\rho = 4\delta^2 r$. We must solve this system of four equations and four unknowns subject to the following constraints

$$r \neq 0, \quad 1 + \rho(1 + r) > 0, \quad 1 + \rho(-1 + r) > 0. \tag{2.6}$$

The first one express that the dimension of the configuration is three and the other two that $s_{ij}^a > 0$.

Solving (2.4) for s_{01} and s_{03} we get

$$s_{01} = \frac{3r^2 - 2r + 1}{8r^2} \quad \text{and} \quad s_{03} = \frac{3r^2 + 2r + 1}{8r^2}. \tag{2.7}$$

To avoid the fractional exponent we square the equations (2.5)

$$\begin{cases} s_{01}^{-3} = (1 + \rho(1 + r))^2 \\ s_{03}^{-3} = (1 + \rho(-1 + r))^2 \end{cases} \tag{2.8}$$

Now, in order to extract an expression of ρ as a function of r we multiply the first one by $(1 - r)^2$ and second one by $(1 + r)^2$. After that, we subtract the first one from the second and substitute in the expressions (2.7) to get

$$(r^2 - 1)\rho = K - r \tag{2.9}$$

where

$$K = \frac{1024r^7(7r^4 - 2r^2 - 1)}{(3r^2 - 2r + 1)^3(3r^2 + 2r + 1)^3}$$

Clearly one can see that $r = \pm 1$ does not solve the system (2.4) and (2.5). We assume then $r \neq \pm 1$ and thus, by (2.9), ρ has a well defined expression in terms of r.

We now insert ρ in the first of the equations (2.8) and, making the simplifications, the final equation is equivalent to the polynomial equation

$$10077696r^{26} - 531441r^{24} - 53208556r^{22} + 29718094r^{20} + 11588260r^{18} - 2955215r^{16}$$
$$-524376r^{14} + 189444r^{12} + 34408r^{10} + 4577r^8 - 188r^6 - 114r^4 - 12r^2 - 1 = 0.$$

Sturm's algorithm indicates six roots $\pm r_1$, $\pm r_2$, and $\pm r_3$. One verifies that r_2 is not compatible with the constraints (2.6). Thus we have four Dziobek configurations with symmetry of type (2.2). Using a computer program for numerical calculations we find that the admissible values for r are

$$r_1 = \pm 0.3872384014\ldots \qquad r_3 = \pm 1.370862272\ldots$$

which produces

$$r_1: \quad \begin{matrix} s_{01} = 0.562994176\ldots \\ s_{03} = 1.854188433\ldots \end{matrix} \quad \begin{matrix} \Delta_0 = +1.235570806\ldots \\ \Delta_1 = -1.106573666\ldots \\ \Delta_3 = +0.488788263\ldots \end{matrix}$$

$$r_3: \quad \begin{matrix} s_{01} = 0.2591484536\ldots \\ s_{03} = 0.6238823954\ldots \end{matrix} \quad \begin{matrix} \Delta_0 = +3.9011371610\ldots \\ \Delta_1 = -1.6867228570\ldots \\ \Delta_3 = -0.2638457233\ldots \end{matrix}$$

The other two are obtained by symmetry, interchanging the indices $1 \longleftrightarrow 3$.

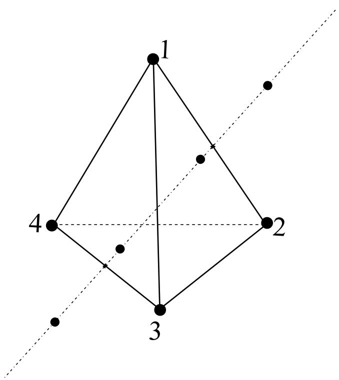

Figure 1. Planar type symmetry.

SYMMETRY (2.3). We have $s_{01} = s_{02} = s_{03}$. Let's make $\Delta_4 = r\Delta_1$ so that $\Delta_0 = -(3+r)\Delta_1$. The equations $t_0 = t_1 = t_4$ are equivalent to

$$\begin{cases} (-s_{01} + s_{04})(3+r) = 1 - r \\ (6+r)s_{01} + rs_{04} = 2 + r \end{cases} \tag{2.10}$$

On the other hand, $S_{0i} = -s_{0i}^{-3/2} + 1$ gives

$$\begin{cases} s_{01}^{-3} = (1+\rho)^2 \\ s_{04}^{-3} = (1+\rho r)^2 \end{cases} \tag{2.11}$$

where we put $\rho = (3+r)\Delta_1^2$. The constraints over the variables s_{0i}, r, ρ are

$$r \neq -3, \quad 1+\rho > 0, \quad 1 + r\rho > 0. \tag{2.12}$$

Solving (2.10) in s_{01} and s_{04} we get

$$s_{01} = \frac{r^2 + 2r + 3}{(3+r)^2} \quad \text{and} \quad s_{03} = \frac{6}{(3+r)^2} \tag{2.13}$$

Now, we multiply the first equation (2.11) by r^2 and deduct the second getting

$$\frac{r^2}{s_{01}^3} - \frac{1}{s_{04}^3} = (r-1)(r+1+2r\rho) \tag{2.14}$$

The value $r = 1$ corresponds to the situation where the body of mass zero is located at the barycenter of the tetrahedron. One assumes $r \neq 1$ and inserts (2.13) into (2.14) to write ρ as a function of r

$$\rho = \frac{K - r - 1}{2r} \tag{2.15}$$

where

$$K = -\frac{1}{216}\frac{(r-1)(r^4 + 8r^3 + 36r^2 + 108r + 27)(3+r)^6}{(r^2+2r+3)^3}$$

The expression (2.15) combined with (2.11) and (2.13) becomes an equation whose polynomial form is

$$\begin{aligned} &191850201 + 2564734266r + 7570731339r^2 + 2940246540r^3 - 23860106577r^4 \\ &- 56617690230r^5 - 64508407371r^6 - 42550400304r^7 - 13108660758r^8 + 4393493460r^9 \\ &+ 8201201886r^{10} + 5687558856r^{11} + 2676120174r^{12} + 957915396r^{13} + 272281338r^{14} \\ &+ 62542800r^{15} + 11647341r^{16} + 1744434r^{17} + 206119r^{18} + 18572r^{19} \\ &+ 1203r^{20} + 50r^{21} + r^{22} = 0. \end{aligned}$$

Again, Sturm's algorithm indicates eight roots among which 5 are ruled out by the constraints (2.12). Using a computer program for numerical calculation we find the following values for r

$r_1 = 1, \quad r_2 = -7.494424564\ldots, \quad r_3 = -1.332058078\ldots, \quad r_4 = 0.4692200276\ldots$

