DOI 10.1007/s10559-023-00559-1
Cybernetics and Systems Analysis, Vol. 59, No. 2, March, 2023

MODIFIED RESOLVING-FUNCTIONS METHOD FOR GAME PROBLEMS OF APPROACH OF CONTROLLED OBJECTS WITH DIFFERENT INERTIA*

A. A. Chikrii[1†] **and I. S. Rappoport**[1‡]

UDC 517.977

Abstract. *The problem of the approach of controlled objects with different inertia in dynamic game problems is considered. Modified sufficient conditions for ending the game in a finite guaranteed time when Pontryagin's condition is not satisfied are formulated. Some shift functions are considered instead of the Pontryagin selector, and special multi-valued mappings are introduced with their help. They generate the upper and lower resolving functions of a special type, and based on them, two types of modified schemes are proposed: the scheme of Pontryagin's first method and the method of resolving functions. This ensures the completion of the conflict-controlled process for objects with different inertia in the class of quasi-strategies and counter-controls. New theoretical results are illustrated by a model example.*

Keywords: *controlled objects with different inertia, quasi-linear differential game, multi-valued mapping, measurable selector, stroboscopic strategy, resolving function.*

INTRODUCTION

We will analyze the problem of the approach of controlled objects with different inertia and target interception in dynamic game problems based on the Pontryagin's first method [1], as well as the method of resolving functions [2] and its modern version [3]. The problem is important due to the need for a theoretical justification of Euler's pursuit curve methods, beam pursuit method, and, in particular, parallel approach, well-known to rocket and space designers. Pontryagin's condition [1] is a key one in the Pontryagin's first method and the method of resolving functions, and if it is not satisfied, these methods fail. Controlled objects with different inertia are characterized by the fact that Pontryagin's condition is not satisfied on some time interval, which significantly complicates the application of the method of resolving functions to this class of dynamic game problems. The problem "boy and crocodile" [2] can be an example.

We will describe the case where Pontryagin's condition is not satisfied, and some shift functions are considered instead of the Pontryagin selector, which are used to introduce special multi-valued mappings. They generate upper and lower resolving functions of a special type, and based on them, modified schemes of the Pontryagin's first method and the method of resolving functions are proposed. This ensures the completion of the conflict-controlled process for objects with different inertia in the class of quasi-strategies and counter-controls. We will illustrate the new theoretical results with a model example.

The article continues the studies [1–5], is related to the publications [6–12], and expands the class of game problems of approach of controlled objects with different inertia, which have a solution.

*The research was partially supported by the National Research Foundation of Ukraine, Grant #2020.02.0121.

[1]V. M. Glushkov Institute of Cybernetics, National Academy of Sciences of Ukraine, Kyiv, Ukraine, [†]*g.chikrii@gmail.com*; [‡]*jeffrappoport@gmail.com*. Translated from Kibernetyka ta Systemnyi Analiz, No. 2, March–April, 2023, pp. 87–103. Original article submitted September 5, 2022.

MODIFIED SCHEME OF THE PONTRYAGIN'S FIRST METHOD

Consider a conflict-controlled process whose evolution can be described by the equality

$$z(t) = g(t) + \int_0^t \Omega(t, \tau)\varphi(u(\tau), v(\tau))\, d\tau, \ \ t \geq 0, \tag{1}$$

where $z(t) \in R^n$, function $g(t)$, $g: R_+ \to R^n$, is Lebesgue measurable [9] and bounded for $t > 0$, the matrix function $\Omega(t, \tau)$, $t \geq \tau \geq 0$, is measurable with respect to t and is summable with respect to τ for each $t \in R_+$. The control unit is specified by the function $\varphi(u, v)$, $\varphi: U \times V \to R^n$, which is assumed to be continuous over the set of variables on the direct product of nonempty compact sets U and V; and m, l, and n are natural numbers.

Players' controls $u(\tau)$, $u: R_+ \to U$, and $v(\tau)$, $v: R_+ \to V$, are measurable functions of time.

Along with process (1), a terminal set M^* of cylindrical form is given:

$$M^* = M_0 + M, \tag{2}$$

where M_0 is a linear subspace from R^n, and M is a compact set from the orthogonal complement L to subspace M_0 in R^n.

The goals of the first (u) and second (v) players are opposite. The first player (pursuer) tends to bring the trajectory of process (1) to the terminal set (2) in the shortest time, and the second player (evader) tries to postpone as much as possible the time when the trajectory hits the set M^* or even to avoid the meeting.

Let us take the side with the first player and assume that when the game (1), (2) continues on the interval $[0, T]$, we choose the control of the first player at time t based on information about $g(T)$ and $v_t(\cdot)$, i.e., in the form of a measurable function

$$u(t) = u(g(T), v_t(\cdot)), \ \ t \in [0, T], \ \ u(t) \in U, \tag{3}$$

where $v_t(\cdot) = \{v(s): s \in [0, t]\}$ is the previous history of control of the second player up to the time t, or in the form of counter-control

$$u(t) = u(g(T), v(t)), \ \ t \in [0, T], \ \ u(t) \in U. \tag{4}$$

If, in particular, $g(t) = e^{At} z_0$, $\Omega(t, \tau) = e^{A(t-\tau)}$, $z(0) = z_0$, and e^{At} is a matrix exponent, then we assume that control $u(t) = u(z_0, v_t(\cdot))$ implements a quasi-strategy [7], and counter-control [6] $u(t) = u(z_0, v(t))$ is a manifestation of the Hajek stroboscopic strategy [8].

Let us formulate the necessary facts from the convex analysis [1, 10] in the form of a lemma.

LEMMA 1. Let $X \in R^n$ be a convex compact set and $\omega(\tau)$ be a nonnegative bounded measurable numerical function. Then $\int_0^T \omega(\tau) X d\tau = \int_0^T \omega(\tau) d\tau X$, $T > 0$. Moreover, if $0 \in X$, $f(\tau) \in \omega(\tau) X$, and $\int_0^T \omega(\tau) d\tau \leq 1$, then $\int_0^T f(\tau) d\tau \in X$, $f(\tau)$ is a measurable function, $\tau \in [0, T]$.

Denote by π the operator of orthogonal projection from R^n into L. Having put $\varphi(U, v) = \{\varphi(u, v): u \in U\}$, we consider the multi-valued mappings

$$W(t, \tau, v) = \pi\Omega(t, \tau)\varphi(U, v), \ \ W(t, \tau) = \bigcap_{v \in V} W(t, \tau, v)$$

on the sets $\Delta_\Theta \times V$ and Δ_Θ, respectively, where $\Delta_\Theta = \{(t, \tau): 0 \leq \tau \leq t \leq \Theta < \infty\}$, Θ is some positive number. Suppose that the multi-valued mapping $W(t, \tau, v)$ has closed values on the set $\Delta_\Theta \times V$.

Pontryagin's Condition. The nulti-valued mapping $W(t, \tau)$ takes non-empty values on the set Δ_Θ, Θ is some positive number.

Taking into account the assumptions about the matrix function $\Omega(t, \tau)$, we may conclude that for any fixed $t > 0$ the vector function $\pi\Omega(t, \tau)\varphi(u, v)$ is $\mathfrak{L} \otimes \mathfrak{B}$-measurable with respect to $(\tau, v) \in [0, t] \times V$ and continuous with respect to $u \in U$. Therefore, based on the direct image theorem [9] for any fixed $t > 0$, the multi-valued mapping $W(t, \tau, v)$ is $\mathfrak{L} \otimes \mathfrak{B}$-measurable with respect to $(\tau, v) \in [0, t] \times V$. If Pontryagin's condition is satisfied, then on the set Δ there exists at least one selector $\gamma_0(t, \tau)$ of the mapping $W(t, \tau)$, $\gamma_0(t, \tau) \in W(t, \tau)$. We will call it Pontryagin's selector. Let us formulate Pontryagin's condition in an equivalent form.

In the set Δ_Θ, where Θ is some positive number, there exists a Pontryagin's selector $\gamma_0(t, \tau)$, for which the inclusion is true:

$$0 \in \bigcap_{v \in V} [W(t, \tau, v) - \gamma_0(t, \tau)].$$

Pontryagin's condition is the key condition for Pontryagin's first method, and if it is not satisfied, the method fails. Let us formulate a modified Pontryagin's condition for controlled objects with different inertia.

Let $\gamma(t, \tau)$, $\gamma: \Delta_\Theta \to L$, be some (almost everywhere) bounded function, measurable with respect to t and summable with respect to τ, $\tau \in [0, t]$, for each $t > 0$, which we will call the shift function, $\Delta_\Theta = \{(t, \tau): 0 \le \tau \le t \le \Theta < \infty\}$, Θ is some positive number. Let M_1 be a convex compact set from the orthogonal complement L to the subspace M_0 in R^n, such that for $m \in M_1$ we get $-m \in M_1$, and $M_2 = M \overset{*}{-} M_1 = \{m \in L: m + M_1 \subset M\} = \bigcap_{m \in M_1} (M - m) \ne \varnothing$, where $\overset{*}{-}$ is the Minkowski geometric difference [1]. The shift function $\gamma(t, \tau)$ and sets M_1 and M_2 for which the above conditions and properties are true will be called admissible.

