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ON THE EXISTENCE OF THE OPTIMAL CONTROL FOR STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS SUBJECT TO EXTERNAL DISTURBANCES

V. K. Yasynskyy[1†] **and I. V. Yurchenko**[1‡]

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Abstract. *The authors discuss the comparison theorem for solutions of stochastic functional differential equations subject to external disturbances and its application to a stochastic control problem.*

Keywords: *comparison theorem, stochastic control, stochastic functional differential equations.*

To the Memory of Mykhailo Leonovych Sverdan
(01.18.1940–11.19.2023)

INTRODUCTION

Within the framework of stochastic control theory, control in systems with random parameters is investigated. This theory is widely applied in many fields, including finance, engineering, economics, etc.

The beginning of stochastic control theory is associated with the analysis of solutions to stochastic differential equations describing system's evolution in time under random disturbances or influence of random factors. Stochastic control theory was developed by improving the methods of solving stochastic differential equations, introducing new approaches to the analysis of random processes, and applying them to such areas as finance, optimal portfolio management, risk management, and many others (see [1–11]).

Modern research in stochastic control theory continues to expand its application to new fields, develop more efficient methods for solving complex problems, and expand its theoretical base.

This article considers the comparison theorem for solutions of stochastic functional differential equations (SFDE) subject to external disturbances and its application to a stochastic control problem, which is a development of the results obtained for one-dimensional Ito processes in [3, 5–7] and for the case of Poisson disturbances in [12–15].

COMPARISON THEOREM FOR SFDE SOLUTIONS

Let (Ω, F, P) be a probability space with a flow of σ-algebras $\{F_t, t \geq 0\}$, $\mathbf{D}$ be the space of right-continuous functions with left-hand boundaries (RCLB) with values from $\mathbf{R}^1$ and with a uniform metric [1–3].

THEOREM 1. Let the following be given:

(i) a strictly increasing function $\{\rho(x), x \in \mathbf{R}_+\}$ such that

$$\rho(0) = 0, \ \int_0^\infty \rho^{-2}(x) dx = \infty;$$

[1]Yuriy Fedkovych Chernivtsi National University, Chernivtsi, Ukraine, [†]*v.yasynskyy@chnu.edu.ua*; [‡]*i.yurchenko@chnu.edu.ua*. Translated from Kibernetyka ta Systemnyi Analiz, No. 3, May–June, 2024, pp. 141–151; DOI 10.34229/KCA2522-9664.24.3.13. Original article submitted January 30, 2024.

(ii) continuous functionals $b:\mathbf{R}_+\times\mathbf{D}\to\mathbf{R}^1$ and $c:\mathbf{R}_+\times\mathbf{D}\times\mathbf{V}\to\mathbf{R}^1$, $\mathbf{V}\subset\mathbf{R}^1$ such that for arbitrary $\varphi,\psi\in\mathbf{D}$

$$|b(t,\varphi)-b(t,\psi)|+\int\limits_{\mathbf{V}}|c(t,\varphi,u)-c(t,\psi,u)|\Pi(du)\le\rho(\|\varphi-\psi\|),\ \ t\ge 0;$$

(iii) two continuous functionals $a_i:\mathbf{R}_+\times\mathbf{D}\to\mathbf{R}^1$, $i=1,2$, such that

$$a_1(t,\varphi)\le a_2(t,\varphi),\ \ t\ge 0,\ \ \varphi\in\mathbf{D};$$

(iv) random processes $\{x_i(t)\equiv x_i(t,\omega),\ t\in\mathbf{R}_+,\ \omega\in\Omega,\ i=1,2\}$, continuous in t and $\{F_t\}$-measurable in ω;

(v) standard Wiener process $w(t)\equiv w(t,\omega)$: $[0,\infty)\times\Omega\subset\mathbf{R}^1$ such that $w(0)=0\ (\mathrm{mod}\ \mathbf{P})$;

(vi) centered Poisson measure $\{\widetilde{\nu}(t,A)=\nu(t,A)-\Pi(A)t,\ A\in\mathbf{V}\subset\mathbf{R}\setminus\{0\}\}$, where $\{w(t)\}$ and $\{\widetilde{\nu}(t,A)\}$ are independent of each other;

(vii) random processes $\{\alpha_i(t)\equiv\alpha_i(t,\omega),\ t\in\mathbf{R},\ \omega\in\Omega\}$ measurable with respect to $\{F_t,\ t\ge 0\}$.

Let also the random processes from (iv)–(vii) of this theorem satisfy the following conditions with probability one:

$$x_i(t)-x_i(0)=\int\limits_0^t\alpha_i(s)ds+\int\limits_0^t b(s,x_i^s)dw(s)+\int\limits_0^t\int\limits_{\mathbf{V}}c(s,x_i^s,v)\widetilde{\nu}(ds,dv),\tag{1}$$

$$x_1(\theta)\le x_2(\theta),\ \ \theta\in(-\infty,0],\ \ \alpha_1(t)\le a_1(t,x_1^t),\ \ t\ge 0,$$

$$\alpha_2(t)\le a_2(t,x_2^t),\ \ t\ge 0,\tag{2}$$

where $\{x^t\equiv x(t+\theta),\theta\in(-\infty,0]\}$.