Such values correspond to the following solutions

$$r_1: \qquad s_{01} = s_{02} = s_{03} = s_{04} = \frac{3}{8} \qquad \begin{array}{l} \Delta_1 = -\frac{1}{6}\sqrt{16\sqrt{6}-9} \\ \Delta_0 = +\frac{2}{3}\sqrt{16\sqrt{6}-9} \end{array}$$

$$r_2: \qquad \begin{array}{l} s_{01} = 2.187023453\ldots \\ s_{04} = 0.297031877\ldots \end{array} \qquad \begin{array}{l} \Delta_1 = +0.392051847\ldots \\ \Delta_4 = -2.938202992\ldots \\ \Delta_0 = +1.762047451\ldots \end{array}$$

$$r_3: \qquad \begin{array}{l} s_{01} = 0.758533292\ldots \\ s_{04} = 2.156698328\ldots \end{array} \qquad \begin{array}{l} \Delta_1 = -0.5549600183\ldots \\ \Delta_4 = +0.7392389753\ldots \\ \Delta_0 = +0.9256410797\ldots \end{array}$$

$$r_4: \qquad \begin{array}{l} s_{01} = 0.3455287919\ldots \\ s_{04} = 0.4985257098\ldots \end{array} \qquad \begin{array}{l} \Delta_1 = -1.0634589110\ldots \\ \Delta_4 = -0.4989962196\ldots \\ \Delta_0 = +3.6893729530\ldots \end{array}$$

The configurations with symmetry of type (2.2) are those where the null mass is on the three straight lines that join the middle points of opposing edges in the tetrahedron. In each of these symmetry axes we have four possible positions totalizing twelve central configurations with planar type symmetry. In the Dziobek configurations of type (2.3) the null mass is located in the axis of symmetry of the tetrahedron that passes through a vertex and crosses the opposing face perpendicularly. There are four axes and four positions for the null mass on each one, such that the barycenter of the tetrahedron is common to the four axes. We have, therefore, thirteen central configurations with axis type symmetry. Now, we can state the

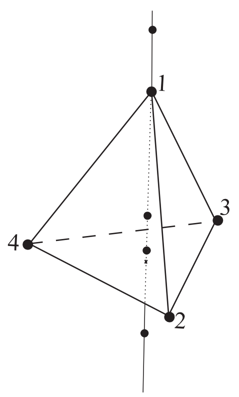

Figure 2. Axis type symmetry.

THEOREM 2.6. *The spatial restricted problem of* 4 + 1 *body with equal masses has* 25 *central configurations among which* 12 *are non-convex*.

3. Symmetry of Central Configurations in the Spatial Five Body Problem

3.1. INTRODUCTION

In his article with J. Llibre (Albouy and Libre, 2002), A. Albouy proved that in the spatial restricted problem of 1 + 4 bodies all central configuration has a plane of symmetry. Central configurations of 1 + 4 bodies are configurations without collision that are a limit of central configurations of five bodies when one of the masses tends to $+\infty$. In part 2, with the simplex method (see Albouy, 1997) adjusted to the situation, we got the same result of symmetry in the restricted problem of 4 + 1 bodies.

When I was his PhD student, Albouy asked me if the result of symmetry in the 1 + 4 body problem could be extended to the case of four equal masses and a much bigger, but finite mass.

A negative reply would give us a warning of that the simplex method is not useful for showing symmetries of non-convex central configurations of five bodies with equal masses. However, we succeeded in proving that the answer to the question raised by Albouy is positive.

3.2. SYMMETRY OF CENTRAL CONFIGURATIONS IN A PROBLEM OF FIVE BODIES

One considers the problem of five bodies in a three-dimensional configuration with four particles with equal mass m and a fifth unitary mass located in the convex hull of those.

We are going to prove the following

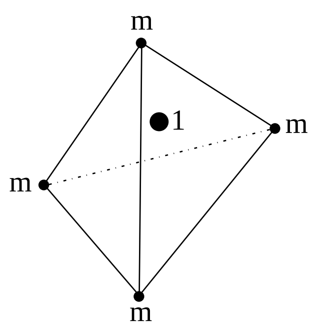

Figure 3. 1 + 4 three-dimensional configuration.

THEOREM 3.1. *If* $0 < m < \frac{1}{16}$ *then every central configuration of this problem has a plane of symmetry.*

Proof. We will use the theory of Dziobek configurations and a strategy to show the incompatibility between the equations $Q_{ijk} = 0$ and the assymmetry of the configurations through the simplex method (for more details about the construction of the simplex see (Albouy, 1997)). We will adopt $m_1 = 1$ as being the unitary central mass and $m_2 = m_3 = m_4 = m_5 = m$ as being the bodies around m_1. Let $(q_1, \ldots, q_5)$ be a non-convex assymmetrical Dziobek configuration. The first (definition 1.7) means that $\Delta_1 < 0$ and $\Delta_i > 0$ for $2 \leqslant i \leqslant 5$ while the absence of symmetry implies that the variables Δ_i are distinct. So, up to reordering, they satisfy the inequalities

$$\Delta_1 < 0 < \frac{\Delta_2}{m} < \frac{\Delta_3}{m} < \frac{\Delta_4}{m} < \frac{\Delta_5}{m} \tag{3.1}$$

which imply that

$$\underbrace{S_{15}}_{S_1} < \underbrace{S_{14}}_{S_2} < \underbrace{S_{13}}_{S_3} < \underbrace{S_{12}}_{S_4} < 0 < \underbrace{S_{23}}_{S_5} < \underbrace{S_{24}}_{S_6} < \underbrace{S_{25}}_{S_7} < \underbrace{S_{34}}_{S_8} < \underbrace{S_{35}}_{S_9} < \underbrace{S_{45}}_{S_{10}} \tag{3.2}$$

or

$$\underbrace{S_{15}}_{S_1} < \underbrace{S_{14}}_{S_2} < \underbrace{S_{13}}_{S_3} < \underbrace{S_{12}}_{S_4} < 0 < \underbrace{S_{23}}_{S_5} < \underbrace{S_{24}}_{S_6} < \underbrace{S_{34}}_{S_7} < \underbrace{S_{25}}_{S_8} < \underbrace{S_{35}}_{S_9} < \underbrace{S_{45}}_{S_{10}} \tag{3.3}$$

where $S_{ij} = \frac{\Delta_i \Delta_j}{m_i m_j}$.

Given the 10 numbers S_l we define the subset of $\mathbb{R}^{10}$

$\{(s_1, \ldots, s_{10}) /$ there exists a convex increasing function ψ s.t. $s_l = \psi(S_l)\}$
Note that the list of 10 variables $s_{ij} = \|q_i - q_j\|^2$ which characterize the central configuration $(q_1, \ldots, q_5)$ belongs to this subset. To see this take $\psi = \varphi^{-1}$.
On this set we can define the linear forms given by the determinants

$$Q_{ijk} = \begin{vmatrix} 1 & 1 & 1 \\ t_i & t_j & t_k \\ \Delta_i & \Delta_j & \Delta_k \end{vmatrix}$$

which are null when $(s_1, \ldots, s_{10})$ is a central configuration and whose sign is invariant by the transformations $s \mapsto \xi s + \eta$, $\xi > 0$ and $\eta \in \mathbb{R}$. Identifying

$(s_1, \ldots, s_{10})$ to all the $(\xi s_1 + \eta, \ldots, \xi s_{10} + \eta)$ with $\xi > 0$ we define a quotient affine space of dimension 8. In this space the simplex with 9 vertices

$$\begin{array}{ll} A_1(0,0,0,0,0,0,0,0,0,1) & B_2(S_8,S_8,S_8,S_8,S_8,S_8,S_8,S_8,S_9,S_{10}) \\ B_3(S_7,S_7,S_7,S_7,S_7,S_7,S_7,S_8,S_9,S_{10}) & B_4(S_6,S_6,S_6,S_6,S_6,S_6,S_7,S_8,S_9,S_{10}) \\ \vdots & \vdots \\ B_8(S_2,S_2,S_3,S_4,S_5,S_6,S_7,S_8,S_9,S_{10}) & B_9(S_1,S_2,S_3,S_4,S_5,S_6,S_7,S_8,S_9,S_{10}) \end{array}$$

contains in its interior the central configuration $(\ldots, s_{ij}, \ldots)$.