Consider the compact-valued multi-valued measurable mapping $\overline{V}(t)$, $\overline{V}(t) \subset V$, and continuous matrix function $B(t)$ with the values of order k, where k is the dimension of the vector $\overline{v} \in \overline{V}(t)$, $t \in [0, \Theta]$, Θ is some positive number. Note that for the mapping $\overline{V}(t)$ there exists a measurable selector $\overline{v}_s(t)$, $\overline{v}_s(t) \in \overline{V}(t)$, $t \in [0, \Theta]$. In what follows, we assume that the multi-valued mapping $\overline{V}(t)$ is a compact-valued measurable mapping, and the matrix function $B(t)$ is a continuous function.

Denote $W_B(t, \tau, v) = \pi\Omega(t, \tau)\varphi(U, B(t - \tau)\overline{v})$, $\varphi_B(t, u, v, \overline{v}) = \varphi(u, B(t, \tau)\overline{v}) - \varphi(u, v)$, $(t, \tau) \in \Delta_\Theta$, $u \in U$, $v \in V$, $\overline{v} \in \overline{V}(t)$ and consider for $(t, \tau) \in \Delta_\Theta$, $v \in V$, $\overline{v} \in \overline{V}(t)$ the multi-valued mapping

$$\Lambda_B(t, \tau, v, \overline{v}) = \{\lambda \ge 0: \pi\Omega(t, \tau)\varphi_B(t, U, v, \overline{v}) \subset \lambda M_1\}.$$

If the condition $\Lambda_B(t, \tau, v, \overline{v}) \ne \varnothing$ is satisfied on the set $\Delta_\Theta \times V \times \overline{V}(t)$, then we consider the scalar function $\lambda_B(t, \tau, v, \overline{v}) = \inf\{\lambda: \lambda \in \Lambda_B(t, \tau, v, \overline{v})\}$, $\tau \in [0, t]$, $v \in V$, $\overline{v} \in \overline{V}(t)$. We will show [12] that the multi-valued mapping $\Lambda_B(t, \tau, v, \overline{v})$ is closed-valued, $\mathfrak{L} \otimes \mathfrak{B} \otimes \mathfrak{B}$-measurable with respect to the set $(\tau, v, \overline{v})$, $\tau \in [0, t]$, $v \in V$, $\overline{v} \in \overline{V}(t)$, and function $\lambda_B(t, \tau, v, \overline{v})$ is $\mathfrak{L} \otimes \mathfrak{B} \otimes \mathfrak{B}$-measurable with respect to the set $(\tau, v, \overline{v})$, $\tau \in [0, t]$, $v \in V$, $\overline{v} \in \overline{V}(t)$; therefore, it is superpositionally measurable [12], i.e., $\lambda_B(t, \tau, v(\tau), \overline{v}(\tau))$ is measurable with respect to τ, $\tau \in [0, t]$, for any measurable functions $v(\tau)$ and $\overline{v}(\tau)$, $v(\cdot) \in V(\cdot)$, $\overline{v}(\cdot) \in \overline{V}_t(\cdot)$, where $V(\cdot)$ is the set of measurable functions $v(\tau)$, $\tau \in [0, +\infty]$, with values from V, $\overline{V}_t(\cdot)$ is the set of measurable functions $\overline{v}(\tau)$, $\tau \in [0, +\infty]$, with values from $\overline{V}(t)$. Note also that the function $\sup_{v \in V, \overline{v} \in \overline{V}(t)} \lambda_B(t, \tau, v, \overline{v})$ is measurable with respect to τ, $\tau \in [0, t]$.

Condition 1. There exist a multi-valued mapping $\overline{V}(t)$, $\overline{V}(t) \subset V$, and a matrix function $B(t)$, $t \in [0, \Theta]$, Θ is some positive number, admissible shift function $\gamma_0(t, \tau)$, and set M_1, for which the relation $\Lambda_B(t, \tau, v, \overline{v}) \ne \varnothing$, $(t, \tau) \in \Delta_\Theta$, $v \in V$, $\overline{v} \in \overline{V}(t)$, inequality $\int_0^t \sup_{v \in V, \overline{v} \in \overline{V}(t)} \lambda_B(t, \tau, v, \overline{v}) d\tau \le 1$, and inclusion

$$0 \in \bigcap_{\overline{v} \in \overline{V}(t)} [W_B(t, \tau, \overline{v}) - \gamma_0(t, \tau)], \ \varphi_B(t, U, V, \overline{V}(t)) \subset \sup_{v \in V, \overline{v} \in \overline{V}(t)} \lambda_B(t, \tau, v, \overline{v}) M_1$$

are true.

Denote $\xi_0(t) = \xi(t, g(t), \gamma_0(t,\cdot)) = \pi g(t) + \int_0^t \gamma_0(t,\tau)d\tau$ and consider the set $\mathrm{P}(g(\cdot), \gamma_0(\cdot,\cdot)) = \{t \in [0, \Theta]: \xi_0(t) \in M_2\}$. If the relation in curly brackets does not hold for any $t \in [0, \Theta]$, then we put $\mathrm{P}(g(\cdot), \gamma_0(\cdot,\cdot)) = \varnothing$.

THEOREM 1. Let for the conflict-controlled process (1), (2) there exist a multi-valued mapping $\overline{V}(t)$, $\overline{V}(t) \subset V$, and matrix function $B(t)$, $t \in [0, \Theta]$, Θ is some positive number, admissible shift function $\gamma_0(t,\tau)$, and sets M_1 and M_2 such that on the set Δ_Θ Condition 1 is satisfied, the set $\mathrm{P}(g(\cdot), \gamma_0(\cdot,\cdot))$ is not empty, and $\mathrm{P} \in \mathrm{P}(g(\cdot), \gamma_0(\cdot,\cdot))$. Then the game can be completed at time P using the control of the form (4).

Proof. Let $\upsilon(\tau)$ be an arbitrary measurable selector of the compact set V, $\tau \in [0, \mathrm{P}]$. Let us specify the method of choosing the control by the pursuer.

For $\overline{\upsilon} \in \overline{V}(t)$, $\tau \in [0, \mathrm{P}]$, consider the compact-valued multi-valued mapping

$$U(\tau, \overline{\upsilon}) = \{u \in U: \pi\Omega(\mathrm{P},\tau)\varphi(u, B(\mathrm{P}-\tau)\overline{\upsilon}) - \gamma(\mathrm{P},\tau) = 0\}.$$

Due to the properties of the parameters of process (1), the compact-valued mapping $U(\tau, \overline{\upsilon})$ is $\mathfrak{L} \otimes \mathfrak{B}$-measurable [12] for $\overline{\upsilon} \in \overline{V}(t)$, $\tau \in [0, \mathrm{P}]$. Therefore, by the theorem on the measurable choice of selector [9], the multi-valued mapping $U(\tau, \overline{\upsilon})$ contains the $\mathfrak{L} \otimes \mathfrak{B}$-measurable selector $u(\tau, \overline{\upsilon})$, which is a superpositionally measurable function [12].

Let $\overline{\upsilon}_s(\cdot)$ be a measurable selector of mapping $\overline{V}(t)$. Put $\tau \in [0, \mathrm{P}]$ for the first player's control $u(\tau) = u(\tau, \overline{\upsilon}(\tau))$, where

$$\overline{\upsilon}(\tau) = \begin{cases} \upsilon(\tau) & \text{if } \upsilon(\tau) \in \overline{V}(t-\tau), \\ \overline{\upsilon}_s(\tau) & \text{if } \upsilon(\tau) \notin \overline{V}(t-\tau). \end{cases}$$

Taking into account formula (1), we obtain

$$\pi z(\mathrm{P}) = -\int_0^{\mathrm{P}} \pi\Omega(\mathrm{P},\tau)\varphi_B(\mathrm{P}, u(\tau), \upsilon(\tau), \overline{\upsilon}(\tau))d\tau$$

$$+\xi_0(\mathrm{P}) + \int_0^{\mathrm{P}} (\pi\Omega(\mathrm{P},\tau)\varphi(u(\tau), B(\mathrm{P}-\tau)\overline{\upsilon}(\tau)) - \gamma_0(\mathrm{P},\tau))d\tau.$$

Then due to Condition 1, regarding the moment P we get

$$\pi\Omega(\mathrm{P},\tau)\varphi_B(\mathrm{P}, u(\tau), \upsilon(\tau), \overline{\upsilon}(\tau)) \in \sup_{\upsilon \in V, \overline{\upsilon} \in \overline{V}(t)} \lambda_B(\mathrm{P}, \tau, \upsilon, \overline{\upsilon}) M_1$$

and

$$\int_0^{\mathrm{P}} \sup_{\upsilon \in V, \overline{\upsilon} \in \overline{V}(t)} \lambda_B(\mathrm{P}, \tau, \upsilon, \overline{\upsilon})\, d\tau \le 1.$$

Therefore, by Lemma 1, the inclusion is true:

$$\int_0^{\mathrm{P}} \pi\Omega(\mathrm{P},\tau)\varphi_B(\mathrm{P}, u(\tau), \upsilon(\tau), \overline{\upsilon}(\tau))d\tau \in M_1;$$

thus,

$$-\int_0^{\mathrm{P}} \pi\Omega(\mathrm{P},\tau)\varphi_B(\mathrm{P}, u(\tau), \upsilon(\tau), \overline{\upsilon}(\tau))d\tau \in M_1.$$

Taking into account the choice of control by the first player, we obtain $\pi z(\mathrm{P}) \in M_1 + \xi_0(\mathrm{P}) \in M_1 + M_2 \subset M$ and $z(\mathrm{P}) \in M^*$, which completes the proof of the theorem.