Then the inequality

$$x_1(t)\le x_2(t)\ \ \forall t\ge 0\tag{3}$$

holds with probability one. If the condition of the uniqueness of strong solutions with probability one holds for at least one of the SFDEs

$$dx_i(t)=a_i(t,x^t)dt+b(t,x^t)dw(t)+\int\limits_{\mathbf{V}}c(t,x^t,v)\widetilde{\nu}(dt,dv),\ \ i=1,2,\tag{4}$$

then (3) also holds under a weaker condition:

$$a_1(t,\varphi)\le a_2(t,\varphi),\ \ t\ge 0,\ \ \varphi\in\mathbf{D}.$$

Proof. Assume that a_i, $i=1,2$, b, and c are bounded (taking into account considerations of localization [3, 7, 13]).

Stage 1. Let the Lipshitz condition be satisfied for $a_1(t,x)$:

$$|a_1(t,\varphi)-a_1(t,\psi)|\le K\|\varphi-\psi\|\ \ \forall\varphi,\psi\in\mathbf{D}.\tag{5}$$

Select a sequence $\{\psi_n(t),\ t\in\mathbf{R}_+,\ n=1,2,\ldots\}$ of continuous functions such that their supports are contained in the intervals

$$(a_n,a_{n-1}),\ \ 1>a_1>\ldots>a_n>0,$$

$$0\le\psi_n(t)\le\frac{2\rho^{-2}(t)}{n},\ \ \int\limits_{a_n}^{a_{n-1}}\psi_n(t)dt=1.$$

Let

$$\varphi_n(t)=\begin{cases}0,\ x\le 0,\\ \int\limits_0^x dy\int\limits_0^y\psi_n(t)dt,\ x>0.\end{cases}$$

For $\varphi_n \in C(\mathbf{R}^1)$, the relations hold:

$$\varphi_n(x)=0 \text{ for } n\le 0,$$

$$\lim_{n\to\infty}\varphi_n(x)=x^+ \text{ for } n\to\infty;$$

$$\varphi'_n(x)=\psi_n(x),\ 0\le\varphi'_n(x)\le 1. \tag{6}$$

Applying the generalized Ito formula [3], we obtain

$$\varphi_n(x_1(t)-x_2(t))=\sum_{i=1}^{5} I_i(n),$$

where

$$I_1(n)\equiv\int_0^t \varphi'_n(x_1(s)-x_2(s))\{b(s,x_1^s)-b(s,x_2^s)\}dw(s),$$

$$I_2(n)\equiv\int_0^t \varphi'_n(x_1(s)-x_2(s))\{\alpha_1(s)-\alpha_2(s)\}ds,$$

$$I_3(n)\equiv\frac{1}{2}\int_0^t \varphi'_n(x_1(s)-x_2(s))\{b(s,x_1^s)-b(s,x_2^s)\}^2 ds,$$

$$I_4(n)\equiv\int_0^t\int_{\mathbf{V}}[\varphi_n(x_1(s)-x_2(s)+c(s,x_1^s,\upsilon)-c(s,x_2^s,\upsilon))-\varphi_n(x_1(s)-x_2(s))]\tilde{\nu}(ds,d\upsilon),$$

$$I_5(n)\equiv\int_0^t\int_{\mathbf{V}}[\varphi_n(x_1(s)-x_2(s)+c(s,x_1^s,\upsilon)-c(s,x_2^s,\upsilon))$$

$$-\varphi_n(x_1(s)-x_2(s))-\varphi'_n(x_1(s)-x_2(s))(c(s,x_1^s,\upsilon)-c(s,x_2^s,\upsilon))]\Pi(d\upsilon)ds.$$

Using the properties of stochastic integrals [3], we obtain

$$E\{I_1(n)\}=0,\ E\{I_4(n)\}=0,$$

$$E\{I_3(n)\}\le\frac{1}{2}E\left\{\int_0^t\varphi'_n(x_1(s)-x_2(s))\rho^2(|x_1(s)-x_2(s)|)ds\right\}\le\frac{1}{2}E\left\{\int_0^t\psi_n(x_1(s)-x_2(s))\rho^2(|x_1(s)-x_2(s)|)ds\right\}$$

$$\le\frac{1}{2}E\left\{\int_0^t\frac{2}{n}\rho^{-2}(|x_1(s)-x_2(s)|)\rho^2(|x_1(s)-x_2(s)|)ds\right\}\le\frac{t}{n}.$$

Taking into account conditions (1), (2), (5), and (6), we obtain

$$I_2(n)\le\int_0^t\varphi'_n(x_1(s)-x_2(s))\{a_1(s,x_1^s)-a_2(s,x_2^s)\}ds\le K\int_0^t\chi_{x_1(s)>x_2(s)}\|x_1^s-x_2^s\|ds\le K\int_0^t(x_1^s-x_2^s)^+ds.$$