REMARK 3.2. The numbers S_l appearing as coordinates of the vertices $A_1, B_2, \ldots$ are those listed in the inequalities (3.2) and (3.3). Note that the linear forms Q_{ijk} and the simplex are parameterized by the five numbers Δ_i. We will now show that, for m in a given interval, at the least one hyperplane, defined by an equation $Q_{ijk} = 0$, does not have intersection with the interior of the simplex. This is made by evaluating the form Q_{ijk} on the vertices of the simplex and showing that $Q_{ijk}(B_i)$ have the same sign for all $1 \leqslant i \leqslant 9$ and consequently on any interior point of the simplex. Thus we conclude that the configurations that correspond to interior points of the simplex do not realize the necessary conditions to be a central configuration which are $Q_{ijk} = 0$ for all $1 \leqslant i < j < k \leqslant 5$. It is a contradiction. Then, for m in a given interval of mass, the Dziobek configurations of this problem must have $\Delta_i = \Delta_j$ for some pair $1 \leqslant i, j \leqslant 5$.

Since the 1-form Q_{123} does not depend on s_{45}, we can discard the variable S_{10} in the vertices coordinates. In this way, we deal with a simplex of dimension 7 defined by 8 vertices.

$$\begin{array}{ll} A_1 = (0,0,0,0,0,0,0,0,0,1) & B_2 = (S_7,S_7,S_7,S_7,S_7,S_7,S_7,S_8,S_9) \\ B_3 = (S_6,S_6,S_6,S_6,S_6,S_6,S_7,S_8,S_9) & B_4 = (S_5,S_5,S_5,S_5,S_5,S_6,S_7,S_8,S_9) \\ \vdots & \vdots \\ B_7 = (S_2,S_2,S_3,S_4,S_5,S_6,S_7,S_8,S_9,) & B_8 = (S_1,S_2,S_3,S_4,S_5,S_6,S_7,S_8,S_9) \end{array}$$

Taking the order (3.2), in which $S_7 = S_{25}$ and $S_8 = S_{34}$, we evaluate the sign of the 1-form Q_{123} on the 8 vertices above by substituting s_{ij} in the expressions of t_i by the coordinates of the vertices. For example, to get $Q_{123}(B_7)$ we make $s_{15} = S_2$, $s_{14} = S_2$, $s_{13} = S_3$ and so on. To make the simplifications we use a program of formal calculus as Maple. We got

$$Q_{123}(A_1) = -\Delta_5(\Delta_2 - \Delta_1)$$

$$Q_{123}(B_2) = -\frac{(2\Delta_2 + \Delta_3 + \Delta_4 + \Delta_5)\left[\Delta_4(\Delta_3\Delta_4 - \Delta_2\Delta_5) + \Delta_5^2(\Delta_3 - \Delta_2)\right]}{m}$$

$$Q_{123}(B_3) = -\frac{(\Delta_3 - \Delta_2)(\Delta_4\Delta_5\Delta_2 + \Delta_4^2\Delta_2 - \Delta_4^2\Delta_1 - \Delta_5^2\Delta_1)}{m}$$

$$Q_{123}(B_4) = -\frac{(\Delta_3 - \Delta_2)(\Delta_2\Delta_3\Delta_4 + \Delta_2\Delta_3\Delta_5 - \Delta_4^2\Delta_1 - \Delta_5^2\Delta_1)}{m}$$

$$Q_{123}(B_5) = -\frac{(\Delta_3 - \Delta_2)[mR_5 + P]}{m} \quad Q_{123}(B_6) = -\frac{(\Delta_3 - \Delta_2)[mR_6 + P]}{m}$$

$$Q_{123}(B_7) = -\frac{(\Delta_3 - \Delta_2)[mR_7 + P]}{m} \quad Q_{123}(B_8) = -\frac{(\Delta_3 - \Delta_2)[mR_8 + P]}{m}$$

where

$$P = (\Delta_5^2 - \Delta_2\Delta_3)(\Delta_2 + \Delta_3) + (\Delta_4^2 - \Delta_2\Delta_3)(\Delta_2 + \Delta_3 + \Delta_4 + \Delta_5) + \Delta_5^2(\Delta_4 + \Delta_5) > 0$$
$$R_5 = -2\Delta_2(\Delta_2 + \Delta_3 + \Delta_4 + \Delta_5)^2 < 0$$
$$R_6 = -\Delta_1(\Delta_1^2 - \Delta_2^2 - \Delta_3^2 - \Delta_3\Delta_4 - \Delta_3\Delta_5) > 0$$
$$R_7 = -\Delta_1(\Delta_1^2 - \Delta_2^2 - \Delta_3^2 - \Delta_4\Delta_5 - \Delta_4^2) > 0$$
$$R_8 = -\Delta_1(\Delta_1^2 - \Delta_2^2 - \Delta_3^2 - \Delta_4^2 - \Delta_5^2) > 0$$

Under the conditions $\sum \Delta_i = 0$, $\Delta_1 < 0 < \Delta_2 < \Delta_3 < \Delta_4 < \Delta_5$ and $\Delta_2\Delta_5 < \Delta_3\Delta_4$ one verifies that the linear form Q_{123} is negative, for any value of m, on all the vertices, except, perhaps, on B_5.

So, let us consider the function $f : (\mathbb{R}_+)^4 \to \mathbb{R}$ given by

$$f(a, b, c, d) = \frac{d^2(c + d)}{2a(a + b + c + d)^2}$$

Under the restriction $0 < a \leqslant b \leqslant c \leqslant d$ we have the estimate

$$f(a, b, c, d) \geqslant \frac{d^2 2c}{2a(4d)^2} \geqslant \frac{1}{16} = f(1, 1, 1, 1) = f_0$$

Thus, if $0 < m < f_0$ we have

$$mR_5 + P = -R_5\left(-m - \frac{P}{R_5}\right) > -R_5(-f_0 + f(\Delta_2, \Delta_3, \Delta_4, \Delta_5)) > 0$$

from which $Q_{123}(B_5) < 0$.

The calculations show that if $0 < m < f_0$ then the configurations inside the simplex, all non-symmetrical of the type (3.2), present $Q_{123} < 0$ and so they cannot be central configurations.