Remark 1. Theorem 1 is an analog of Pontryagin's first direct method [1] for controlled objects with different inertia. Note that the admissible shift function $\gamma_0(t,\tau)$ in Condition 1 is a Pontryagin's selector for the multi-valued mapping $\overline{V}(t)$, $\overline{V}(t) \subset V$. If Pontryagin's condition is satisfied, then we put $\overline{V}(t) = V$, $B(t) = E$, E is a unit matrix, and transform Condition 1 into Pontryagin's condition. Therefore, we will consider Condition 1 a modified Pontryagin's condition of the first type for controlled objects with different inertia.

Denote $\xi(t) = \xi(t, g(t), \gamma(t, \cdot)) = \pi g(t) + \int_0^t \gamma(t, \tau) d\tau$ and consider the multi-valued mapping for $(t, \tau) \in \Delta_\Theta$, $\Theta > 0$, $\bar{v} \in \bar{V}(t)$:

$$\mathfrak{A}(t, \tau, \bar{v}) = \{\alpha \geq 0 : [W_B(t, \tau, \bar{v}) - \gamma(t, \tau)] \cap \alpha[M_2 - \xi(t)] \neq \varnothing\}. \quad (5)$$

If the condition $\mathfrak{A}(t, \tau, \bar{v}) \neq \varnothing$ is satisfied on the set $\Delta_\Theta \times \bar{V}$, then consider the upper and lower scalar resolving functions [4]

$$\alpha^*(t, \tau, \bar{v}) = \sup\{\alpha : \alpha \in \mathfrak{A}(t, \tau, \bar{v})\},$$

$$\alpha_*(t, \tau, \bar{v}) = \inf\{\alpha : \alpha \in \mathfrak{A}(t, \tau, \bar{v})\}, \; (t, \tau) \in \Delta_\Theta, \; \bar{v} \in \bar{V}(t).$$

It can be shown [12] that the multi-valued mapping $\mathfrak{A}(t, \tau, \bar{v})$ is closed-valued, $\mathfrak{L} \otimes \mathfrak{B}$-measurable with respect to the set of (τ, v), $(t, \tau) \in \Delta_\Theta$, $\bar{v} \in \bar{V}(t)$, and the upper and lower resolving functions are $\mathfrak{L} \otimes \mathfrak{B}$-measurable with respect to the set of (τ, v), $(t, \tau) \in \Delta_\Theta$, $\bar{v} \in \bar{V}(t)$; therefore, they are superpositionally measurable [12], i.e., $\alpha^*(t, \tau, \bar{v}(\tau))$ and $\alpha_*(t, \tau, \bar{v}(\tau))$ are measurable with respect to τ, $(t, \tau) \in \Delta_\Theta$, for any measurable function $\bar{v}(\cdot) \in \bar{V}_t(\cdot)$. Note also that the upper resolving function is upper semi-continuous, the lower one is lower semi-continuous in terms of the variable $\bar{v}$, and functions $\inf_{\bar{v} \in \bar{V}(t)} \alpha^*(t, \tau, \bar{v})$ and $\sup_{\bar{v} \in \bar{V}(t)} \alpha_*(t, \tau, \bar{v})$ are measurable in τ, $(t, \tau) \in \Delta_\Theta$.

Condition 2. There exist multi-valued mapping $\bar{V}(t)$, $\bar{V}(t) \subset V$, and matrix function $B(t)$, $t \in [0, \Theta]$, Θ is some positive number, admissible shift function $\gamma(t, \tau)$ and sets M_1 and M_2, for which there hold the relations $\Lambda_B(t, \tau, v, \bar{v}) \neq \varnothing$, $(t, \tau) \in \Delta_\Theta$, $v \in V$, $\bar{v} \in \bar{V}(t)$, the inequalities $\int_0^t \sup_{v \in V, \bar{v} \in \bar{V}(t)} \lambda_B(t, \tau, v, \bar{v}) d\tau \leq 1$, $\int_0^t \sup_{\bar{v} \in \bar{V}(t)} \alpha_*(t, \tau, \bar{v}) d\tau < 1$, and inclusion

$$\varphi_B(t, U, V, \bar{V}(t)) \subset \sup_{v \in V, \bar{v} \in \bar{V}(t)} \lambda_B(t, \tau, v, \bar{v}) M_1,$$

$$0 \in \bigcap_{\bar{v} \in \bar{V}(t)} \{[W_B(t, \tau, \bar{v}) - \gamma(t, \tau)] - \mathfrak{A}(t, \tau, \bar{v})[M_2 - \xi(t)]\}.$$

Consider the set

$$\mathrm{P}_*(g(\cdot), \gamma(\cdot, \cdot)) = \{t \in [0, \Theta] : \xi(t) \in M_2\}. \quad (6)$$

If the inclusion in curly brackets in relation (6) does not hold for any $t \in [0, \Theta]$, then put $\mathrm{P}_*(g(\cdot), \gamma(\cdot, \cdot)) = \varnothing$.

THEOREM 2. Let for the conflict-controlled process (1), (2) there exist a multi-valued mapping $\bar{V}(t)$, $\bar{V}(t) \subset V$, and matrix function $B(t)$, $t \in [0, \Theta]$, Θ is some positive number, the admissible shift function $\gamma(t, \tau)$, and sets M_1 and M_2 such that Condition 2 is satisfied on the set Δ_Θ, the set $\mathrm{P}_*(g(\cdot), \gamma(\cdot, \cdot))$ is not empty, and $\mathrm{P}_* \in \mathrm{P}_*(g(\cdot), \gamma(\cdot, \cdot))$. Then the game can be ended at time P_* using control of the form (4).

Proof. Let $v(\tau)$ be an arbitrary measurable selector of the compact set V, $\tau \in [0, \mathrm{P}_*]$. Let us specify the method of choosing the control by the pursuer.

For $\bar{v} \in \bar{V}(t)$ and $\tau \in [0, \mathrm{P}_*]$, consider the compact-valued multi-valued mapping

$$U_*(\tau, \bar{v}) = \{u \in U : \pi\Omega(\mathrm{P}_*, \tau)\varphi(u, B(\mathrm{P}_* - \tau)\bar{v}) - \gamma(\mathrm{P}_*, \tau) \in \alpha_*(\mathrm{P}_*, \tau, \bar{v})[M_2 - \xi(\mathrm{P}_*)]\}.$$

Due to the properties of the parameters of process (1) and the lower resolving function $\alpha_*(\mathrm{P}_*, \tau, \bar{v})$, the compact-valued mapping $U_*(\tau, \bar{v})$ is $\mathfrak{L} \otimes \mathfrak{B}$-measurable [12] with respect to $\bar{v} \in \bar{V}(t)$ and $\tau \in [0, \mathrm{P}_*]$. Therefore, by the theorem on the measurable choice of selector [9], the multi-valued mapping $U_*(\tau, \bar{v})$ contains the $\mathfrak{L} \otimes \mathfrak{B}$-measurable selector $u_*(\tau, \bar{v})$, which is a superpositionally measurable function [12].

Let $\bar{v}_s(\cdot)$ be a measurable selector of mapping $\bar{V}(t)$. For $\tau \in [0, \mathrm{P}_*]$, let us put the control of the first player $u_*(\tau) = u_*(\tau, \bar{v}(\tau))$, where

$$\bar{v}(\tau) = \begin{cases} v(\tau) & \text{if } v(\tau) \in \bar{V}(t - \tau), \\ \bar{v}_s(\tau) & \text{if } v(\tau) \notin \bar{V}(t - \tau). \end{cases}$$

Taking formula (1) into account, we obtain

$$\pi z(\mathrm{P}_*) = -\int_0^{\mathrm{P}_*} \pi\Omega(\mathrm{P}_*,\tau)\varphi_B(\mathrm{P}_*, u_*(\tau), v(\tau), \overline{v}(\tau))d\tau$$

$$+\xi(\mathrm{P}_*) + \int_0^{\mathrm{P}_*} (\pi\Omega(\mathrm{P}_*,\tau)\varphi(u_*(\tau), B(\mathrm{P}_* - \tau)\overline{v}(\tau)) - \gamma(\mathrm{P}_*,\tau))d\tau. \quad (7)$$

As a result of Condition 2 regarding the moment P_*, we get

$$0 \in M_1,\ \pi\Omega(\mathrm{P}_*,\tau)\varphi_B(\mathrm{P}_*, u_*(\tau), v(\tau), \overline{v}(\tau)) \in \sup_{v\in V,\, \overline{v}\in\overline{V}(t)} \lambda_B(\mathrm{P}_*, \tau, v, \overline{v})M_1,$$

$$\int_0^{\mathrm{P}_*} \sup_{v\in V,\, \overline{v}\in\overline{V}(t)} \lambda_B(\mathrm{P}_*, \tau, v, \overline{v})\,d\tau \le 1.$$