From (6) it follows that

$$\lim_{n\to\infty}E\{I_5(n)\}=E\left\{\int_0^t\int_V\lim_{n\to\infty}[\varphi_n(x_1(s)-x_2(s)+c(s,x_1^s,\upsilon)-c(s,x_2^s,\upsilon))]\right.$$

$$\left.-\lim_{n\to\infty}\varphi_n(x_1(s)-x_2(s))-\lim_{n\to\infty}\varphi'_n(x_1(s)-x_2(s))(c(s,x_1^s,\upsilon)-c(s,x_2^s,\upsilon))\right\}\Pi(d\upsilon)ds$$

$$=\int_0^t\int_{\mathbf{V}}\lim_{n\to\infty}(c(s,x_1^s,\upsilon)-c(s,x_2^s,\upsilon))[1-\varphi'_n(x_1(s)-x_2(s))]\Pi(d\upsilon)ds=0.$$

From the limit equality
$$\lim_{n\to\infty}\frac{\varphi_n(x_1(t)-x_2(t))}{x_1(t)-x_2(t)}=1$$
the inequality follows:
$$E\{(x_1(t)-x_2(t))^+\}\le K\int_0^t E\{(x_1(s)-x_2(s))^+\}ds.$$
From here, we obtain that $E\{(x_1(t)-x_2(t))^+\}=0\ \forall t\ge 0,$ i.e., $P\{(x_1(t)\le x_2(t))^+\}=1\ \forall t\ge 0.$

Since the conditions of the theorems on the existence and uniqueness of the solution from [3, 7, 13] are satisfied, we obtain the RCLB of the trajectory and conclude that (3) holds. Note that if the Lipshitz condition (5) is satisfied for $a_2(t,\varphi)$, and from a similar reasoning regarding Stage 1, we can obtain (3).

Stage 2. In the general case, choose $a(t,\varphi)$ in such a way that
$$a_1(t,\varphi)< a(t,\varphi)< a_2(t,\varphi),\ t\ge 0,\ \varphi\in\mathbf{D},$$
and so that condition (3) holds for $a(t,\varphi)$.

Let $\{x(t)\}$ be the unique solution of the SFDE
$$dx(t)=a(t,x^t)dt+b(t,x^t)dw(t)+\int_{\mathbf{V}} c(t,x^t,\upsilon)\tilde{\nu}(dt,d\upsilon),\ t\ge 0,$$
with the initial conditions
$$x(\theta)=\varphi_2(\theta),\ -\infty<\theta\le 0.$$
Then, by the results of Stage 1, we get $x(t)\le x_2(t)$ and $x_1(t)\le x(t)$ for $t\ge 0$ almost surely. Therefore, (3) holds.

Stage 3. Let there exist a unique, up to stochastic equivalence, solution $\{x(t)\}$ to Eq. (4) provided that $i=1$:
$$dx(t)=a_1(t,x^t)dt+b(t,x^t)dw(t)+\int_{\mathbf{V}} c(t,x^t,\upsilon)\tilde{\nu}(dt,d\upsilon), \tag{7}$$
$$x(\theta)=\varphi_1(\theta),\ -\infty<\theta\le 0.$$

For $\varepsilon>0$, we get two solutions $x^{(\pm\varepsilon)}(t)$ of the equations
$$dx(t)=(a_1(t,x^t)\pm\varepsilon)dt+b(t,x^t)dw(t)+\int_{\mathbf{V}} c(t,x^t,\upsilon)\tilde{\nu}(dt,d\upsilon),$$
$$x(\theta)=\varphi_1(t),\ -\infty<t\le 0,$$
for which it follows from Stages 1, 2 that
$$x^{(-\varepsilon)}(t)\le x(t)\le x^{(+\varepsilon)}(t)\ \ \forall t\ge 0.$$
If $0<\varepsilon_2<\varepsilon_1$, then
$$x^{(-\varepsilon_1)}(t)\le x^{(-\varepsilon_2)}(t),\ x^{(+\varepsilon_2)}(t)\le x^{(+\varepsilon_1)}(t)\ \ \forall t\ge 0.$$
Thus, due to the continuity of $a_1(t,\varphi)$ and uniqueness of the solution for (7), we obtain
$$\lim_{\varepsilon\downarrow 0} x^{(-\varepsilon)}(t)=\lim_{\varepsilon\downarrow 0} x^{(+\varepsilon)}(t)=x(t)\ \ \forall t\ge 0.$$
Given the inequalities $\alpha_1\le a_1(t,x_1^t),\ a_1(t,x_1^t)\le a_1(t,x_1^t)+\varepsilon,$ applying the results proved above for $x_2(t)$ and $x^{(+\varepsilon)}(t)$, we can write
$$x_1(t)\le x^{(+\varepsilon)}(t)\ \ \forall t\ge 0.$$
For $\varepsilon\downarrow 0^+$, we obtain
$$x_1(t)\le x(t)\ \ \forall t\ge 0. \tag{8}$$