We take now, the order (3.3), in the which $S_7 = S_{34}$ and $S_8 = S_{25}$. By evaluating the sign of the 1-form Q_{123} on the vertices we get

$$Q_{123}(A_1) = -\Delta_5(\Delta_2 - \Delta_1) \qquad Q_{123}(B_2) = -\frac{\Delta_5(\Delta_3 - \Delta_2)(-\Delta_1\Delta_5 + \Delta_3\Delta_4)}{m}$$
$$Q_{123}(B_3) = -\frac{(\Delta_3 - \Delta_2)(\Delta_2\Delta_4\Delta_5 + \Delta_2\Delta_4^2 - \Delta_4^2\Delta_1 - \Delta_5^2\Delta_1)}{m}$$
$$Q_{123}(B_4) = -\frac{(\Delta_3 - \Delta_2)(\Delta_2\Delta_3\Delta_4 + \Delta_2\Delta_3\Delta_5 - \Delta_4^2\Delta_1 - \Delta_5^2\Delta_1)}{m}$$
$$Q_{123}(B_5) = -\frac{(\Delta_3 - \Delta_2)[mR_5 + P]}{m} \quad Q_{123}(B_6) = -\frac{(\Delta_3 - \Delta_2)[mR_6 + P]}{m}$$
$$Q_{123}(B_7) = -\frac{(\Delta_3 - \Delta_2)[mR_7 + P]}{m} \quad Q_{123}(B_8) = -\frac{(\Delta_3 - \Delta_2)[mR_8 + P]}{m}$$

By the same way, one verifies that if $0 < m < f_0$ one has $Q_{123}(A_1) < 0$ and $Q_{123}(B_i) < 0$ for all i.

This shows that if $0 < m < f_0$ there do not exist Dziobek configurations without symmetry, that is, with $\Delta_1 < 0 < \Delta_2 < \Delta_3 < \Delta_4 < \Delta_5$. In this case, $\Delta_2 = \Delta_3$ is a necessary, but not sufficient, condition for a Dziobek configuration. So the Dziobek configurations have at the least one plane of symmetry. □

Following Albouy and Libre (2002), we can consider non-convex configurations with $\Delta_1 < 0 < \Delta_2 = \Delta_3 < \Delta_4 < \Delta_5$, which gives us

$$S_{15} < S_{14} < S_{13} < 0 < S_{33} < S_{34} < S_{35} < S_{45}.$$

Now, the simplex is defined by the vertices

	s_{15}	s_{14}	s_{13}	s_{12}	s_{23}	s_{24}	s_{34}	s_{35}	s_{45}
A_1 :	0	0	0	0	0	0	0	0	1
B_2 :	S_{34}	S_{34}	S_{34}	S_{34}	S_{34}	S_{34}	S_{34}	S_{35}	S_{45}
B_3 :	S_{33}	S_{33}	S_{33}	S_{33}	S_{33}	S_{34}	S_{34}	S_{35}	S_{45}
B_4 :	S_{13}	S_{13}	S_{13}	S_{13}	S_{33}	S_{34}	S_{34}	S_{35}	S_{45}
B_5 :	S_{14}	S_{14}	S_{13}	S_{13}	S_{33}	S_{34}	S_{34}	S_{35}	S_{45}
B_6 :	S_{15}	S_{14}	S_{13}	S_{13}	S_{33}	S_{34}	S_{34}	S_{35}	S_{45}

Making $\Delta_2 = \Delta_3$ in Q_{134} and evaluating it on the 6 vertices above we get

$$Q_{134}(A_1) = \Delta_5(\Delta_1 - \Delta_3) \quad Q_{134}(B_2) = \frac{\Delta_5(\Delta_4 - \Delta_3)(\Delta_1\Delta_5 - \Delta_3\Delta_4)}{m}$$
$$Q_{134}(B_3) = \frac{(\Delta_3 - \Delta_4)(6\Delta_3^3 + 3\Delta_5\Delta_3^2 + 2\Delta_3(\Delta_5^2 - \Delta_4^2) + \Delta_5\Delta_4(\Delta_5 - \Delta_3) + \Delta_5^3)}{m}$$
$$Q_{134}(B_4) = \frac{(\Delta_3 - \Delta_4)[mT_4 + J]}{m} \quad Q_{134}(B_5) = \frac{(\Delta_3 - \Delta_4)[mT_5 + J]}{m}$$
$$Q_{134}(B_6) = \frac{(\Delta_3 - \Delta_4)[mT_6 + J]}{m}$$

where

$$J = \Delta_5\Delta_4(\Delta_5 - \Delta_3) + (\Delta_5^3 - \Delta_4\Delta_3^2) + 2\Delta_3(\Delta_5^2 - \Delta_4^2) + \Delta_3^2(\Delta_5 - \Delta_4) + 2\Delta_3^3 > 2\Delta_3^3$$

$$T_4 = -2\Delta_3(2\Delta_3 + \Delta_4 + \Delta_5)^2 < 0$$

$$T_5 = -\Delta_1(\Delta_1^2 - 2\Delta_3^2 - \Delta_4^2 - \Delta_5\Delta_4) > 0$$

$$T_6 = -\Delta_1(\Delta_1^2 - 2\Delta_3^2 - \Delta_4^2 - \Delta_5^2) > 0$$

Except on B_4, the sign of Q_{134} is negative on all vertices. As before, consider the function $g\colon (\mathbb{R}_+)^3 \to \mathbb{R}$ given by

$$g(a,b,c) = \frac{bc(c-a) + 2a^3}{2a(2a+b+c)^2}$$

On the domain $0 < a \leqslant b \leqslant c$ we have two estimates, to know

$$g(a,b,c) \geqslant \frac{bc^2}{4a(2a+b+c)^2} \geqslant \frac{1}{64} \quad \text{if } a < \frac{c}{2}$$

$$g(a,b,c) \geqslant \frac{2\frac{c^3}{8}}{2a(2a+b+c)^2} \geqslant \frac{1}{128} = g_0 \quad \text{if } a \geqslant \frac{c}{2}$$

By restricting $0 < m < g_0$ we will have

$$mT_4 + J = -T_4\left(-m - \frac{J}{T_4}\right) > -T_4(-g_0 + g(\Delta_3, \Delta_4, \Delta_5)) > 0$$

from which $Q_{134}(B_4) < 0$.

These last calculations show that a non-convex configuration with $0 < \Delta_3 < \Delta_4$ cannot be central. By adding this to the previous result we have the

PROPOSITION 3.3. *Under the same hypothesis as Theorem* (3.1), *if* $0 \leqslant m < 1/128$ *then the central configurations have an axis of symmetry.*

REMARK 3.4. Note that $1/128$ is not a sharp value for estimating the symmetry of the configuration. We must remember that m is the ratio between the external mass and the central mass.

4. Bifurcation of the Regular Tetrahedron

4.1. SOME CONCEPTS AND RESULTS ON BIFURCATION THEORY WITH SYMMETRY

The best reference for what follows below is (Golubitsky et al., 1998). Let Γ be a Lie Group and $\mathbb{V}$ a finite dimensional vector space.

DEFINITION 4.1. We say that Γ acts linearly on $\mathbb{V}$ if there is a continuous mapping

$$\Gamma \times \mathbb{V} \to \mathbb{V}$$
$$(\gamma, \mathbf{v}) \mapsto \gamma \cdot \mathbf{v}$$

such that

(i) if $\gamma_1, \gamma_2 \in \Gamma$ then $\gamma_1 \cdot (\gamma_2 \cdot \mathbf{v}) = (\gamma_1 \cdot \gamma_2) \cdot \mathbf{v}$
(ii) for each $\gamma \in \Gamma$ the mapping $\rho_\gamma : \quad \mathbb{V} \to \mathbb{V}$ defined by $\rho_\gamma(\mathbf{v}) = \gamma \cdot \mathbf{v}$ is linear.

The mapping $\rho : \Gamma \to GL(\mathbb{V})$ is called a *representation of* Γ on $\mathbb{V}$.