Then, given Lemma 1, the inclusion is true:

$$\int_0^{\mathrm{P}_*} \pi\Omega(\mathrm{P}_*,\tau)\varphi_B(\mathrm{P}_*, u_*(\tau), v(\tau), \overline{v}(\tau))d\tau \in M_1$$

and by assumption

$$-\int_0^{\mathrm{P}_*} \pi\Omega(\mathrm{P}_*,\tau)\varphi_B(\mathrm{P}_*, u_*(\tau), v(\tau), \overline{v}(\tau))d\tau \in M_1.$$

Taking into account the choice of control and from the definition of time P_*, we get

$$0 \in M_2 - \xi(\mathrm{P}_*),$$

$$\pi\Omega(\mathrm{P}_*,\tau)\varphi(u_*(\tau), B(\mathrm{P}_* - \tau)\overline{v}(\tau)) - \gamma(\mathrm{P}_*,\tau) \in \alpha_*(\mathrm{P}_*, \tau, \overline{v}(\tau))[M_2 - \xi(\mathrm{P}_*)],$$

$$\int_0^{\mathrm{P}_*} \sup_{\overline{v}\in\overline{V}(t)} \alpha_*(\mathrm{P}_*, \tau, \overline{v})\,d\tau \le 1.$$

Then, given Lemma 1, the inclusion is true:

$$\int_0^{\mathrm{P}_*} (\pi\Omega(\mathrm{P}_*,\tau)\varphi(u_*(\tau), B(\mathrm{P}_* - \tau)\overline{v}(\tau)) - \gamma(\mathrm{P}_*,\tau))d\tau \in M_2 - \xi(\mathrm{P}_*).$$

Therefore, relation (7) determines

$$\pi z(\mathrm{P}_*) \in M_1 + \xi(\mathrm{P}_*) + M_2 - \xi(\mathrm{P}_*) = M_1 + M_2 \subset M$$

and $z(\mathrm{P}_*) \in M^*$, which completes the proof of the theorem.

LEMMA 2. For the conflict-controlled process (1), (2), Condition 1 is satisfied if and only if there exist a multi-valued mapping $\overline{V}(t)$, $\overline{V}(t) \subset V$, and a matrix function $B(t)$, $t \in [0, \Theta]$, Θ is some positive number, admissible shift function $\gamma(t,\tau)$, and sets M_1 and M_2, for which Condition 2 is true and $0 \in \mathfrak{A}(t, \tau, \overline{v})$ on the set $\Delta_\Theta \times \overline{V}(t)$.

Proof. Let for the conflict-controlled process (1), (2) there exist multi-valued mapping $\overline{V}(t)$, $\overline{V}(t) \subset V$, and matrix function $B(t)$, $t \in [0, \Theta]$, Θ is some positive number, admissible shift function $\gamma(t,\tau)$ and sets M_1 and M_2 such that Condition 2 is satisfied and $0 \in \mathfrak{A}(t, \tau, \overline{v})$ on the set $\Delta_\Theta \times \overline{V}(t)$. Then zero value of α ensures a non-empty intersection in expression (5) and therefore, with regard for the condition $\Lambda(t, \tau, v, \overline{v}) \ne \varnothing$, $(t,\tau) \in \Delta_\Theta$, $v \in V$, $\overline{v} \in \overline{V}(t)$, we get $0 \in W_B(t, \tau, \overline{v}) - \gamma(t,\tau)$, $(t, \tau, \overline{v}) \in \Delta_\Theta \times \overline{V}(t)$.

It follows herefrom that for $(t,\tau)\in\Delta_\Theta$ we get $0\in\bigcap_{\overline{v}\in\overline{V}(t)}[W_B(t,\tau,\overline{v})-\gamma(t,\tau)]$, i.e., Condition 1 is true. Considering the reverse order, we can make the desired conclusion.

Remark 2. Given a multi-valued mapping $\overline{V}(t)$, $\overline{V}(t)\subset V$, and matrix function $B(t)$, $t\in[0,\Theta]$, Θ is some positive number, admissible shift function $\gamma(t,\tau)$, sets M_1 and M_2 for which Condition 1 is satisfied, then using Lemma 2 we obtain $\alpha_*(t,\tau,\overline{v})=\inf\{\alpha:\alpha\in\mathfrak{A}(t,\tau,\overline{v})\}=0$ on the set $\Delta_\Theta\times\overline{V}(t)$.

A MODIFIED SCHEME OF THE METHOD OF RESOLVING FUNCTIONS

We will consider Condition 2 to be a modified Pontryagin's condition of the second type. It is a key one in the modified scheme of the method of resolving functions for controlled objects with different inertia. Let us formulate a modified scheme of the method of resolving functions.

Condition 3. There exist multi-valued mappings $\overline{V}(t)$, $\overline{V}(t)\subset V$, and matrix function $B(t)$, $t\in[0,\Theta]$, Θ is some positive number, admissible shift function $\gamma(t,\tau)$, and sets M_1 and M_2 such that on the set Δ_Θ Condition 2 is satisfied and the inclusion is true:

$$0\in\bigcap_{\overline{v}\in\overline{V}(t)}\left\{[W_B(t,\tau,\overline{v})-\gamma(t,\tau)]-\sup_{\overline{v}\in\overline{V}(t)}\alpha_*(t,\tau,\overline{v})[M_2-\xi(t)]\right\}.$$

Remark 3. If there exist multi-valued mapping $\overline{V}(t)$, $\overline{V}(t)\subset V$, and matrix function $B(t)$, $t\in[0,\Theta]$, Θ is some positive number, admissible shift function $\gamma(t,\tau)$, and sets M_1 and M_2 for which Condition 1 is satisfied, then with the use of Lemma 2, Condition 3 is satisfied and the equality $\sup_{\overline{v}\in\overline{V}(t)}\alpha_*(t,\tau,\overline{v})=0$ holds.

Consider the set

$$T(g(t),\gamma(\cdot,\cdot))=\left\{t\in[0,\Theta]:\int_0^t\inf_{\overline{v}\in\overline{V}(t)}\alpha^*(t,\tau,\overline{v})\,d\tau\ge 1\right\}. \quad (8)$$

If $\alpha^*(t,\tau,\overline{v})\equiv+\infty$ for $(t,\tau)\in\Delta_\Theta$, $\overline{v}\in\overline{V}(t)$, then it is natural to assume that the value of the corresponding integral in the curly brackets in (8) is $+\infty$ and $t\in T(g(t),\gamma(\cdot,\cdot))$. When the inequality in (8) does not hold for all $t\in[0,\Theta]$, we put $T(g(t),\gamma(\cdot,\cdot))=\varnothing$.

THEOREM 3. Let for the conflict-controlled process (1), (2) there exist a multi-valued mapping $\overline{V}(t)$, $\overline{V}(t)\subset V$, and matrix function $B(t)$, $t\in[0,\Theta]$, Θ is some positive number, admissible shift function $\gamma(t,\tau)$, and sets M_1 and M_2 such that Condition 3 is satisfied, set $T(g(t),\gamma(\cdot,\cdot))$ is not empty, and $T\in T(g(t),\gamma(\cdot,\cdot))$. Then the game can be ended at time T with the use of control (3).

Proof. Let $v(\cdot)$ be an arbitrary measurable selector of the compact set V, and $\overline{v}_s(\cdot)$ be a measurable selector of mapping $\overline{V}(t)$. Let us put for $\tau\in[0,T]$

$$\overline{v}(\tau)=\begin{cases}v(\tau) & \text{if } v(\tau)\in\overline{V}(t-\tau),\\ \overline{v}_s(\tau) & \text{if } v(\tau)\notin\overline{V}(t-\tau).\end{cases}$$

Let us specify the method of choosing the control by the pursuer.

First, consider the case $\xi(T,g(T),\gamma(\cdot,\cdot))\notin M_2$ and introduce a control function

$$h(t)=1-\int_0^t\alpha^*(T,\tau,\overline{v}(\tau))d\tau-\int_t^T\sup_{\overline{v}\in\overline{V}(t)}\alpha_*(T,\tau,\overline{v})d\tau,\ t\in[0,T].$$

By the definition of T, we get

$$h(0)=1-\int_0^T\sup_{\overline{v}\in\overline{V}(t)}\alpha_*(T,\tau,\overline{v})\,d\tau>0,$$

$$h(T)=1-\int_0^T\alpha^*(T,\tau,\overline{v}(\tau))d\tau\le 1-\int_0^T\inf_{\overline{v}\in\overline{V}(t)}\alpha^*(T,\tau,\overline{v})\,d\tau\le 0.$$

Since the function $h(t)$ is continuous, there exists a time t_*, $t_* \in (0, T]$, such that $h(t_*) = 0$. Note that switching time t_* depends on the previous history of control of the second player $v_{t_*}(\cdot) = \{v(s) : s \in [0, t_*]\}$.