From a similar reasoning in the case $\alpha_2(t) \geq a_2(t, x_2^t)$ and $a_2(t, x^t) > a_1(t, x^t) - \varepsilon$, we get the inequality $x^{(-\varepsilon)}(t) \leq x_2(t)$ with probability one. Under the condition $\varepsilon \downarrow 0^+$, we get

$$x(t) \leq x_2(t) \forall t \geq 0. \tag{9}$$

Comparing inequalities (8) and (9), we get

$$x_1(t) \leq x(t) \leq x_2(t) \ \ \forall t \geq 0,$$

which completes the proof of the theorem. ■

Remark 1. Theorem 1 also holds for non-linear SFDEs of the form

$$dx(t,\omega) = x(\theta) + \sum_{j=1}^{r} \alpha_j(t,\omega) a_j(t, x(t+\theta,\omega))dt + \sum_{j=r+1}^{2r+1} \alpha_j(t,\omega) b_j(s, x(t+\theta,\omega))dw_j(t,\omega)$$

$$+ \sum_{j=2r+2}^{3r+2} \int_{\mathbf{V}} \alpha_j(t,\omega) c_j(t, x(s+\theta,\omega), v)\widetilde{\nu}_j(dt, dv, \omega) \tag{10}$$

provided that

$$x(t+\theta,\omega)\big|_{t=0} = \varphi(\theta); \ \ \alpha_j(t,\omega)\big|_{t=0} = \beta(\theta), \ \ j = \overline{1, 3r+2}, \tag{11}$$

where $\alpha_j(t,\omega){:}[0,T]\times\Omega \to \mathbf{R}^n \times \mathbf{R}^n$ are external random processes pairwise independent with each other and with the Wiener processes, and coefficients a_j, b_j, and c_j satisfy the Lipshitz properties and conditions of Theorem 1.

PROBLEM OF THE EXISTENCE OF OPTIMAL CONTROL OF SFDE SOLUTIONS AMONG THE FEASIBLE CONTROLS

Let us consider a stochastic optimization problem that can be solved using Theorem 1. This optimization problem is an example of proving the existence of stochastic control for the class of SFDE subject to external disturbances of the type of random processes. The obtained results are a development of the research carried out in [7, 13, 15] to the case of the action of external disturbances of the type of random processes.

Let $k(z)$ be a nondecreasing non-negative function defined on $[0, \infty)$.

Definition 1. A system of random processes

$$\{\alpha_j(t,\omega),\ j = \overline{1, 3r+1},\ w_l(t,\omega),\ l = \overline{r+1, 2r+1},\ \widetilde{\nu}_k(t, v, \omega),$$

$$k = \overline{2r+2, 3r+2},\ \ u(t,\omega),\ \ t \geq 0\} \tag{12}$$

is called a feasible system or feasible control if:

(i) it is defined in space $(\Omega, F, \{F_t, t \geq 0\}, \mathfrak{R}_y(\mathbf{Y}), P)$, where $\{F_t, t \geq 0\}$ is the flow of σ-algebras and $\mathfrak{R}_y(\mathbf{Y})$ is σ-algebra of the set $\mathbf{Y} \equiv \{y_1, y_2, \ldots, y_r\} \subset \mathbf{R}^r$;

(ii) $\alpha_j(t,\omega){:}\ [0,T]\times\Omega \to \mathbf{R}^n \times \mathbf{R}^n$ are diagonal matrices, measurable with respect to the minimum σ-algebra $F_t \cap \mathfrak{R}_y(\mathbf{Y})$, pairwise independent of each other and of n-measurable Wiener processes $w_j(t,\omega)$ and centered Poisson measures $\widetilde{\nu}_j(t, v, \omega)$, with $\alpha_j(t,\omega) \in \mathbf{C}([-\Delta, \infty])$;

(iii) $u(t)$ is an n-measurable $F_t \cap \mathfrak{R}_y(\mathbf{Y})$ random process such that $|u(t,\omega)| \leq 1$ for all $t \geq 0$ with probability one (almost everywhere);

(iv) $x(t,\omega) \in \mathbf{C}([-\Delta, \infty])$ is a given and fixed random process $x(t,\omega){:}\ [-\Delta, \infty]\times\Omega \to \mathbf{R}^n$.

For the feasible system (12) and its coefficients, the conditions of Theorem 1 (comparison theorem) are satisfied. Risk $x^u(t,\omega)$ for (12) is determined by the equality

$$x^u(t,\omega)=x(\theta)+\int_0^t \sum_{j=r+1}^{2r+1} \alpha_j(s,\omega) b(s, y, x^u(s+\theta)) dw_j(s)$$

$$+\int_0^t \int_{\mathbf{V}} \sum_{j=2r+2}^{3r+2} \alpha_j(s,\omega)\, c(s, x^u(s+\theta,\omega), v)\widetilde{\nu}(ds, dv, \omega)+\int_0^t u(s)ds \tag{13}$$

with the initial conditions
$$x^u(t+\theta,\omega)\big|_{t=0}=x_0^u(\theta),$$

$$\alpha_j(t,\omega)\big|_{t=0}=\alpha_j^0(\omega),\ \ j=\overline{1,3r+2}. \tag{14}$$

To make the proof shorter, we assume that $\alpha_j(t,\omega)\equiv 0,\ j=\overline{1,r},\ x_t^u\equiv\{x^u(t+\theta,\omega),\theta\in[-\Delta,0],\Delta>0\}$.