EXAMPLE 4.2. The action of the permutation group S_n on $\mathbb{R}^n$ is given by

$$\sigma \cdot (x_1, x_2, \ldots, x_n) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$$

Such action is linear and ρ_σ is represented by an elementary matrix obtained from the identity matrix exchanging the rows according to σ. For example, in the action of S_3 on $\mathbb{R}^3$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \Leftrightarrow \quad \rho_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

In what follows, we consider that the action of Γ on $\mathbb{V}$ is always linear.

DEFINITION 4.3. if Σ is a subgroup of Γ then we define

$$\mathrm{Fix}(\Sigma) = \{\mathbf{v} \in \mathbb{V} / \sigma\mathbf{v} = \mathbf{v} \text{ for all } \sigma \in \Sigma\} \tag{4.1}$$

DEFINITION 4.4. The isotropy subgroup of $\mathbf{v} \in \mathbb{V}$ is the set

$$\Sigma_{\mathbf{v}} = \{\gamma \in \Gamma : \quad \gamma\mathbf{v} = \mathbf{v}\} \tag{4.2}$$

DEFINITION 4.5. A subspace $\mathbb{W} \subset \mathbb{V}$ is called Γ-invariant if $\gamma\mathbf{w} \in \mathbb{W}$ for all $\mathbf{w} \in \mathbb{W}$ and $\gamma \in \Gamma$.

PROPOSITION 4.6. *Let Γ be a compact Lie group acting on $\mathbb{V}$. Then there exists an inner product $\langle\ ,\ \rangle_\Gamma$ on $\mathbb{V}$ such that for all $\gamma \in \Gamma$, ρ_γ is orthogonal.*

EXAMPLE 4.7. If $\Gamma = S_n$ then $\langle\ ,\ \rangle_\Gamma = \langle\ ,\ \rangle$ is the canonical inner product on $\mathbb{R}^n$.

PROPOSITION 4.8. *If Γ is a compact group acting on $\mathbb{V}$ and if $\mathbb{W}$ is Γ-invariant subspace then there exists a Γ-invariant complementary subspace $\mathbb{U}$ such that*

$$\mathbb{V} = \mathbb{W} \oplus \mathbb{U}$$

DEFINITION 4.9. The action of Γ on $\mathbb{V}$ is irreducible if there is no Γ-invariant subspaces except $\{0\}$ and $\mathbb{V}$. A subspace $\mathbb{W} \subset \mathbb{V}$ is said to be Γ-irreducible if $\mathbb{W}$ is Γ-invariant and the action of Γ on $\mathbb{W}$ is irreducible.

DEFINITION 4.10. A mapping $g\colon \mathbb{V} \to \mathbb{V}$ is called Γ-equivariant if $g(\gamma\mathbf{v}) = \gamma g(\mathbf{v})$ for all $\gamma \in \Gamma$ and $\mathbf{v} \in \mathbb{V}$.

LEMMA 4.11. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a S_n-equivariant linear map. Then the matrix of A with respect to canonical basis of $\mathbb{R}^n$ has the form*

$$A = \begin{pmatrix} x & y & \dots & y \\ y & x & \dots & y \\ \vdots & \vdots & \ddots & \vdots \\ y & y & \dots & x \end{pmatrix}_{n\times n} \tag{4.3}$$

Proof. Indeed, by labelling $(\sigma A)_{ij} = b_{ij}$ and $(A\sigma)_{ij} = c_{ij}$ we have

$$b_{ij} = a_{\sigma(i)j} \quad \text{and} \quad c_{ij} = a_{i\sigma^{-1}(j)}$$

So, the equation $A = \sigma A \sigma^{-1}$ implies that $a_{ij} = a_{\sigma(i)\sigma(j)}$. Making the index i, j run over $\{1, 2, \dots, n\}$ and the permutation σ over S_n one see that A has the form (4.3). $\square$

By induction on n it's easy to prove that the determinant of A is

$$\det(A) = (x + (n-1)y)(x-y)^{n-1} \tag{4.4}$$

DEFINITION 4.12. A representation of a group Γ on a vector space $\mathbb{V}$ is absolutely irreducible if the only Γ-equivariant linear mappings on $\mathbb{V}$ are scalar multiples of the identity.

PROPOSITION 4.13. *If Γ is compact and the Γ-action on $\mathbb{V}$ is absolutely irreducible then it is irreducible.*

EXAMPLE 4.14. The action of S_4 on $\mathbb{R}^3$: let us consider $\mathbb{R}^3 \approx \mathbb{W} \subset \mathbb{R}^4$ where $\mathbb{W}$ is the S_4-invariant linear subspace

$$\mathbb{W} = \{\mathbf{x} \in \mathbb{R}^4 : \quad \sum x_i = 0\}$$

The isomorphism is given by

$$\mathbb{R}^3 \to \mathbb{W}$$
$$(y_1, y_2, y_3) \mapsto \sum y_j \mathbf{u}_j$$

where

$$\mathbf{u_1} = (-1, 1, -1, 1), \quad \mathbf{u_2} = (-1, -1, 1, 1), \quad \mathbf{u_3} = (1, -1, -1, 1)$$

form an orthogonal basis $\mathcal{B}$ for $\mathbb{W}$. With respect to this basis the action of S_4 on $\mathbb{W}$ is given by multiplying the column vectors $[\mathbf{u}]_{\mathcal{B}}$ by matrices of the type

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{etc} \ldots$$

that is, 3×3 matrices whose rows are permutations of those of the identity matrix changing two signs or none. This action is absolutely irreducible. Indeed, the 3×3 matrix which commutes with the elements of $S_3 \subset S_4$ are of the type

$$\begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix} \tag{4.5}$$

By requiring that this matrix commutes with, for example,

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4.6}$$

we have that $b = -b$.

LEMMA 4.15. *Let* $g : \mathbb{V} \to \mathbb{V}$ *be a* Γ*-equivariant mapping. If* $\Sigma \subset \Gamma$ *is a subgroup then*

$$g(\mathrm{Fix}(\Sigma)) \subseteq \mathrm{Fix}(\Sigma) \tag{4.7}$$

PROPOSITION 4.16. *If* $\mathbf{x} \in \mathrm{Fix}(\Gamma)$ *and* g *is a* Γ*-equivariant mapping then*

(a) $Dg(\mathbf{x})$ *is a* Γ*-equivariant linear mapping,*
(b) $\ker\{Dg(\mathbf{x})\}$ *and* $Im\{Dg(\mathbf{x})\}$ *are* Γ*-invariant subspaces.*

The main result in this section is the *equivariant branching lemma* which we will discuss now. First, let $g : \mathbb{V} \times \mathbb{R} \to \mathbb{V}$ be a Γ-equivariant mapping such that

$$g(0, \epsilon) = 0 \quad \text{and} \quad D_{\mathbf{x}} g(0, 0) = 0 \tag{4.8}$$

If Γ is a Lie group acting absolutely irreducibly on $\mathbb{V}$ then by (4.16a) we have that

$$D_{\mathbf{x}} g(0, \epsilon) = c(\epsilon)\mathrm{Id}$$

and, by (4.8), $c(0) = 0$.

THEOREM 4.17. (Vanderbauwhede – 1980). *Under the hypothesis above, if* $\Sigma \subset \Gamma$ *is an isotropy subgroup satisfying*

$$\dim(\mathrm{Fix}(\Sigma)) = 1$$

and $c'(0) \neq 0$ *then there exists a unique smooth solution branch to* $g(\mathbf{x}, \epsilon) = 0$ *such that* $\Sigma_{\mathbf{x}} = \Sigma$.

