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CHEBYSHEV APPROXIMATION OF MULTIVARIABLE FUNCTIONS BY A LOGARITHMIC EXPRESSION

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Abstract. *The method for constructing the Chebyshev approximation of multivariable functions by a logarithmic expression with an absolute error is proposed. It implies constructing an intermediate Chebyshev approximation of the values of the exponent of an approximated function by a polynomial with the relative error. The construction of the Chebyshev approximation by a polynomial is based on calculating the boundary mean-power approximation by an iterative scheme based on the least squares method with properly formed values of variable weight function. The presented results of test examples' solving confirm the fast convergence of the method in calculating the parameters of the Chebyshev approximation of the functions of one, two, and three variables by the logarithmic expression.*

Keywords: *Chebyshev approximation of the multivariable functions, logarithmic expression, mean-power approximation, least squares method, variable weight function.*

INTRODUCTION

Using logarithmic dependences is expedient for the analysis of values exponentially increasing with time or dependences described by products. In this case, logarithmic transformations allow reducing the analysis of these dependences to linear models [1, 2]. The Chebyshev approximation by non-linear expressions is used for describing special mathematical functions [3, 4] and modeling the special classes of physical, chemical, and economic processes [5–9]. The approximation by logarithmic expressions is used for signal and image processing in telecommunication and biomedical systems [10, 11]. The change in the radiant energy flux in the upper atmosphere caused by some greenhouse gases depends logarithmically on the concentrations of these gases [12, 13]. The logarithmic dependence is also used to describe the characteristics of polymer-based capacitive humidity sensors at low humidity [14], as well as to determine functional dependences of initial conditions in lag problems arising in the infectious disease processes modeling based on the diffusion disturbances, the body temperature response, and other influences [15, 16].

PROBLEM STATEMENT

Let a function of n variables $f(X)$, where X is a vector $X=(x_1, x_2, \dots, x_n)$, be continuous in some bounded domain D, $D \subset R^n$, where R^n is an n-dimensional vector space. Let us approximate the function $f(X)$ defined on a point set $\Omega = \{X_j\}_{j=1}^{s}$, $\Omega \in D$, by a logarithmic expression

$$L_m(a; X) = a_0 + \ln(P_m(a; X)), \quad m+1 < s, \tag{1}$$

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where $P_m(a;X)$ is a generalized polynomial

$$P_m(a;X) = 1 + \sum_{i=1}^{m} a_i \varphi_i(X) \tag{2}$$

with respect to the system of linearly independent and continuous on D real functions $\varphi_i(X)$, $i = \overline{1, m}$, and a_i, $i = \overline{0, m}$, are unknown parameters: $\{a_i\}_{i=0}^{m} \in A$, $A \subseteq R^{m+1}$. We call the expression $L_m(a^*; X)$ Chebyshev approximation of function $f(X)$ with an absolute error on the point set Ω if it satisfies the condition

$$\max_{X \in \Omega} |f(X) - L_m(a^*; X)| = \min_{a \in A} \max_{X \in \Omega} |f(X) - L_m(a; X)|. \tag{3}$$

The properties and the method of calculating the parameters of the Chebyshev approximation of functions of one variable by logarithmic expression (1) with an absolute error are determined in [5], where construction of the Chebyshev approximation is reduced to calculating the parameters of the Chebyshev approximation by a polynomial with a relative error. Similarly to the method described in [5], we reduce Chebyshev approximation by logarithmic expression (1) with an absolute error to Chebyshev approximation by the generalized polynomial

$$\bar{P}_m(a; X) = e^{a_0} P_m(a; X) = \bar{a}_0 \left(1 + \sum_{i=1}^{m} a_i \varphi_i(X)\right) \tag{4}$$

of the function $f_e(X) = \exp(f(X))$ on the point set Ω with the relative error with respect to the unknown parameters $\bar{a}_0$ and a_i, $i = \overline{1, m}$. We calculate the Chebyshev approximation of multivariable functions $f_e(X)$ by polynomial (4) as the boundary approximation in the norm of space L^p as $p \to \infty$ using the method described in [17–19].

METHOD FOR CALCULATING THE PARAMETERS OF THE CHEBYSHEV APPROXIMATION OF A MULTIVARIABLE FUNCTION BY A LOGARITHMIC EXPRESSION

Paper [4] theoretically proves that one can reduce the Chebyshev approximation of a function of one variable by the logarithmic expression (1) with an absolute error to the Chebyshev approximation of the exponent of this function by the polynomial with a relative error. This feature of the Chebyshev approximation by a logarithmic expression is also valid in the case of multivariable function approximation.