The problem of minimizing the expectation $E\{k\|x_t^u\|\}$ is formulated with respect to all possible systems (12). We will solve this problem according to the technique described in [7, 13].

Let $U(l)$ be defined as follows:
$$U(l)=\begin{cases}\dfrac{-l}{|l|},\ \ l\in\mathbf{R}^n\setminus\{0\},\\ 0,\ \ l=0.\end{cases}$$

Consider the SFDE of the form
$$dx(t,\omega)=\sum_{j=r+1}^{2r+1}\alpha_j(t,\omega)b(t,x_t)dw_j(t,\omega)+\int_{\mathbf{V}}\sum_{j=2r+1}^{3r+2}\alpha_j(t,\omega)\,c(t,x_t,v)\widetilde{\nu}(dt,dv,\omega)+U(x(t,\omega))dt \tag{15}$$

with the initial conditions
$$x(t+\theta,\omega)\big|_{t=0}=\varphi(\theta),\ \ \theta\in[-\Delta,0),\ \Delta>0; \tag{16}$$

$$x(t+\theta,\omega)\Big|_{\substack{t=0\\ \theta=0}}=x,\ \ \alpha_j(t,\omega)\big|_{t=0}=\alpha_j,\ \ j=1,2. \tag{17}$$

It is known [7, 13] that the solution (15)–(17) exists and is unique with probability one under the conditions of Theorem 1.

Let
$$u^0(s)\equiv U^0(x(t,\omega)),$$

then the feasible system
$$\{\alpha_j^0(t,\omega),\ j=\overline{1,3r+1},\ w_l^0(t,\omega),\ l=\overline{r+1,2r+1},$$

$$\widetilde{\nu}_k^0(t,v,\omega),\ k=\overline{2r+2,3r+2},\ u^0(t,\omega),\ t\ge 0\}$$

specifies the optimal control, i.e., for an arbitrary feasible system (12) we get

$$E\{k(|x(t,\omega)|)\}\le E\{k(|x^u(t,\omega)|)\}.$$

LEMMA 1. Let system (12) be the set of *n*-measurable $\{F_t,\mathfrak{R}_y(\mathbf{Y})\}$-coordinated processes defined on the modified space $(\Omega, F\cap\mathfrak{R}, P)$ with the flow $\{F_t\cap\mathfrak{R}(x)\}$; similar set

$$\{\widetilde{\alpha}_j(t,\omega),\ j=\overline{1,3r+1},\ \widetilde{w}_l(t,\omega),\ l=\overline{r+1,2r+1},\ \widetilde{\widetilde{\nu}}_k(t,v,\omega),\ k=\overline{2r+2,3r+2},\ \widetilde{u}(t,\omega)\}$$

is defined on another modified space $(\widetilde{\Omega},\widetilde{F}\cap\widetilde{\mathfrak{R}},\widetilde{P})$ with the flow $\{\widetilde{F}_t\cap\widetilde{\mathfrak{R}}(x)\}$.

Then there is a modified space $(\hat{\Omega},\hat{F}\cap\hat{\mathfrak{R}},\hat{P})$ with the flow $\{\hat{F}_t\cap\hat{\mathfrak{R}}(x)\}$ and the set

$$\{\hat{\alpha}_j(t,\omega),\ j=\overline{1,3r+1},\ \hat{w}_l(t,\omega),\ l=\overline{r+1,2r+1},\ \hat{\widetilde{\nu}}_k(t,v,\omega),\ k=\overline{2r+2,3r+2},\ \hat{u}(t,\omega)\}$$

of n-dimensional $\{\widetilde{F}_t \cap \widetilde{\mathfrak{R}}(x)\}$-coordinated processes such that:

(i) $\{x(t,\omega),\ \alpha_j(t,\omega),\ j=\overline{1,3r+1},\ w_k(t,\omega),\ k=\overline{r+1,2r+1},\ \widetilde{\nu}_l(t,A,\omega),\ l=\overline{2r+2,3r+2},\ A\in\mathfrak{R}\} \overset{L}{\approx} \{\hat{x}(t,\omega);$ $\hat{\alpha}_j(t,\omega),\ j=\overline{1,3r+1},\ \hat{w}_l(t,\omega),\ l=\overline{r+1,2r+1},\ \hat{\widetilde{\nu}}_k(t,A,\omega),\ k=\overline{2r+2,3r+2},\ A\in\hat{\mathfrak{R}}\}$;

(ii) $\{\widetilde{x}(t,\omega),\ \widetilde{\alpha}_j(t,\omega),\ j=\overline{1,3r+1},\ \widetilde{w}_k(t,\omega),\ k=\overline{r+1,2r+1},\ \widetilde{\widetilde{\nu}}_l(t,A,\omega),\ l=\overline{2r+2,3r+2},\ A\in\widetilde{\mathfrak{R}}\} \overset{L}{\approx} \{\hat{x}(t,\omega);$ $\hat{\alpha}_j(t,\omega),\ j=\overline{1,3r+1},\ \hat{w}_l(t,\omega),\ l=\overline{r+1,2r+1},\ \hat{\widetilde{\nu}}_k(t,\upsilon,\omega),\ k=\overline{2r+2,3r+2},\ A\in\hat{\mathfrak{R}}\}$;

(iii) $\{\hat{w}_k(t,\omega),\ k=\overline{r+1,2r+1}\}$ are n-dimensional Wiener processes;

(iv) $\{\hat{\widetilde{\nu}}_k(t,\upsilon,\omega),\ k=\overline{2r+2,3r+2},\ A\in\hat{\mathfrak{R}}\}$ are $n\times n$-dimensional centered Poisson measures.