The proof can be found in (Golubitsky et al., 1988). It shows that the solution branch is given in the form $(\mathbf{x}, \epsilon(\mathbf{x}))$.

4.2. THE PROBLEM

Let $q(m, 1, 1, 1, 1)$ be a spatial configuration consisting of four unitary masses located at the vertices of a regular tetrahedron and one variable mass located at the barycenter.

Such a configuration is Dziobek for any value of m and it's described by the coordinates

$$\begin{cases} s_{ij} = s, & \text{for } 1 \leqslant qi < j \leqslant 4 \\ s_{0j} = s', & \text{for } 1 \leqslant j \leqslant 4 \\ \Delta = \delta(m)(-4, 1, 1, 1, 1) \end{cases} \tag{4.9}$$

where

$$\frac{s'}{s} = \frac{3}{8} \quad \text{and} \quad \delta(m) = \sqrt{\frac{(\beta - 1)m}{(4 + m)(\beta m + 4)}} \quad \text{with} \quad \beta = \left(\frac{3}{8}\right)^a$$

REMARK 4.18. The fraction $3/8$ is given by the geometry of q while the expression for $\delta(m)$ is calculated by substituting s_{ij} given for (1.10) into the fraction $s'/s = 3/8$.

Now, we transform the equations (1.8), (1.9) and (1.10) into a 4×4 system which describes all spatial Dziobek configurations with four equal masses. By labelling $x_i = \Delta_i$ for $i = 1, \ldots, 4$ and defining

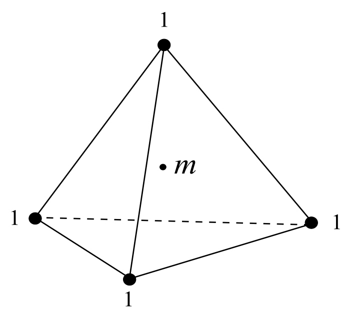

Figure 4. Configuration $q(m, 1, 1, 1, 1)$.

$$\psi(S) = \left(-S + \frac{1}{4+m}\right)^{\frac{1}{a}}$$

we rewrite the four equations $t_0 - t_k$, $k = 1, \ldots, 4$, in the form

$$f_k(\mathbf{x}, m) = \sum_{\substack{j=1 \\ j \neq k}}^{4} x_j \left\{ \psi\left(\frac{x_0 x_j}{m}\right) - \psi\left(x_k x_j\right) \right\} + (x_k - x_0)\psi\left(\frac{x_0 x_k}{m}\right) \tag{4.10}$$

where we have put $x_0 = -x_1 - x_2 - x_3 - x_4$.

So, if $F : \mathbb{R}^4 \times \mathbb{R}_+ \to \mathbb{R}^4$ is given by $F = (f_1, f_2, f_3, f_4)$ then the equation

$$F(x_1, x_2, x_3, x_4, m) = (0, 0, 0, 0) \tag{4.11}$$

defines the Dziobek configurations for the masses $(m, 1, 1, 1, 1)$.

PROPOSITION 4.19. *The mapping F is S_4 equivariant.*

Proof. Let $\sigma \in S_4$. Being $\sigma \cdot \mathbf{x} = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$ for $\mathbf{x} \in \mathbb{R}^4$, we write

$$f_k(\sigma \cdot \mathbf{x}, m) = \sum_{\substack{j=1 \\ j \neq k}}^{4} x_{\sigma(j)} \left\{ \psi\left(\frac{x_0 x_{\sigma(j)}}{m}\right) - \psi\left(x_{\sigma(k)} x_{\sigma(j)}\right) \right\} + (x_{\sigma(k)} - x_0)\psi\left(\frac{x_0 x_{\sigma(k)}}{m}\right)$$

$$= \sum_{\substack{j=1 \\ j \neq \sigma(k)}}^{4} x_j \left\{ \psi\left(\frac{x_0 x_j}{m}\right) - \psi\left(x_{\sigma(k)} x_j\right) \right\} + (x_{\sigma(k)} - x_0)\psi\left(\frac{x_0 x_{\sigma(k)}}{m}\right) = f_{\sigma(k)}(\mathbf{x}, m)$$

from where $F(\sigma \cdot \mathbf{x}, m) = \sigma \cdot F(\mathbf{x}, m)$. □

The vector $\bar{\mathbf{x}} = (\delta(m), \delta(m), \delta(m), \delta(m))$ is a solution of the system (4.11), contained in $\mathrm{Fix}(S_4)$. According to (4.16a), the Jacobian matrix has the structure presented in (4.11), so that the determinant of $D_x F(\bar{x}(m), m)$ is

$$\det(D_{\mathbf{x}} F(\bar{\mathbf{x}}(m), m)) = (\eta_1 + 3\eta_2)(\eta_1 - \eta_2)^3$$

where $\eta_1 = D_{x_1} f_1(\bar{\mathbf{x}}, m)$ and $\eta_2 = D_{x_1} f_2(\bar{\mathbf{x}}, m)$.

After a brief calculation one see that

$$\eta_1 + 3\eta_2 = \frac{6(\beta - 1)(4 + m\beta)}{a\beta(4 + m)} \psi\left(\delta(m)^2\right) < 0 \quad \text{for all} \quad a < -1 \tag{4.12}$$

$$\eta_1 - \eta_2 = \frac{m\beta(a - 2 + 2\beta) + 6(\beta - 1) + 4a\beta}{a\beta(4 + m)} \psi\left(\delta(m)^2\right) \tag{4.13}$$

So, the determinant is zero if, and only if,

$$m = m_c = \frac{6 - 6\beta - 4a\beta}{(a - 2 + 2\beta)\beta} \tag{4.14}$$

REMARK 4.20. Using derivatives with respect to a, one can see that the numerator and the denominator of m_c are both positive for $a < -1$. For Newton's case where $a = -3/2$ we have

$$m_c = \frac{10368 + 1701\sqrt{6}}{54952} \tag{4.15}$$

found by Schmidt (1988).

Calling $\epsilon = m - m_c$, the problem is posed as a bifurcation problem with symmetry

$$F(\mathbf{x}, \epsilon) = \mathbf{0} \tag{4.16}$$

where F is S_4-equivariant and $\bar{\mathbf{x}}(\epsilon) \in \mathrm{Fix}(S_4)$ is a non-degenerate solution for all $\epsilon \neq 0$. Note that, if $\mathrm{L} = \mathrm{D}_{\mathbf{x}}F(\bar{\mathbf{x}}(0), 0)$ then

$$\ker\{\mathrm{L}\} = \{\mathbf{x} \in \mathbb{R}^4 : \quad \sum x_i = 0\} \quad \text{and} \quad \mathrm{Im}\{\mathrm{L}\} = \{(\kappa, \kappa, \kappa, \kappa) : \quad \kappa \in \mathbb{R}\}.$$