THEOREM 1. Let a function of n variables $f(X)$, continuous in some bounded domain D, be defined on the point set $\Omega = \{X_j\}_{j=1}^{s}$ $(\Omega \in D)$. Chebyshev approximation, by logarithmic expression (1), of the function $f(X)$ with an absolute error on the point set Ω can be determined by the Chebyshev approximation, by the generalized polynomial (4), of the function $f_e(X) = \exp(f(X))$ on the point set Ω with a relative error. The values of the parameters a_i, $i = \overline{1, m}$, of the Chebyshev approximation by expression (1) coincide with the values of the same-name parameters of the Chebyshev approximation, by the generalized polynomial $\bar{P}_m(a; X)$ (4), of the function $f_e(X)$ on the point set Ω with an absolute error, and the value of the parameter a_0 can be calculated by the formula

$$a_0 = (\mu_{\max} + \mu_{\min})/2, \tag{5}$$

where

$$\mu_{\max} = \max_{X \in \Omega} (f(X) - \bar{L}_m(a; X)), \quad \mu_{\min} = \min_{X \in \Omega} (f(X) - \bar{L}_m(a; X)),$$

$$\bar{L}_m(a; X) = \ln(P_m(a; X)).$$

Proof. In [20], it is theoretically substantiated for multivariable functions that points of the H-set of the Chebyshev approximation by generalized polynomial (4) coincide with the points of the H-set of the Chebyshev approximation by the expression $\psi(\bar{P}_m(a; X))$, where $\psi(z)$ is the strictly monotonic function. Since a logarithm is a strictly monotonic function, this feature holds for expression (1). According to [20], points of the H-set of the

Chebyshev approximation of multivariable functions are analogous to the points of alternation of the Chebyshev approximation of functions of one variable. At points of the H-set, the error of the Chebyshev approximation is of the greatest absolute value. Based on the results of [20] regarding the above-mentioned property of points of the H-set of the Chebyshev approximation, we assume that the H-set of points of the Chebyshev approximation by the logarithmic expression (1) consists of $(m+2)$ points

$$H=\{H_i^+,\ i=\overline{1,k},\ H_i^-,\ i=\overline{1,l},\ k+l=m+2\}, \tag{6}$$

where H^+ are the points at which the approximation error is positive and H^- are the points at which the approximation error is negative. According to the characteristic feature of the Chebyshev approximation, parameters of the Chebyshev approximation by the logarithmic expression (1) of the function $f(X)$ with an absolute error satisfy the system of equations

$$\begin{cases} f(H_i^+)-a_0-\ln(P_m(a;H_i^+))=|\mu|, & i=\overline{1,k}, \\ f(H_i^-)-a_0-\ln(P_m(a;H_i^-))=-|\mu|, & i=\overline{1,l}, \end{cases} \tag{7}$$

where $|\mu|$ is the error of the Chebyshev approximation. Sequentially subtracting the $(k-1)$th equation of system (7) from the kth one, the $(k-2)$th equiation from the $(k-1)$th one, etc., and, similarly, the $(l-1)$th equation from the lth one, etc., we exclude the unknowns a_0 and μ from the system of equations (7):

$$\begin{cases} \ln(P(a;H_{i+1}^+))-\ln(P(a;H_i^+))=f(H_{i+1}^+)-f(H_i^+), & i=\overline{1,k-1}, \\ \ln(P(a;H_{i+1}^-))-\ln(P(a;H_i^-))=f(H_{i+1}^-)-f(H_i^-), & i=\overline{1,l-1}. \end{cases} \tag{8}$$

Getting rid of the logarithms in the left-hand sides of the system of equations (8), we obtain

$$\begin{cases} P(a;H_{i+1}^+)f_e(H_i^+)=P(a;H_i^+)f_e(H_{i+1}^+), & i=\overline{1,k-1}, \\ P(a;H_{i+1}^-)f_e(H_i^-)=P(a;H_i^-)f_e(H_{i+1}^-), & i=\overline{1,l-1}, \end{cases} \tag{9}$$

where $f_e(X)=\exp(f(X))$.