Here, symbol $\overset{L}{\approx}$ means that the processes have the same distribution laws.

The proof of Lemma 1 is similar to the proof of Lemma VI–2.1 [7].

Based on the above assumptions, the following theorem is true.

THEOREM 2. Let (12) be an arbitrary given feasible system, and for a given $x\in\mathbf{R}^n$ solution $\{x^u(t,\omega)\}$ be determined by the SFDE (13). Then on some modified probability space it is possible to generate $\mathbf{R}^n$-dimensional processes $\{\widetilde{x}^u(t,\omega)\}$, $\{\widetilde{x}^0(t,\omega)\}$ and $\{\widetilde{\alpha}_j^u(t,\omega),\ j=\overline{1,3r+2}\}$, $\{\widetilde{\alpha}_j^0(t,\omega),\ j=\overline{1,3r+2}\}$ such that

(i) $\{x^u(t,\omega)\} \overset{L}{\approx} \{\widetilde{x}^u(t,\omega)\}$;

(ii) $\{x^0(t,\omega)\} \overset{L}{\approx} \{\widetilde{x}^0(t,\omega)\}$;

(iii) $|\widetilde{x}^0(t,\omega)| \le |\widetilde{x}^u(t,\omega)|$ for an arbitrary $t\ge 0$ with probability one;

(iv) $\{\widetilde{\alpha}_j^0(t,\omega),\ j=\overline{1,3r+2}\} \overset{L}{\approx} \{\widetilde{\alpha}_j^u(t,\omega),\ j=\overline{1,3r+2}\}$ with probability one.

Proof. Let (12) be a given feasible system for which (13), (14) is a solution of Eq. (15)–(17).

Denote by $\{x^0(t),\alpha^0(t),w^0(t),\widetilde{\nu}^0(t,A)\}$ the solution of Eq. (15)–(17) and omit the corresponding indices for convenience.

Select an $o(n)$-dimensional Borel function $\{p_{ij}(x)\}$ such that

$$p_{ij}(x)=\begin{cases}\dfrac{x^i}{|x|}, & x\equiv(x^1,x^2,\ldots,x^n)\ne 0,\\ \delta_{1j}, & x=0.\end{cases}$$

Let

$$\overline{\alpha}^0(t,\omega)=\int_0^t p(x^0(s))ds,\ \ \overline{w}(t)=\int_0^t p(x^u(s))dw(s),\ \ \overline{w}^0(t)=\int_0^t p(x^0(s))dw^0(s),$$

$$\overline{\widetilde{\nu}}(t,\omega)=\int_0^t\int_{\mathbf{V}} p(x^u(s,\omega))\widetilde{\nu}^0(dv,ds,\omega),\ \ \overline{\widetilde{\nu}}^0(t,\omega)=\int_0^t\int_{\mathbf{V}} p(x^0(s,\omega))\widetilde{\nu}^0(dv,ds,\omega).$$

Then we obtain

$$x_u(t)=x+\int_0^t\sum_{j=r+1}^{2r+1}\alpha_j^{-1}(s,\omega)p^{-1}(x^u(s,\omega))d\,\overline{w}(t,\omega)$$

$$+\int_0^t\int_{\mathbf{V}}\sum_{k=2r+2}^{3r+2}\alpha_k^{-1}(s,\omega)p^{-1}(x^u(s,\omega))\widetilde{\nu}(dv,ds,\omega)+\int_0^t u(s)ds,$$

$$x^0(t,\omega)=x+\int_0^t\sum_{j=r+1}^{2r+1}\alpha_j^{-1}(s,\omega)p^{-1}(x^0(s,\omega))d\overline{w}^0(t,\omega)$$

$$+\int_0^t\int_{\mathbf{V}}\sum_{k=2r+2}^{3r+2}\alpha_k^{-1}(s,\omega)\,p^{-1}(x^0(s,\omega))\overline{\widetilde{\nu}}^0(dv,ds,\omega)+\int_0^t U(x^0(s,\omega))ds.$$