We will call the time intervals $[0, t_*)$ and $[t_*, T]$ active and passive, respectively. Let us represent the method of control of the first player on each interval. To this end, consider the compact mappings

$$U^*(\tau, \overline{v}) = \{u \in U : \pi\Omega(T, \tau)\varphi(u, B(T-\tau)\overline{v}) - \gamma(T, \tau))$$

$$\in \alpha^*(T, \tau, \overline{v})[M_2 - \xi(T)]\}, \ \tau \in [0, t_*), \tag{9}$$

$$U_*(\tau, v) = \left\{ u \in U : \pi\Omega(T, \tau)\varphi(u, B(T-\tau)\overline{v}) - \gamma(T, \tau)) \right.$$

$$\left. \in \sup_{\overline{v} \in \overline{V}(t)} \alpha_*(T, \tau, \overline{v})[M_2 - \xi(T)] \right\}, \ \tau \in [t_*, T]. \tag{10}$$

The multi-valued mappings $U^*(\tau, \overline{v})$ and $U_*(\tau, \overline{v})$ have non-empty images. Taking into account the properties of parameters of process (1), functions $\alpha^*(T, \tau, \overline{v})$ and $\sup_{\overline{v} \in \overline{V}(t)} \alpha_*(T, \tau, \overline{v})$, the compact-valued mappings $U^*(\tau, \overline{v})$, $\tau \in [0, t_*)$, and $U_*(\tau, \overline{v})$, $\tau \in [t_*, T]$, for $\overline{v} \in \overline{V}(t)$ are $\mathfrak{L} \otimes \mathfrak{B}$-measurable [12]. Therefore, by the theorem on the measurable choice of selector [9], each of them has at least one $\mathfrak{L} \otimes \mathfrak{B}$-measurable selector $u^*(\tau, \overline{v})$ and $u_*(\tau, \overline{v})$, which are superpositionally measurable functions [12]. Let the control of the first player on the active interval be $u^*(\tau) = u^*(\tau, \overline{v}(\tau))$, $\tau \in [0, t_*)$, and on the passive interval be $u_*(\tau) = u_*(\tau, \overline{v}(\tau))$, $\tau \in [t_*, T]$.

Based on formula (1), for the controls we obtain

$$\pi z(T) = -\left[\int_0^{t_*} \pi\Omega(T, \tau)\varphi_B(T, u^*(\tau), v(\tau), \overline{v}(\tau))d\tau + \int_{t_*}^{T} \pi\Omega(T, \tau)\varphi_B(T, u_*(\tau), v(\tau), \overline{v}(\tau))d\tau \right]$$

$$+ \xi(T) + \int_0^{t_*} (\pi\Omega(T, \tau)\varphi(u^*(\tau), B(T-\tau)\overline{v}(\tau)) - \gamma(T, \tau))d\tau$$

$$+ \int_{t_*}^{T} (\pi\Omega(T, \tau)\varphi(u_*(\tau), B(T-\tau)\overline{v}(\tau)) - \gamma(T, \tau))d\tau. \tag{11}$$

As a result of Condition 2, we get

$$0 \in M_1, \ \int_0^T \sup_{v \in V, \overline{v} \in \overline{V}(t)} \lambda_B(T, \tau, v, \overline{v})\, d\tau \le 1,$$

$$\pi\Omega(T, \tau)\varphi_B(T, u^*(\tau), v(\tau), \overline{v}(\tau)) \in \sup_{v \in V, \overline{v} \in \overline{V}(t)} \lambda_B(T, \tau, v, \overline{v})M_1, \ \tau \in [0, t_*),$$

$$\pi\Omega(T, \tau)\varphi_B(T, u_*(\tau), v(\tau), \overline{v}(\tau)) \in \sup_{v \in V, \overline{v} \in \overline{V}(t)} \lambda_B(T, \tau, v, \overline{v})M_1, \ \tau \in [t_*, T].$$

Then, taking into account Lemma 1, we obtain

$$\int_0^{t_*} \pi\Omega(T, \tau)\varphi_B(T, u^*(\tau), v(\tau), \overline{v}(\tau))d\tau + \int_{t_*}^{T} \pi\Omega(T, \tau)\varphi_B(T, u_*(\tau), v(\tau), \overline{v}(\tau))d\tau \in M_1$$

and by assumption we get

$$-\left[\int_0^{t_*} \pi\Omega(T, \tau)\varphi_B(T, u^*(\tau), v(\tau), \overline{v}(\tau))d\tau + \int_{t_*}^{T} \pi\Omega(T, \tau)\varphi_B(T, u_*(\tau), v(\tau), \overline{v}(\tau))\, d\tau \right] \in M_1.$$

With regard for the last inclusion, from (9)–(11) we get

$$\pi z(T) \in M_1 + \xi(T) + \int_0^{t_*} \alpha^*(T,\tau,\bar{\upsilon}(\tau))[M_2 - \xi(T)]d\tau + \int_{t_*}^{T} \sup_{\bar{\upsilon}\in\bar{V}(t)} \alpha_*(T,\tau,\bar{\upsilon})[M_2 - \xi(T)]d\tau$$

$$= M_1 + \xi(T) + \int_0^{t_*} \alpha^*(T,\tau,\bar{\upsilon}(\tau))d\tau[M_2 - \xi(T)] + \int_{t_*}^{T} \sup_{\bar{\upsilon}\in\bar{V}(t)} \alpha_*(T,\tau,\bar{\upsilon})d\tau[M_2 - \xi(T)]$$

$$= M_1 + \xi(T)\left[1 - \int_0^{t_*} \alpha^*(T,\tau,\bar{\upsilon}(\tau))d\tau - \int_{t_*}^{T} \sup_{\bar{\upsilon}\in\bar{V}(t)} \alpha_*(T,\tau,\bar{\upsilon})d\tau\right]$$

$$+\left[\int_0^{t_*} \alpha^*(T,\tau,\bar{\upsilon}(\tau))d\tau + \int_{t_*}^{T} \sup_{\bar{\upsilon}\in\bar{V}(t)} \alpha_*(T,\tau,\bar{\upsilon})d\tau\right] M_2 = M_1 + M_2 \subset M.$$

We took into account Lemma 1, the equality $h(t_*) = 0$, and inclusion $M_1 + M_2 \subset M$.

For the case $\xi(T, g(T), \gamma(\cdot,\cdot)) \in M_2$, it will suffice to apply Theorem 2.

Condition 4. There exist multi-valued mappings $\bar{V}(t)$, $\bar{V}(t) \subset V$, and matrix function $B(t)$, $t \in [0, \Theta]$, Θ is some positive number, admissible shift function $\gamma(t,\tau)$, and sets M_1 and M_2 such that on the set Δ_Θ Condition 2 is satisfied and the inclusion is true:

$$0 \in \bigcap_{\upsilon\in V} \left\{ [W_B(t,\tau,\upsilon) - \gamma(t,\tau)] - \inf_{\upsilon\in V} \alpha^*(t,\tau,\upsilon)[M_2 - \xi(t)] \right\}.$$

THEOREM 4. Let for the conflict-controlled process (1), (2) there exist a multi-valued mapping $\bar{V}(t)$, $\bar{V}(t) \subset V$, and matrix function $B(t)$, $t \in [0, \Theta]$, Θ is some positive number, admissible shift function $\gamma(t,\tau)$, and sets M_1 and M_2 such that Conditions 3 and 4 are satisfied, set $T(g(t), \gamma(\cdot,\cdot))$ is not empty, and $T \in T(g(t), \gamma(\cdot,\cdot))$. Then the game can be ended at time T with the use of control (4).

Proof. Let $\upsilon(\cdot)$ be an arbitrary measurable selector of the compact set V and $\bar{\upsilon}_s(\cdot)$ be a measurable selector of mapping $\bar{V}(t)$. Let us put for $\tau \in [0, T]$

$$\bar{\upsilon}(\tau) = \begin{cases} \upsilon(\tau) & \text{if } \upsilon(\tau) \in \bar{V}(t-\tau), \\ \bar{\upsilon}_s(\tau) & \text{if } \upsilon(\tau) \notin \bar{V}(t-\tau). \end{cases}$$

Let us specify the method of choosing the control by the pursuer.

First, consider the case $\xi(T, g(T), \gamma(\cdot,\cdot)) \notin M_2$ and introduce the control function

$$h(t) = 1 - \int_0^{t} \inf_{\bar{\upsilon}\in\bar{V}(t)} \alpha^*(T,\tau,\bar{\upsilon})d\tau - \int_{t}^{T} \sup_{\bar{\upsilon}\in\bar{V}(t)} \alpha_*(T,\tau,\bar{\upsilon})d\tau, \ t \in [0, T].$$

By the definition of T, we get

$$h(0) = 1 - \int_0^{T} \sup_{\bar{\upsilon}\in\bar{V}(t)} \alpha_*(T,\tau,\bar{\upsilon})\,d\tau > 0, \ h(T) = 1 - \int_0^{T} \inf_{\bar{\upsilon}\in\bar{V}(t)} \alpha^*(T,\tau,\bar{\upsilon})\,d\tau \le 0.$$

Since the function $h(t)$ is continuous, there exists time t_*, $t_* \in (0, T]$, such that $h(t_*) = 0$. Note that switching time t_* does not depend on the previous history of control of the second player $\upsilon_{t_*}(\cdot) = \{\upsilon(s) : s \in [0, t_*]\}$.