To make the Liapunov-Schmidt reduction, consider the change of variables $\mathbf{x} \to \mathbf{y}$ given by

$$\mathbf{x} = \sum_{i=1}^{4} y_i \mathbf{u}_i \tag{4.17}$$

where $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ are exhibited in the example (4.14) and $\mathbf{u}_4 = (1, 1, 1, 1)$. In substituting (4.17) into (4.16), the new equations, labelled by $G(\mathbf{y}, \epsilon) = \mathbf{0}$, have $\bar{\mathbf{y}}(\epsilon) = (0, 0, 0, \delta(\epsilon))$ as a trivial solution. Let $P : \mathbb{R}^4 \to \mathrm{Im}\{\mathrm{L}\}$ be the canonical projection. Then, the equations $G(\mathbf{y}, \epsilon) = \mathbf{0}$ are equivalent to the system

$$PG(\mathbf{y}, \epsilon) = \langle (1, 1, 1, 1), G(\mathbf{y}, \epsilon) \rangle \cdot (1, 1, 1, 1) = (0, 0, 0, 0) \tag{4.18}$$

$$(\mathrm{Id} - P)G(\mathbf{y}, \epsilon) = \sum_{i=1}^{3} \langle \mathbf{u}_i, G(\mathbf{y}, \epsilon) \rangle \mathbf{u}_i = (0, 0, 0, 0) \tag{4.19}$$

The equation (4.18) can be solved for y_4 in terms of $(y_1, y_2, y_3, \epsilon)$ in a neighborhood of $(0, 0, 0, 0)$. We write $y_4 = W(y_1, y_2, y_3, \epsilon)$ and insert it in the equations (4.19) to get the S_4-equivariant bifurcation problem

$$g_i(y_1, y_2, y_3, \epsilon) = \langle \mathbf{u}_i, G(y_1, y_2, y_3, W(y_1, y_2, y_3, \epsilon), \epsilon) \rangle = 0, \quad i = 1, 2, 3 \tag{4.20}$$

with $g_i(0, 0, 0, \epsilon) = 0$ and $\mathrm{D}_{\mathbf{y}} g(0, 0, 0, 0) = \mathbf{0}$. From now, we set $\mathbf{y} = (y_1, y_2, y_3)$.

REMARK 4.21. Considering the $\mathcal{S}_4$ action on $\mathbb{R}^3$ given in the example (4.14), the function $W(\mathbf{y}, \epsilon)$ is $\mathcal{S}_4$-invariant, that is, $W(\sigma \cdot \mathbf{y}, \epsilon) = W(\mathbf{y}, \epsilon)$ for all $\sigma \in \mathcal{S}_4$. Thus

$$D_{\mathbf{y}} W(\mathbf{0}, \epsilon) \in \mathrm{Fix}(S_4) = \{(0, 0, 0)\}.$$

Since the $\mathcal{S}_4$ action on $\mathbb{R}^3$ is absolutely irreducible the matrix $\mathrm{D}_{\mathbf{y}}g(\mathbf{0}, \epsilon)$ is a multiple of $\mathrm{Id}_{3\times 3}$. The derivative of g at the solution $(\mathbf{0}, \epsilon)$ is given by the product matrix

$$\begin{pmatrix} -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \eta_1 & \eta_2 & \eta_2 & \eta_2 \\ \eta_2 & \eta_1 & \eta_2 & \eta_2 \\ \eta_2 & \eta_2 & \eta_1 & \eta_2 \\ \eta_2 & \eta_2 & \eta_2 & \eta_1 \end{pmatrix} \cdot \begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which results in $\mathrm{D}_y g(\mathbf{0}, \epsilon) = 4(\eta_1 - \eta_2)\mathrm{Id}_{3\times 3}$. The function $c(\epsilon) = 4(\eta_1 - \eta_2)$ is given by (4.13) with $m = \epsilon + m_c$. Its derivative with respect to ϵ at $\epsilon = 0$ is

$$c'(0) = \frac{2\beta(a - 2 + 2\beta)^2}{a(\beta - 1)(4\beta - 3)} \left(\frac{1}{2} \frac{(a - 2 + 2\beta)}{(\beta - 1)(1 - 2a)} \right)^{\frac{1}{a}} \neq 0 \quad \text{for all} \quad a < -1$$

The hypotheses of the equivariant branching lemma are fully satisfied, so for each isotropy subgroup of S_4 whose subspace fixed is one-dimensional there exists only one solution of the bifurcation problem $g(\mathbf{y}, \epsilon) = \mathbf{0}$ contained in its fixed subspace. This gives us

THEOREM 4.22. *If* $q(m, 1, 1, 1, 1)$ *is the tetrahedral non-convex central configuration of five bodies then there exists a value* $m_c > 0$, *which depends on the exponent* a, *such that the configuration is non-degenerate for any* $m \neq m_c$. *Furthermore, defined for* m *near* m_c, *there are at least seven families of central configurations which bifurcate from the trivial solution* $q(m, 1, 1, 1, 1)$. *Among them, four present an axis type symmetry and three present a planar type symmetry.*

Proof. The $\mathcal{S}_4$-action on $\mathbb{R}^3$ has seven isotropy subgroups. The first four

$$\Sigma_{(1,1,1)} \quad \Sigma_{(-1,-1,1)} \quad \Sigma_{(-1,1,-1)} \quad \Sigma_{(1,-1,-1)} \tag{4.21}$$

produce solutions with planar type symmetry ($\Delta_1 = \Delta_2 = \Delta_3$ for example) and the last three

$$\Sigma_{(1,0,0)} \qquad \Sigma_{(0,1,0)} \qquad \Sigma_{(0,0,1)} \tag{4.22}$$

produce solutions with axis type symmetry ($\Delta_1 = \Delta_2$ and $\Delta_3 = \Delta_4$ for example). □

The relation between the solutions of $g(\mathbf{y}, \epsilon) = \mathbf{0}$ and $F(\mathbf{x}, \epsilon) = \mathbf{0}$ is made explicit below:

Axis type symmetry

$$(y, y, y) \in \mathrm{Fix}(\Sigma_{(1,1,1)}) \quad \Longrightarrow \quad (x, x, x, z) \tag{4.23}$$

$$(-y, -y, y) \in \mathrm{Fix}(\Sigma_{(-1,-1,1)}) \quad \Longrightarrow \quad (z, x, x, x) \tag{4.24}$$

$$(-y, y, -y) \in \mathrm{Fix}(\Sigma_{(-1,1,-1)}) \quad \Longrightarrow \quad (x, x, z, x) \tag{4.25}$$

$$(y,-y,-y) \in \mathrm{Fix}(\Sigma_{(1,-1,-1)}) \quad \Longrightarrow \quad (x,z,x,x) \tag{4.26}$$

Planar type symmetry

$$(y,0,0) \in \mathrm{Fix}(\Sigma_{(1,0,0)}) \quad \Longrightarrow \quad (r,s,r,s) \tag{4.27}$$

$$(0,y,0) \in \mathrm{Fix}(\Sigma_{(0,1,0)}) \quad \Longrightarrow \quad (r,r,s,s) \tag{4.28}$$

$$(0,0,y) \in \mathrm{Fix}(\Sigma_{(0,0,1)}) \quad \Longrightarrow \quad (s,r,r,s) \tag{4.29}$$

4.3. COMPUTATION OF THE BIFURCATION

4.3.1. *Axis Type Symmetry*

We suppose the solutions have y as formal power series of ϵ.