Applying Lemma 1 to $\{x^u(t,\omega), \overline{w}(t,\omega), \overline{\tilde{\nu}}(t,A,\omega)\}$ and $\{x^0(t,\omega), \overline{w}^0(t,\omega), \overline{\tilde{\nu}}^0(t,A,\omega)\}$, we obtain the set $\{\hat{x}^u(t,\omega), \hat{x}^0(t,\omega), \hat{w}(t,\omega), \hat{\tilde{\nu}}(t,A,\omega)\}$ of the corresponding processes on the probability space with the flow $\{\hat{F}_t \cap \hat{\mathfrak{R}}(x)\}$, such that $\{\hat{w}(t,\omega)\}$ is an $n\times r$-dimensional $\hat{F}_t \cap \hat{\mathfrak{R}}(x)$ Wiener process, $\{\hat{\tilde{\nu}}(t,A,\omega)\} \in \hat{F}_t \cap \hat{\mathfrak{R}}(x)$ is an $n\times r$-dimensional centered Poisson measure, and the following relations hold for the set $A \in \mathfrak{R}$:

$$\{x^u(t,\omega), \overline{w}(t,\omega), \overline{\tilde{\nu}}(t,A,\omega)\} \overset{L}{\approx} \{\hat{x}_u^t(t,\omega), \overline{w}(t,\omega), \hat{\tilde{\nu}}(t,A,\omega)\},$$

$$\{x^0(t,\omega), w^0(t,\omega), \overline{\tilde{\nu}}^0(t,A,\omega)\} \overset{L}{\approx} \{\hat{x}^0(t,\omega), w^0(t,\omega), \hat{\tilde{\nu}}^0(t,A,\omega)\}.$$

Obviously, there exists an $\hat{F}_t \cap \hat{\mathfrak{R}}(x)$-measurable n-dimensional process $\{u(t)\}$ such that $|\hat{u}(t)| \le 1$ for an arbitrary $t \ge 0$ and

$$\hat{x}_u(t,\omega) = x + \int_0^t \sum_{j=r+1}^{2r+1} \alpha_j^{-1}(s,\omega) p^{-1}(\hat{x}^u(s,\omega)) d\hat{w}(s,\omega)$$

$$+ \int_0^t \int_{\mathbf{V}} \sum_{k=2r+2}^{3r+2} \alpha_k^{-1}(s,\omega) p^{-1}(\hat{x}_u(s,\omega)) \hat{\tilde{\nu}}(dv, ds, \omega) + \int_0^t \hat{u}(s,\omega) ds.$$

To simplify the computation, put $\sum_{j=r+1}^{2r+1} \alpha_j^{-1}(t,\omega) = 1$, $\sum_{k=2r+2}^{3r+2} \alpha_k^{-1}(t,\omega) = 1$.

Applying the Ito–Skorohod formula to $x_1(t,\omega) = |\hat{x}^0(t,\omega)|^2$ and $x_2(t,\omega) = |\hat{x}^u(t,\omega)|^2$, we obtain

$$dx_2(t) = 2\hat{x}^u(t,\omega) p^{-1}(\hat{x}^u(t,\omega)) dw(t,\omega) + \int_{\mathbf{V}} [|\hat{x}^u(t,\omega) + p^{-1}(\hat{x}^u(t,\omega))|^2 - |\hat{x}^u(t,\omega)|^2]_1 \tilde{\nu}(dv, dt)$$

$$+ \left\{ 2\hat{x}^u(t,\omega)\hat{u}(t,\omega) + n + \int_{\mathbf{V}} [|\hat{x}^u(t,\omega) + p^{-1}(\hat{x}^u(t,\omega))|^2 - |\hat{x}^u(t,\omega)|^2 - 2\hat{x}^u(t,\omega) p^{-1}(\hat{x}^u(t,\omega))]_2 \Pi(du) \right\} dt$$

$$= 2|\hat{x}^u(t,\omega)| d\hat{w}^1(t,\omega) + \int_{\mathbf{V}} [|\hat{x}^u(t,\omega) + p^{-1}(\hat{x}^u(t,\omega))|^2 - |\hat{x}^u(t,\omega)|^2]_1 \tilde{\nu}(dv, dt)$$

$$+ \left\{ 2\hat{x}^u(t,\omega)\hat{u}(t,\omega) + n + \int_{\mathbf{V}} [|\hat{x}^u(t,\omega) + p^{-1}(\hat{x}^u(t,\omega))|^2 - |\hat{x}^u(t,\omega)|^2 - 2\hat{x}^u(t,\omega) p^{-1}(\hat{x}^u(t,\omega))]_2 \Pi(dv) \right\} dt$$

$$= 2\sqrt{x_2(t,\omega)}\, d\hat{w}^1(t,\omega) + \int_{\mathbf{V}} [|\hat{x}^u(t,\omega) + p^{-1}(\hat{x}^u(t,\omega))|^2 - |\hat{x}^u(t,\omega)|^2]_1 \tilde{\nu}(dv, dt)$$

$$+ \left\{ 2\hat{x}^u(t,\omega)\hat{u}(t,\omega) + n + \int_{\mathbf{V}} [|\hat{x}^u(t,\omega) + p^{-1}(\hat{x}^u(t,\omega))|^2 - |\hat{x}^u(t,\omega)|^2 \right.$$

$$\left. - 2\hat{x}^u(t,\omega) p^{-1}(\hat{x}^u(t,\omega))]_2 \Pi(dv) \right\} dt,$$