We will call the time intervals $[0, t_*)$ and $[t_*, T]$ active and passive, respectively. Let us specify the method of control of the first player on each interval. To this end, consider the compact mappings

$$\tilde{U}^*(\tau,\bar{\upsilon}) = \left\{ u \in U : \pi\Omega(T,\tau)\varphi(u, B(T-\tau)\bar{\upsilon}) - \gamma(T,\tau) \right.$$

$$\left. \in \inf_{\bar{\upsilon}\in\bar{V}(t)} \alpha^*(T,\tau,\bar{\upsilon})[M_2 - \xi(T)] \right\}, \ \tau \in [0, t_*), \tag{12}$$

$$\widetilde{U}_*(\tau,\overline{v}) = \left\{ u \in U : \pi\Omega(T,\tau)\varphi(u, B(T-\tau)\overline{v}) - \gamma(T,\tau) \right.$$

$$\left. \in \sup_{\overline{v}\in\overline{V}(t)} \alpha_*(T,\tau,\overline{v})[M_2 - \xi(T)] \right\},\ \tau\in[t_*,T]. \tag{13}$$

The multi-valued mappings $\widetilde{U}^*(\tau,\overline{v})$ and $\widetilde{U}_*(\tau,\overline{v})$ have non-empty images. Taking into account the properties of the parameters of process (1), functions $\inf_{\overline{v}\in\overline{V}(t)} \alpha^*(T,\tau,\overline{v})$ and $\sup_{\overline{v}\in\overline{V}(t)} \alpha_*(T,\tau,\overline{v})$, the compact-valued mappings $\widetilde{U}^*(\tau,\overline{v})$, $\tau\in[0,t_*)$, and $\widetilde{U}_*(\tau,\overline{v})$, $\tau\in[t_*,T]$, for $\overline{v}\in\overline{V}(t)$ are $\mathfrak{L}\otimes\mathfrak{B}$-measurable [12]. Therefore, by the theorem on the measurable choice of selector [9], each of them contains at least one $\mathfrak{L}\otimes\mathfrak{B}$-measurable selector $\widetilde{u}^*(\tau,\overline{v})$ and $\widetilde{u}_*(\tau,\overline{v})$, which are superpositionally measurable functions [12]. Let the control of the first player on the active interval be $\widetilde{u}^*(\tau)=\widetilde{u}^*(\tau,\overline{v}(\tau))$, $\tau\in[0,t_*)$, and on the passive interval be $\widetilde{u}_*(\tau)=\widetilde{u}_*(\tau,\overline{v}(\tau))$, $\tau\in[t_*,T]$.

Based on formula (1) for the selected controls, we obtain

$$\pi z(T) = -\left[\int_0^{t_*} \pi\Omega(T,\tau)\varphi_B(T,\widetilde{u}^*(\tau),v(\tau),\overline{v}(\tau))d\tau + \int_{t_*}^{T} \pi\Omega(T,\tau)\varphi_B(T,\widetilde{u}_*(\tau),v(\tau),\overline{v}(\tau))\,d\tau\right]$$

$$+\xi(T) + \int_0^{t_*} (\pi\Omega(T,\tau)\varphi(\widetilde{u}^*(\tau), B(T-\tau)\overline{v}(\tau)) - \gamma(T,\tau))d\tau$$

$$+\int_{t_*}^{T} (\pi\Omega(T,\tau)\varphi(\widetilde{u}_*(\tau), B(T-\tau)\overline{v}(\tau)) - \gamma(T,\tau))d\tau. \tag{14}$$

As a result of Condition 2, we get

$$0\in M_1,\ \int_0^T \sup_{v\in V,\,\overline{v}\in\overline{V}(t)} \lambda_B(T,\tau,v,\overline{v})\,d\tau \le 1,$$

$$\pi\Omega(T,\tau)\varphi_B(T,\widetilde{u}^*(\tau),v(\tau),\overline{v}(\tau)) \in \sup_{v\in V,\,\overline{v}\in\overline{V}(t)} \lambda_B(T,\tau,v,\overline{v})M_1,\ \tau\in[0,t_*),$$

$$\pi\Omega(T,\tau)\varphi_B(T,\widetilde{u}_*(\tau),v(\tau),\overline{v}(t)) \in \sup_{v\in V,\,\overline{v}\in\overline{V}(t)} \lambda_B(T,\tau,v,\overline{v})M_1,\ \tau\in[t_*,T].$$

Then, taking into account Lemma 1, we obtain

$$\int_0^{t_*} \pi\Omega(T,\tau)\varphi_B(T,\widetilde{u}^*(\tau),v(\tau),\overline{v}(t))d\tau + \int_{t_*}^{T} \pi\Omega(T,\tau)\varphi_B(T,\widetilde{u}_*(\tau),v(\tau),\overline{v}(\tau))d\tau \in M_1;$$

therefore, we get

$$-\left[\int_0^{t_*} \pi\Omega(T,\tau)\varphi_B(T,\widetilde{u}^*(\tau),v(\tau),\overline{v}(\tau))d\tau + \int_{t_*}^{T} \pi\Omega(T,\tau)\varphi_B(T,\widetilde{u}_*(\tau),v(\tau),\overline{v}(\tau))\,d\tau\right] \in M_1.$$

With regard for the last inclusion, relations (12)–(14) determine

$$\pi z(T) \in M_1 + \xi(T) + \int_0^{t_*} \inf_{\overline{v}\in\overline{V}(t)} \alpha^*(T,\tau,\overline{v})[M_2 - \xi(T)]d\tau + \int_{t_*}^{T} \sup_{\overline{v}\in\overline{V}(t)} \alpha_*(T,\tau,\overline{v})[M_2 - \xi(T)]d\tau$$

$$= M_1 + \xi(T) + \int_0^{t_*} \inf_{\bar{v} \in \bar{V}(t)} \alpha^*(t, \tau, \bar{v})\, d\tau [M_2 - \xi(T)] + \int_{t_*}^{T} \sup_{\bar{v} \in \bar{V}(t)} \alpha_*(T, \tau, \bar{v}) d\tau [M_2 - \xi(T)]$$

$$= M_1 + \xi(T)\left[1 - \int_0^{t_*} \inf_{\bar{v} \in \bar{V}(t)} \alpha^*(t, \tau, \bar{v}) d\tau - \int_{t_*}^{T} \sup_{\bar{v} \in \bar{V}(t)} \alpha_*(T, \tau, \bar{v}) d\tau\right]$$

$$+ \left[\int_0^{t_*} \inf_{\bar{v} \in \bar{V}(t)} \alpha^*(t, \tau, \bar{v}) d\tau + \int_{t_*}^{T} \sup_{\bar{v} \in \bar{V}(t)} \alpha_*(T, \tau, \bar{v}) d\tau\right] M_2 = M_1 + M_2 \subset M.$$

We have taken into account Lemma 1, the equality $h(t_*) = 0$, and inclusion $M_1 + M_2 \subset M$.

In the case of $\xi(T, g(T), \gamma(T, \cdot)) \in M_2$, it will suffice to apply Theorem 2.

MODEL EXAMPLE ("BOY AND CROCODILE")

Let us determine the dynamics of the pursuer and evader by the equations

$$\ddot{x} = u,\ x \in R^n,\ n \geq 2,\ u \in S_r^{\rho},$$

$$\dot{y} = v,\ y \in R^n,\ n \geq 2,\ v \in S_0^{\sigma},\ \rho > \sigma > r > 0, \tag{15}$$

respectively, $S_b^a \subset R^n$ is a ring centered at zero, with the outer radius a and inner radius b.

The pursuit will be considered complete if $\|x - y\| \leq \varepsilon$.