The system of equations (9) coincides with the system of equations for finding the parameters a_i, $i=\overline{1,m}$, of the Chebyshev approximation of the function $f(X)$ by generalized polynomial (4) with a relative error. Based on the fact that points of the H-set of the Chebyshev approximation by logarithmic expression (1) coincide with the points of the H-set of the Chebyshev approximation by generalized polynomial (4), parameters of the Chebyshev approximation by generalized polynomial (4) of the function $f(X)$ with a relative error satisfy the system of equations

$$\begin{cases} 1-\bar{a}_0P(a;H_i^+)/f_e(H_i^+)=|\bar{\mu}|, & i=\overline{1,k}, \\ 1-\bar{a}_0P(a;H_i^-)/f_e(H_i^-)=-|\bar{\mu}|, & i=\overline{1,l}, \end{cases} \tag{10}$$

where $|\bar{\mu}|$ is the error of the Chebyshev approximation and $\bar{a}_0=e^{a_0}$. Similarly to deleting the unknowns a_0 and μ from the system of equations (7), we delete the unknowns $\bar{a}_0$ and $\bar{\mu}$ from the system of equations (10). After deleting these unknowns, we obtain the system of equations that coincides with the system of equations (9).

Since the system of equations for calculating the values of parameters a_i, $i=\overline{1,m}$, of the Chebyshev approximation of the function $f(X)$ by logarithmic expression (1) on the point set Ω with an absolute error, coincides with the system of equations for the values of the same-name parameters of the Chebyshev approximation of the function $f_e(X)$ by polynomial (4) on the point set Ω with a relative error, we can reduce the calculation of values of the parameter a_i, $i=\overline{1,m}$, of the Chebyshev approximation by logarithmic expression (1) to calculating the values of the same-name parameters of the Chebyshev approximation by polynomial (4).

The value of the parameter a_0 can be determined as a solution of the one-parameter problem of the Chebyshev approximation of the function $f(X)$ by the expression $a_0+\bar{L}_m(a;X)$ on the point set Ω with the absolute error

$$\max_{X\in\Omega}|f(X)-a_0+\bar{L}_m(a;X)|\xrightarrow[a_0]{}\min. \tag{11}$$

The solution of problem (11) with respect to the value of the parameter a_0 can be calculated by formula (5).

Thus, the values of parameters a_i, $i=\overline{1,m}$, of the Chebyshev approximation by expression (1) with a relative error coincide with the values of the same-name parameters of the Chebyshev approximation by the generalized polynomial $\overline{P}_m(a;X)$ of the function $f_e(X)$ on the point set Ω with an absolute error, and the value of the parameter a_0 can be calculated by formula (5). The theorem is proved.

According to Theorem 1, the construction of the Chebyshev approximation by logarithmic expression (1) is reduced to calculating the parameters of the Chebyshev approximation of the function $f_e(X)$ by polynomial (4). To calculate the parameters of the Chebyshev approximation by polynomial (4) of the multivariable function $f_e(X)$ we can use the method described in [18]. The method sequentially constructs mean-power approximations using iterative process based on the least squares method

$$\sum_{X\in\Omega}\rho_r(X)(f_e(X)-\overline{P}_m(a;X))^2 \xrightarrow[a\in A]{} \min, \ r=0,1,\dots, \tag{12}$$

with the weight function

$$\rho_r(X)=\rho_{r-1}(X)\,|\Delta_r(X)|=\rho_0(X)\prod_{i=1}^{r}|\Delta_i(X)|, \ r=1,\dots,p-2, \ p=3,4,\dots, \tag{13}$$

where

$$\rho_0(X)=\frac{1}{f_e^2(X)}, \ \Delta_k(X)=\frac{f_e(X)-\overline{P}_{m,k-1}(a;X)}{f_e(X)}, \ k=\overline{1,r},$$

and $\overline{P}_{m,k}(a;X)$ is the approximation of the function $f_e(X)$ by the least squares method with the weight function $\rho_k(X)$, which corresponds to the mean-power approximation of power $k+2$.

Completion of the construction of the mean-power approximations by iterations (12) can be monitored by achieving some prescribed accuracy ε

$$|\mu_{r-1}-\mu_r|\le\varepsilon\mu_r, \tag{14}$$

where $\mu_r=\max\limits_{X\in\Omega}|f_e(X)-\overline{P}_{m,r}(a;X)|$.

According to Theorem 1, the values of the parameters a_i, $i=\overline{1,m}$, of the Chebyshev approximation of the function $f(X)$ by logarithmic expression (1) on the point set Ω with an absolute error coincide with the values of the same-name parameters of the obtained approximation $\overline{P}_{m,r}(a;X)$. The value of the parameter a_0 can be determined by formula (5).

The results of calculating the parameters of the Chebyshev approximation by logarithmic expression (1) with an absolute error for test examples confirm good convergence in the case of approximation of functions of one, two, and three variables.