$$dx_1(t,\omega) = 2\hat{x}^0(t,\omega) p^{-1}(\hat{x}^0(t,\omega)) d\hat{w}(t,\omega) + \int_{\mathbf{V}} [|\hat{x}^0(t,\omega) + p^{-1}(\hat{x}^0(t,\omega))|^2 - |\hat{x}^0(t,\omega)|^2]_1 \tilde{\nu}^0(dv, dt)$$

$$+ \left\{ 2\hat{x}^0(t,\omega) U(x^0(t,\omega)) + n + \int_{\mathbf{V}} [|\hat{x}^0(t,\omega) + p^{-1}(\hat{x}^0(t,\omega))|^2 \right.$$

$$\left. - |\hat{x}^0(t,\omega)|^2 - 2\hat{x}^0(t,\omega) p^{-1}(\hat{x}^0(t,\omega))]_2 \Pi(dv) \right\} dt$$

$$= 2|\hat{x}^0(t,\omega)| d\hat{w}^1(t,\omega) + \int_{\mathbf{V}} [|\hat{x}^0(t,\omega) + p^{-1}(\hat{x}^0(t,\omega))|^2 - |\hat{x}^0(t,\omega)|^2]_1 \hat{\tilde{\nu}}^0(dv, dt)$$

$$+\left\{-2|\hat{x}^{0}(t,\omega)|+n+\int_{\mathbf{V}}[|\hat{x}^{0}(t,\omega)+p^{-1}(\hat{x}^{0}(t,\omega))|^{2}-|\hat{x}^{0}(t,\omega)|^{2}-2\hat{x}^{0}(t,\omega)p^{-1}(\hat{x}^{0}(t,\omega))]_{2}\Pi(dv)\right\}dt$$

$$=2\sqrt{x_1(t,\omega)}d\hat{w}^{1}(t,\omega)+\int_{\mathbf{V}}[|\hat{x}^{0}(t,\omega)+p^{-1}(\hat{x}^{0}(t,\omega))|^{2}-|\hat{x}^{0}(t,\omega)|^{2}]_{1}\overline{\overline{\nu}}^{0}(dv,dt)$$

$$+\left\{-2\sqrt{x_1(t,\omega)}+n+\int_{\mathbf{V}}[|\hat{x}^{0}(t,\omega)+p^{-1}(\hat{x}^{0}(t,\omega))|^{2}-|\hat{x}^{0}(t,\omega)|^{2}-2\hat{x}^{0}(t,\omega)p^{-1}(\hat{x}^{0}(t,\omega))]_{2}\Pi(dv)\right\}dt,$$

where $\hat{w}(t)=(\hat{w}^{1}(t),\hat{w}^{2}(t),\ldots,\hat{w}^{n}(t))$.

Note that

$$[xp^{-1}(x)]_i=\sum_j x_j(p^{-1}(x_j))_1=\sum_j x_j p_{ij}(x)=\delta_{i1}|x|.$$

Let

$$b(t,n)=2\sqrt{x\vee 0};$$

$$b_1(t,x)=b_2(t,x)=-2\sqrt{x\vee 0}+n+\int_{\mathbf{V}}[|\hat{x}^{u}(t,\omega)+p^{-1}(\hat{x}^{u}(t,\omega))|^{2}-|\hat{x}^{u}(t,\omega)|^{2}-2\hat{x}^{u}(t,\omega)p^{-1}(\hat{x}^{u}(t,\omega))]_{2}\Pi(dv),$$

$$\beta_2(t,\omega)=2\hat{x}^{u}(t,\omega)\hat{u}(t,\omega)+n+\int_{\mathbf{V}}[|\hat{x}^{u}(t,\omega)+p^{-1}(\hat{x}^{u}(t,\omega))|^{2}-|\hat{x}^{u}(t,\omega)|^{2}-2\hat{x}^{u}(t,\omega)p^{-1}(\hat{x}^{u}(t,\omega))]_{2}\Pi(dv).$$

Then obviously $\beta_1(t,\omega)=b_1(t,x_1(t,\omega))$ and

$$\beta_2(t,\omega)\geq-2|\hat{x}^{u}(t,\omega)|+n+\int_{\mathbf{V}}[|\hat{x}^{u}(t,\omega)+p^{-1}(\hat{x}^{u}(t,\omega))|^{2}$$

$$-|\hat{x}^{u}(t,\omega)|^{2}-2\hat{x}^{u}(t,\omega)p^{-1}(\hat{x}^{u}(t,\omega))]_{2}\Pi(dv)=b_2(t,x_2(t,\omega)).$$

The trajectory-wise uniqueness of the solution for the SFDE (10), (11) takes place [7, 13, 15]. Therefore, it is possible to apply the second statement of Theorem 1 and obtain the inequality $x_1(t,\omega)\leq x_2(t,\omega)\ (\mathrm{mod}\,P)$ for an arbitrary $t\geq 0$. Theorem 2 is proved.

CONCLUSIONS

We have proved that there is a minimum value of expectation $E\{||x_t^u||\}$ for feasible systems of the form (12). We have considered the comparison theorem for the SFDE subject to external disturbances of the type of random processes and established the fact of the existence of optimal control for this class of systems.

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