$$y(\epsilon) = b\epsilon + \mathcal{O}(\epsilon^2) \tag{4.30}$$

Expanding each equation $g_i = 0$ in power series and substituting $y_i = y(\epsilon)$ we obtain

$$g_i(y(\epsilon),\epsilon) = (p \cdot b + q \cdot b^2)\epsilon^2 + \mathcal{O}(\epsilon^3) = 0$$

from where the non-trivial solutions have the first order coefficient $b = -\frac{p}{q}$. The computations show that

$$b = -\frac{1}{24}\frac{a\beta}{\xi(a)}\sqrt{\frac{(6-6\beta-4a\beta)(a-2+2\beta)^5(4\beta-3)}{(\beta-1)^3(1-2a)}}$$

where

$$\xi(a) = a^3 - 8a^2\beta^2 + 10a^2\beta - 5a^2 - 8a\beta^2 + 9a\beta - a + 8\beta - 4\beta^2 - 4$$

4.3.2. *Planar Type Symmetry*

Writing the possible solutions as a formal power series in terms of ϵ, we only obtain the solutions with axis type symmetry. According the proof of theorem (4.17), the solutions are given as $(\mathbf{x}, \epsilon(\mathbf{x}))$ with $\mathbf{x} \in \mathrm{Fix}(\Sigma)$ which is a one dimensional vector space. In view of this, one cannot hope that the solution with planar type symmetry arise as a series in powers of ϵ^n with $n \in \mathbb{Z}_+$. So we look for solutions (4.27)–(4.29) where

$$y(\epsilon) = d(-\epsilon)^{1/2} + \mathcal{O}(((-\epsilon)^{1/2})^2) \tag{4.31}$$

with $\epsilon < 0$, that is, $m < m_c$.

Now we expand g in power series until the third order. Substituting $(y(\epsilon),0,0)$ in g_1, g_2 and g_3 we have that g_2 and g_3 vanish fully while

$$g_1(y(\epsilon),0,0,\epsilon) = y(\epsilon)((p - d^2\tilde{q})\epsilon + \ldots)$$

Thus, $g_1 = 0$ until the third order if

$$d = \sqrt{\frac{p}{\tilde{q}}} = \sqrt{-\frac{3}{8}\frac{a^2\beta(6-6\beta-4a\beta)(a-2+2\beta)^3(4\beta-3)}{(1-2a)(\beta-1)\zeta(a)}}$$

where

$$\begin{aligned}\zeta(a) &= 96+36a-432a\beta+32a^5\beta^4-30a^3+120a^2-408a^2\beta-12a^5\beta+\\ &48a^5\beta^2-80a^4\beta^4+8a^3\beta^4+324a^2\beta^4+324a\beta^4+1080a\beta^2+96a^3\beta^2-384\beta^3+\\ &780a^2\beta^2+27a^4-6a^5-1008a\beta^3-32a^3\beta^3-42a^3\beta-816a^2\beta^3+96a^4\beta^3-\\ &18a^4\beta+96\beta^4+576\beta^2-24a^4\beta^2-64a^5\beta^3-384\beta\end{aligned}$$

REMARK 4.23. We have $\zeta(a) < 0$ for all $a < -1$. In effect, $\zeta(a)$ is a polynomial in a and β. Expanding it in Taylor series at the point $a = -1$ and next writing the coefficients in terms of β as a Taylor series at the point $\beta = 2$ we have

$$\begin{aligned}\zeta(a) &= -(357+660(\beta-2)+468(\beta-2)^2+192(\beta-2)^3+24(\beta-2)^4)\\ &+[2156+4494(\beta-2)+3408(\beta-2)^2+1264(\beta-2)^3+180(\beta-2)^4]\cdot(a+1)\\ &-[4668+10846(\beta-2)+9156(\beta-2)^2+3504(\beta-2)^3+500(\beta-2)^4]\cdot(a+1)^2\\ &+[4230+10662(\beta-2)+9888(\beta-2)^2+4128(\beta-2)^3+648(\beta-2)^4]\cdot(a+1)^3\\ &-[1427+3702(\beta-2)+3528(\beta-2)^2+1504(\beta-2)^3+240(\beta-2)^4]\cdot(a+1)^4\\ &+[162+436(\beta-2)+432(\beta-2)^2+192(\beta-2)^3+32(\beta-2)^4](a+1)^5\end{aligned}$$

which is clearly negative for $a < -1$. (Remember that $\beta = \left(\frac{3}{8}\right)^a > 2$ for all $a < -1$.)

To close the question it would be enough to prove the uniqueness of these seven bifurcations. Up to now, we did not succeed in this task.

Acknowledgements

I am indebt to my advisor M. Alain Albouy for his support and encouragement at the difficult moments of my doctorate. Not less important were Prof. Hildeberto Cabral, Claudio Vidal and Eduardo Shirlippe all from the department of Mathematics of Recife for helpful conversations and important remarks about the problems.

Special acknowledgment. To Prof. Danilo Felizardo Barboza who brought me from the engineering course to the mathematic one. Thank you for helping me to discover my vocation.

References

Albouy, A.: 1997, 'Recherches sur le problème des N corps', *Notes Scientifiques et Techniques du Bureau des Longitudes*, pp. 69–94.

Albouy, A.: 1995, 'Symétrie des configurations centrales de quatre corps', *C.R. Acad. Sci. Paris* **320**, 217–220.

Albouy, A.: 1996, 'The symmetric central configurations of four equal masses', *Contemporary Math.* **198**, 131–135.

Albouy, A. and Chenciner, A.: 1998, 'Le problème des N corps et les distances mutuelles', *Inventiones Math.* **131**, 151–184.

Albouy, A. and Libre, J.: 2002, 'Spatial central configurations for the $1+4$ body problem', *Proceedings of an International Conference dedicated to D. Saari, Contemporary Math.* **292**, 1–16.

Albouy, A.: 2004, 'On a paper of Moeckel on Central Configurations', *Regular and Chaotic Dynamics*, in press

Saari, D.: 1980, 'On the role and properties of n body central configurations', *Celestial Mech.* **21**, 9–20.

Long, Y. and Sun, S.: 2002, 'Four-Body central configurations with some equal masses', *Arch. Rational Mech. Anal.* **162**, 25–44.

Pedersen, P.: 1944, 'Librationspunke im restringierten Vierköperproblem', *Dan. Mat. Fys Medd.* **21** (6).

Meyer, Kenneth R., Schmidt and Dieter, S.: 1988, 'Bifurcations of relative equilibria in the 4- and 5-body problem', *Ergodic Theory Dyn. Syst.* **8***, 215–225.

Arenstorf, R.F.: 1982, 'Central configurations of four bodies with one inferior mass', *Celestial Mech.* **28**, 9–15.

Simó, C.: 1978, 'Relative equilibrium solutions in the four body problem', *Celestial Mech.* **18**, 165–184.

Schmidt, D.S.: 1988, 'Central configurations in $\mathbb{R}^2$ and $\mathbb{R}^3$,' *Contemporary Math.* **81**, 59–76.

Chow, S.-N. and Hale, J.K.: 1982, '*Methods of Bifurcation Theory*', Springer-Verlag, New York.

Golubitsky, M. and Schaeffer, D.G.: 1985, '*Singularities and Groups in Bifurcation Theory*', Vol. I, Springer-Verlag, New York.

Golubitsky, M., Stewart, I. and Schaeffer, D.: 1988, '*Singularities and Groups in Bifurcation Theory*', Vol. II, Springer-Verlag, New York.