Example 1. Find the Chebyshev approximation with an absolute error by the logarithmic expression $L_2(a;x) = a_0+\ln(1+a_1+a_2x^2)$ of the function of one variable $y(x)=\sqrt{1+2x+0.3x^3}$ given at the points x_i, $i=\overline{0,20}$, where $x_i=0.1i$.

Using the proposed method for $\varepsilon=0.003$ in condition (14), in six iterations (12), (13) we have obtained Chebyshev approximation of the function $y_e(x)=\exp(y(x))$ by the polynomial

$$\overline{P}_2(a;x)=2.351184246x^2+1.112030207x+2.842810617. \tag{15}$$

This polynomial provides the relative approximation error of 4.9115%. Accordingly, the Chebyshev approximation of the function $y(x)$ by the logarithmic expression

$$L_2(a;x)=0.0011628185+\ln(2.351184246x^2+1.112030207x+2.842810617) \tag{16}$$

provides the absolute approximation error of 0.049110222.

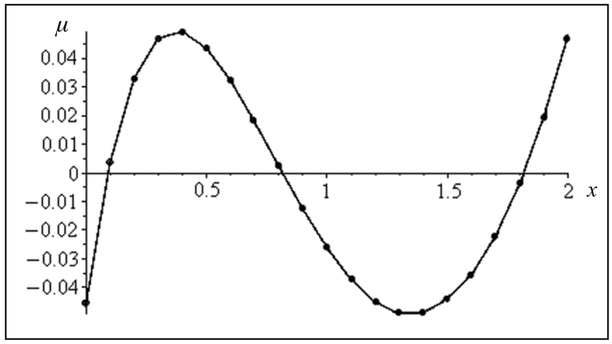

Fig. 1. The curve of the error of the approximation of the function $y(x)$ by the logarithmic expression (16) with an absolute error.

The Chebyshev approximation of the function $y(x)$ by the logarithmic expression

$$L_2(a;x) = 0.001161662 + \ln(2.849228545 + 1.10539099x + 2.34832149x^2), \tag{17}$$

obtained by the Remez iterative scheme [5, 21] with refinement of the points of alternation according to the Vallée–Poussin algorithm provides the approximation error of 0.04821. The error of approximation by logarithmic expression (16) exceeds the error of the Chebyshev approximation obtained according to the Remez scheme by 0.000900285. The error of the approximation by logarithmic expression (16) exceeds the error of the Chebyshev approximation (17) obtained by the Remez scheme by 1.23%.

Figure 1 shows the curve of the error of approximation. It corresponds to the characteristic feature of the Chebyshev approximation and has four extremum points

$$\begin{gathered}(0,\ -0.045956036),\ (0.4,\ 0.049110222),\\ (1.4,\ -0.049110221),\ (2.0,\ 0.046942622),\end{gathered} \tag{18}$$

at which the absolute values of the approximation error coincide within the given accuracy, and the deviation sign alternates at these points [5, 21]. These extremum points coincide with the points of alternation obtained in the approximation by the Remez scheme (17). Moreover, according to the statement [20] regarding the H-set points of the Chebyshev approximation by the polynomial (4), these points also coincide with the extremum approximation error points (15).

We can reduce the divergence of the values of the approximation error at extremum points (18) by increasing the accuracy of calculation of the Chebyshev approximation ε in condition (14). The Chebyshev approximation of the function $y(x)$ by the logarithmic expression $L_2(a;x)$ for $\varepsilon = 0.00003$

$$L_2(a;x) = 0.001168037 + \ln(2.348419563x^2 + 1.105368579x + 2.849149494) \tag{19}$$

provides the absolute error of reconstructing the values of the function $y(x)$, which is equal to 0.04822. Approximation (19) was obtained in 79 iterations. At the extremum points, the absolute approximation error (19) has the following values:

$$\begin{gathered}(0,\ -0.048188564),\ (0.4,\ 0.048223284),\\ (1.4,\ -0.048181811),\ (2.0,\ 0.048184975).\end{gathered} \tag{20}$$

It follows from (18) and (20) that the extremum points of the error of approximating the function $y(x)$ by the logarithmic expression did not change, and the divergence of the absolute approximation error values at these points has decreased.