The original problem (15) is reduced to a conflict-controlled process in the following way. Let us introduce new variables

$$(z_1, z_2) = z,$$

$$z_1 = x - y,\ \dot{z}_2 = \dot{x}. \tag{16}$$

Differentiating relation (16) with respect to time and taking into account the original equations (15), we obtain

$$\dot{z}_1 = z_2 - v,$$

$$\dot{z}_2 = u. \tag{17}$$

The terminal set is $M^* = \{z\colon z_1 \in S_0^{\varepsilon}\}$, $M_0 = \{z\colon z_1 = 0\}$, $M = \{z\colon z_1 \in S_0^{\varepsilon}, z_2 = 0\}$. Denote

$$M_1 = \varepsilon_1 S_0^1,\ M_2 = \varepsilon_2 S_0^1 = M \overset{*}{-} M_1 = \varepsilon S_0^1 \overset{*}{-} \varepsilon_1 S_0^1 = (\varepsilon - \varepsilon_1) S_0^1,\ \varepsilon_2 = \varepsilon - \varepsilon_1,\ \varepsilon > \varepsilon_1.$$

Then

$$L = \{z\colon z_2 = 0\},\ \pi = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix},\ \pi z = z_1,\ A = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}.$$

The control domains have the form

$$U = \left\{\begin{pmatrix} 0 \\ u \end{pmatrix}\colon u \in S_r^{\rho}\right\},\ V = \left\{\begin{pmatrix} v \\ 0 \end{pmatrix}\colon v \in S_0^{\sigma}\right\},\ \bar{V} = \left\{\begin{pmatrix} \bar{v} \\ 0 \end{pmatrix}\colon \bar{v} \in \bar{S}\right\},\ \bar{S} = S_{\sigma r/\rho}^{\sigma},\ \bar{V} \subset V.$$

The fundamental matrix of the homogeneous system (17) is $e^{At} = \begin{pmatrix} E & tE \\ 0 & 0 \end{pmatrix}$. Then we obtain

$$\pi e^{At} U = t S_r^{\rho},\ \pi e^{At} V = S_0^{\sigma},\ \pi e^{At} \bar{V} = \bar{S},\ M = S_0^{\varepsilon} \subset L.$$

Pontryagin's condition is not satisfied on the interval $[0, \sigma/\rho)$:

$$\pi e^{At} U \overset{*}{-} \pi e^{At} V = t S_r^{\rho} \overset{*}{-} S_0^{\sigma} = \varnothing,\ t \in [0, \sigma/\rho).$$

Let us determine the shift function $\gamma(t) \equiv 0$ and put

$$B(t) = \begin{cases} (\rho/\sigma)tE, & 0 \le t \le \sigma/\rho, \\ E, & t > \sigma/\rho, \end{cases} \quad \bar{V}(t) = \begin{cases} \bar{V}, & 0 \le t \le \sigma/\rho, \\ V, & t > \sigma/\rho. \end{cases}$$

Then we get

$$W_B(t,\tau,\bar{v}) = \pi e^{At}[U - B(t)\bar{v}], \ \bar{v} \in \bar{S},$$

$$0 \in \bigcap_{\bar{v} \in \bar{S}} [W_B(t,\tau,\bar{v}) - \gamma(t,\tau)] = \pi e^{At} U \overset{*}{-} \pi e^{At} B(t)\bar{V}(t) = \{0\}, \ t \in [0,\sigma/\rho]. \quad (18)$$

As a result of simple calculations, we obtain

$$\lambda(t,v,\bar{v}) = \begin{cases} \dfrac{(\|v\| - (\rho/\sigma)t)\|\bar{v}\|}{\varepsilon_1}, & t \in [0,\sigma/\rho], \\ 0, & t > \sigma/\rho, \end{cases}$$

$$\max_{v \in S_0^\sigma, \bar{v} \in \bar{S}} \lambda(t,v,\bar{v}) = \begin{cases} \dfrac{\sigma - \rho t}{\varepsilon_1}, & t \in [0,\sigma/\rho], \\ 0, & t > \sigma/\rho. \end{cases}$$

Since

$$\varphi_B(t,U,V,\bar{V}(t)) = \pi e^{At}(V - B(t)\bar{V}(t)) = \begin{cases} S_0^\sigma - \bar{S}, & t \in [0, \sigma/\rho], \\ \{0\}, & t > \sigma/\rho, \end{cases}$$

we get

$$\varphi_B(t,U,V,\bar{V}(t)) \subset \max_{v \in S_0^\sigma, \bar{v} \in \bar{S}} \lambda(t,v,\bar{v}) S_0^{\varepsilon_1}. \quad (19)$$

If $\varepsilon_1 \ge \sigma^2/2\rho$, then for all $t \ge 0$ the inequality holds:

$$(\rho t^2/2) - \sigma t + \varepsilon_1 \ge 0. \quad (20)$$

Therefore, for $\varepsilon_1 \ge \sigma^2/2\rho$ the inequality holds for all $t \ge 0$:

$$\int_0^t \max_{v \in S_0^\sigma, \bar{v} \in \bar{S}} \lambda(\tau,v,\bar{v}) d\tau \le \frac{\sigma t - (\rho t^2/2)}{\varepsilon_1} \le 1. \quad (21)$$

Therefore, as a result of relations (18), (19), and (21), Condition 1 is satisfied. Put $\xi(t) = \pi e^{At} z = z_1 + t z_2$. Since Condition 1 is satisfied, Conditions 2 and 3 are true, and for $t \in [0,\sigma/\rho]$, $\bar{v} \in \bar{S}$ we get $\alpha_*(t,\tau,\bar{v}) = \sup_{\bar{v} \in \bar{S}} \alpha_*(t,\tau,\bar{v}) = 0$. If $\xi(t) \in \varepsilon_2 S_0^1$, then as a result of Theorem 1 or Theorem 2, the game can be finished at time t with the use of control (4). Moreover, with regard for inequality (20), the smallest instant of time satisfies the equation

$$\| z_1 + t z_2 \| = (\rho t^2/2) - \sigma t + \varepsilon, \ t \le \sigma/\rho.$$

Let $\xi(t) \notin \varepsilon_2 S_0^1$. Then for $t - \tau \le \sigma/\rho$ the upper resolving function $\alpha^*(t,\tau,\bar{v})$ can be found from the relation

$$\alpha^*(t,\tau,\bar{v}) = \sup\{\alpha \ge 0 : \alpha[\varepsilon_2 S_0^1 - \xi(t)] \cap (t-\tau)[S_r^\rho - \bar{v}]\}$$

$$= \sup\{\alpha \ge 0 : \|\rho(t-\tau)\bar{v} - \alpha\xi(t)\| = \alpha\varepsilon_2 + \rho(t-\tau)\}, \ \bar{v} \in \bar{S},$$

and function $\alpha^*(t,\tau,\bar{v})$ is the larger positive root of the quadratic equation

$$(\|\xi(t)\|^2 - (\varepsilon_2)^2)\alpha^2 - 2[(\bar{v},\xi(t)) + \rho(t-\tau)\varepsilon_2]\alpha - [\rho^2(t-\tau)^2(\sigma^2 - \|\bar{v}\|^2)] = 0$$

with respect to α when $\xi(t) \notin \varepsilon_2 S$, $\bar{v} \in \bar{S}$.

As a result of calculations, we obtain

$$\min_{\overline{v}\in\overline{S}} \alpha^*(t,\tau,\overline{v})=0,\ t-\tau\le\sigma/\rho, \tag{22}$$

moreover, the minimum is attained for the vector $\overline{v}=-\sigma(\xi(t)/\|\xi(t)\|)$.

Let $\xi(t)\notin\varepsilon_2 S$ and $t-\tau>\sigma/\rho$. Then a multi-valued mapping $\mathfrak{A}(t,\tau,v)$ has the form

$$\mathfrak{A}(t,\tau,v)=\{\alpha\ge 0:\alpha[\varepsilon_2 S_0^1-\xi(t)]\cap[(t-\tau)S_r^\rho - v]\},\ v\in S_0^\sigma.$$

Obviously, $\mathfrak{A}(t,\tau,v)\ne\varnothing$; therefore, the inclusion is true:

$$0\in\bigcap_{v\in S_0^\sigma}\{[(t-\tau)S_r^\rho-v]-\mathfrak{A}(t,\tau,v)[\varepsilon_2 S_0^1-\xi(t)]\}. \tag{23}$$

The upper resolving function $\alpha^*(t,\tau,v)$ can be found from the relation

$$\alpha^*(t,\tau,v)=\sup\{\alpha\ge 0:\alpha[\varepsilon_2 S_0^1-\xi(t)]\in(t-\tau)S_r^\rho-v\}$$

$$=\sup\{\alpha\ge 0:\|v-\alpha\xi(t)\|=\alpha\varepsilon_2+\rho(t-\tau)\},\ v\in S_0^\sigma,$$

and function $\alpha^*(t,\tau,v)$ is the larger positive root of the quadratic equation

$$(\|\xi(t)\|^2-(\varepsilon_2)^2)\alpha^2-2[(v,\xi(t))+\rho(t-\tau)\varepsilon_2]\alpha-[\rho^2(t-\tau)^2-\|v\|^2]=0$$

with respect to α for $\xi(t)\notin\varepsilon_2 S,\ v\in S_0^\sigma$.