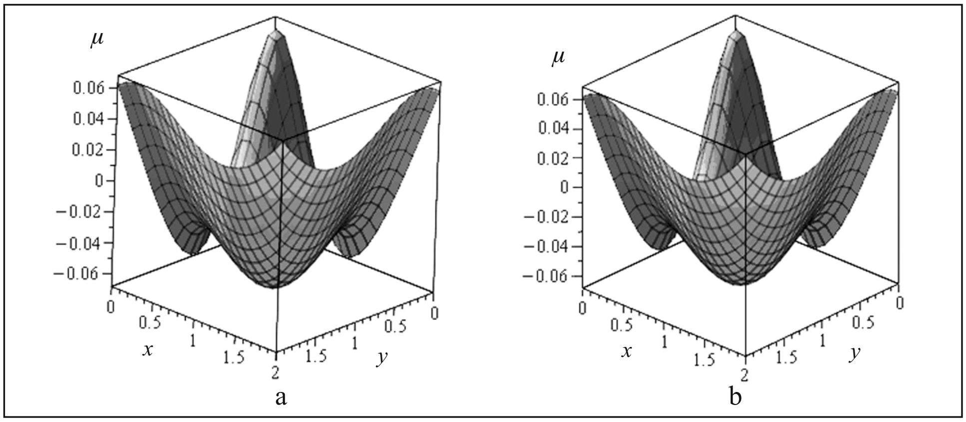

Fig. 2. Surfaces of the error of approximation: the approximation of the function $\exp(z_2(x, y))$ by polynomial (23) with a relative error (a), and approximation of the function $z_2(x, y)$ by logarithmic expression (24) with an absolute error (b).

Example 2. Let us find the Chebyshev approximation of the two-variable function

$$z_2(x, y) = \sqrt{2 + 2.9(x^2 + y^2) + 1.3(x^3 + y^3)}, \tag{21}$$

given at the points (x_i, y_j), $i = \overline{0, 20}$, $j = \overline{0, 20}$, where $x_i = 0.1i$, $y_j = 0.1j$ by the logarithmic expression

$$L_{2,2}(a; x, y) = a_0 + \ln(1 + a_1(x + y) + a_2(x^2 + y^2)) \tag{22}$$

with an absolute error.

Using the proposed method for the function $z_2(x, y)$ in ten iterations (12), (13) for $\varepsilon = 0.003$, we have obtained the intermediate approximation of the function $\exp(z_2(x, y))$ by the polynomial

$$P_{2,2}(a; x, y) = 2.128112043 - 1.212702233(x + y) + 5.701012697(x^2 + y^2) \tag{23}$$

with the relative error of 6.8365%. Accordingly, the logarithmic expression

$$L_{2,2}(a; x, y) = 0.0022649401 + \ln(P_{2,2}(a; x, y)) = 0.0022649401$$
$$+ \ln(2.128112043 - 1.212702233(x + y) + 5.701012697(x^2 + y^2)) \tag{24}$$

provides the approximation of the function $z_2(x, y)$ with the absolute error of 0.06839447.

Figure 2 shows the surfaces of the error of approximation. It confirms the theoretical result [20]. According to this result, the extremum points (the points of the H-set) of the error of approximating the values of the function $\exp(z_2(x, y))$ by polynomial (23) with a relative error coincide with the extremum points of the error of approximating the values of the function $z_2(x, y)$ by logarithmic expression (24) with an absolute error.

Example 3. Let us find the Chebyshev approximation of the three-variable function

$$z_3(x, y, t) = \ln(4.7 + 2.567(x^2 + y^2 + t^2)),$$

given at the points (x_i, y_j, t_r), $i = \overline{0, 10}$, $j = \overline{0, 10}$, $r = \overline{0, 10}$, where $x_i = 0.1i$, $y_j = 0.1j$, $t_r = 0.1r$ by the logarithmic expression

$$L_{3,1}(a; x, t) = a_0 + \ln(1 + a_1(x + y + t))$$

with an absolute error.

Using the proposed method for $\varepsilon = 0.003$, in 11 iterations we have obtained for the function $z_3(x, y, t)$ an approximation by the logarithmic expression

$$L_{3,1}(a; x, t) = 0.009280081 + \ln(4.081427175 + 2.212641011(x + y + t)),$$

which provides the absolute approximation error of 0.136446495.

CONCLUSIONS

The method of constructing the Chebyshev approximation of tabular multivariable functions $f(X)$ by logarithmic expression (1) with an absolute error is implemented using the intermediate Chebyshev approximation by generalized polynomial (4) of the function $f_e(X) = \exp(f(X))$ with a relative error. The Chebyshev approximation by a polynomial is calculated as the boundary mean-power approximation based on the least squares method with the properly formed values of a variable weight function. The method proposed allows constructing the Chebyshev approximation by a logarithmic expression with required accuracy. The results of solving the test examples confirm a fast convergence of the method when constructing the Chebyshev approximation by a logarithmic expression with an absolute error for the functions of one, two, and three variables.

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