As a result of calculations, for $t-\tau>\sigma/\rho$ we obtain $\min_{v\in S_0^\sigma}\alpha^*(t,\tau,v)=\dfrac{\rho(t-\tau)-\sigma}{\|\xi(t)\|-\varepsilon_2}$ and the minimum is attained for the vector

$$v=-\sigma\frac{\xi(t)}{\|\xi(t)\|}. \tag{24}$$

Let us determine the time of the game end for $t>\sigma/\rho$. Put

$$\pi e^{At}\overline{V}(t)=\begin{cases}\overline{S}, & 0\le t\le\sigma/\rho,\\ S_0^\sigma, & t>\sigma/\rho.\end{cases}$$

With regard for Eqs. (22), (24),

$$\int_0^t \min_{\overline{v}\in\pi e^{At}\overline{V}(t)}\alpha^*(t,\tau,\overline{v})\,d\tau$$

$$=\int_0^{t-(\sigma/\rho)}\min_{\overline{v}\in\overline{S}}\alpha^*(t,\tau,\overline{v})\,d\tau+\int_{t-(\sigma/\rho)}^t\min_{v\in S_0^\sigma}\alpha^*(t,\tau,v)\,d\tau=\int_0^{t-(\sigma/\rho)}\frac{\rho(t-\tau)-\sigma}{\|\xi(t)\|-\varepsilon_2}\,d\tau=1.$$

From the last equality, we obtain the equation

$$\|z_1+tz_2\|=(\rho t^2/2)-\sigma t+\varepsilon-\varepsilon_1. \tag{25}$$

The smallest positive root of Eq. (25) is the time of the game end. For $t=0$, the left-hand side of Eq. (25) is $\|z_1\|$ and grows linearly as t grows, and the right-hand side is equal to $\varepsilon-\varepsilon_1$ and grows quadratically. Since $\|z_1\|>\varepsilon-\varepsilon_1$, for any z_1 and z_2, the time of the game end is final. The equality $z_1+tz_2=0$ can only hold later, provided that Eq. (25) holds; therefore, we do not consider this case.

Let $\xi(t) \notin \varepsilon_2 S$ and $\sigma / R < t-\tau \le \sigma / r$. Then for $v \in S_0^\sigma$ the lower resolving function $\alpha_*(t, \tau, v)$ can be found from the relation

$$\alpha_*(t, \tau, v) = \inf\{\alpha \ge 0 : \alpha[\varepsilon_2 S_0^1 - \xi(t)] \cap [(t-\tau) S_r^\rho - v]\}.$$

However, for $v \in (t-\tau) S_0^r$ we get

$$\alpha_*(t, \tau, v) = \alpha_*^r(t, \tau, v) = \sup\{\alpha > 0 : [v - \alpha\xi(t)] \in (t-\tau) S_0^r + \alpha\varepsilon_2 S_0^1\}$$

$$= \sup\{\alpha > 0 : \| v - \alpha\xi(t) \| = (t-\tau) r + \alpha\varepsilon_2\}$$

and function $\alpha_*^r(t, \tau, v)$ is the larger positive root of the quadratic equation

$$(\|\xi(t)\|^2 - (\varepsilon_2)^2)\alpha^2 - 2[(v, \xi(t)) + r(t-\tau)\varepsilon_2]\alpha - [r^2(t-\tau)^2 - \|v\|^2] = 0$$

with respect to α when $\xi(t) \notin \varepsilon_2 S$, $\sigma / R < t-\tau \le \sigma / r$, $v \in (t-\tau) S_0^r$.

Since the relation $(t-\tau) S_r^\rho \overset{*}{-} S_{(t-\tau)r}^\sigma = S_0^{(t-\tau)\rho-\sigma}$ holds, for $v \in S_{(t-\tau)r}^\sigma$ we get

$$\alpha_*(t, \tau, v) = \max_{v \in S_{(t-\tau)r}^\sigma} \alpha_*(t, \tau, v) = 0.$$

Therefore, the equality $\alpha_*(t, \tau, v) = \alpha_*^r(t, \tau, v)$ holds for $\sigma / R < t-\tau \le \sigma / r$, $v \in S_0^\sigma$ and at the same time we get

$$\max_{v \in S_0^\sigma} \alpha_*(t, \tau, v) = \max_{v \in (t-\tau) S_0^r} \alpha_*^r(t, \tau, v) = \frac{r(t-\tau) + r}{\|\xi(t)\| + \varepsilon_2}$$

and the maximum is attained for the vector $v = r \dfrac{\xi(t)}{\|\xi(t)\|}$.

Let $\xi(t) \notin \varepsilon_2 S$, $t-\tau > \sigma / r$, $v \in S_0^\sigma$. Then we get $\max\limits_{v \in S_0^\sigma} \alpha_*(t, \tau, v) = \dfrac{r(t-\tau) + \sigma}{\|\xi(t)\| + \varepsilon_2}$ and the maximum is attained for the vector $v = \sigma \dfrac{\xi(t)}{\|\xi(t)\|}$.

If the game parameters for $t-\tau > \sigma / \rho$, $\rho > \sigma > r > 0$, satisfy the inequality

$$\frac{r(t-\tau) + \sigma}{\|\xi(t)\| + \varepsilon_2} < \frac{\rho(t-\tau) - \sigma}{\|\xi(t)\| - \varepsilon_2}, \tag{26}$$

we get

$$\max_{v \in S_0^\sigma} \alpha_*(t, \tau, v) < \min_{v \in S_0^\sigma} \alpha^*(t, \tau, v)$$

and therefore the inequality holds:

$$\int_0^T \max_{v \in S_0^\sigma} \alpha_*(t, \tau, v)\, d\tau < \int_0^T \min_{v \in S_0^\sigma} \alpha^*(t, \tau, v)\, d\tau = 1. \tag{27}$$

Therefore, with regard for relations (19), (21), (23), and (27), if the game parameters for $t-\tau > \sigma / \rho$ satisfy inequality (26), Condition 2 is satisfied.

By construction, for all $v \in S_0^\sigma$ the inclusion holds:

$$\alpha_*(t, \tau, v)[\varepsilon_2 S_0^1 - \xi(t)] \in (t-\tau) S_r^\rho - v.$$

Therefore, taking into account inequality (26), we get

$$\max_{v \in S_0^\sigma} \alpha_*(t, \tau, v)[\varepsilon_2 S_0^1 - \xi(t)] \in (t-\tau) S_r^\rho - v.$$

Thus, the inclusion

$$0 \in \bigcap_{v \in S_0^\sigma} \left[[(t-\tau)S_r^\rho - v] - \max_{v \in S_0^\sigma} \alpha_*(t,\tau,v)[\varepsilon_2 S_0^1 - \xi(t)] \right]$$

holds and Condition 3 is true. The smallest positive root of Eq. (25) is the moment of the game end by Theorem 3 with the use of control (3).

Let us test Condition 4. By construction, for all $v \in S_0^\sigma$ the inclusion holds:

$$\alpha^*(t,\tau,v)[\varepsilon_2 S_0^1 - \xi(t)] \in (t-\tau)S_r^\rho - v.$$

Therefore, taking into account inequality (26), we get

$$\inf_{v \in S_0^\sigma} \{\alpha^*(t,\tau,v)[\varepsilon_2 S_0^1 - \xi(t)]\} \in (t-\tau)S_r^\rho - v.$$

Therefore, the inclusion

$$0 \in \bigcap_{v \in S_0^\sigma} \left[[(t-\tau)S_r^\rho - v] - \inf_{v \in S_0^\sigma} \alpha_*(t,\tau,v)[\varepsilon_2 S_0^1 - \xi(t)] \right]$$

holds and Condition 4 is true. The smallest positive root of Eq. (25) is the moment of the game end by Theorem 4 with the use of control (4).

CONCLUSIONS

We have considered the problem of the approach of controlled objects with different inertia in dynamic game problems and have formulated the sufficient conditions for the game end in a guaranteed finite time in the case where Pontryagin's condition is not satisfied. We have introduced the upper and lower resolving functions of a special type, and based on them, proposed modified schemes of the Pontryagin method and of the method of resolving functions, which ensures the completion of the conflict-controlled process for objects with different inertia in the class of quasi-strategies and counter-controls. We have also provided a model example.

REFERENCES

1. L. S. Pontryagin, Selected Scientific Works, Vol. 2 [in Russian], Nauka, Moscow (1988).
2. A. A. Chikrii, Conflict Controlled Processes, Springer Sci. and Business Media, Boston–London–Dordrecht (2013).
3. A. A. Chikrii, "An analytical method in dynamic pursuit games," Proc. Steklov Inst. of Mathematics, Vol. 271, 69–85 (2010).
4. A. A. Chikrii and V. K. Chikrii, "Image structure of multivalued mappings in game problems of motion control," J. Autom. Inform. Sci., Vol. 48, Iss. 3, 20–35 (2016).
5. I. S. Rappoport, "Method of resolving functions for game problems of approach of controlled objects with different inertia," Cybern. Syst. Analysis, Vol. 57, No. 2, 296–312 (2021). https://doi.org/10.1007/s10559-021-00355-9.
6. N. N. Krasovskii and A. I. Subbotin, Positional Differential Games [in Russian], Nauka, Moscow (1974).
7. A. I. Subbotin and A. G. Chentsov, Guarantee Optimization in Control Problems [in Russian], Nauka, Moscow (1981).
8. O. Hajek, Pursuit Games, Vol. 12, Acad. Press, New York (1975).
9. J.-P. Aubin and H. Frankowska, Set-Valued Analysis, Birkhauser, Boston–Basel–Berlin (1990).
10. R. T. Rockafellar, Convex Analysis, Princeton Univ. Press, Princeton (1970).
11. A. D. Ioffe and V. M. Tikhomirov, Theory of Extremum Problems [in Russian], Nauka, Moscow (1974).
12. A. A. Chikrii and I. S. Rappoport, "Method of resolving functions in the theory of conflict-controlled processes," Cybern. Syst. Analysis, Vol. 48, No. 4, 512–531 (2012). https://doi.org/10.1007/s10559-012-9430-y.