DOI 10.1007/s10559-023-00552-8
Cybernetics and Systems Analysis, Vol. 59, No. 1, January, 2023

CHEBYSHEV APPROXIMATION OF MULTIVARIABLE FUNCTIONS BY A CONSTRAINED RATIONAL EXPRESSION

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UDC 519.65

Abstract. *The authors propose a method for constructing the Chebyshev approximation of multivariable functions by the rational expression with the interpolation condition. The idea of the method is based on constructing the limiting power mean approximation by a rational expression with an interpolation condition in the norm of space L^p as $p \to \infty$. To construct such an approximation, an iterative scheme based on the least squares method with two variable weight functions is used. One weight function ensures the construction of a power mean approximation with the interpolation condition, and the second one specifies the parameters of the rational expression according to its linearization scheme. The convergence of the method is provided by the original method of sequential specification of the values of weight functions, which takes into account the approximation results at previous iterations. The results of test examples confirm the fast convergence of the proposed method of constructing the Chebyshev approximation by a rational expression with a condition.*

Keywords: *Chebyshev approximation by a rational expression, constrained Chebyshev approximation, multivariable functions, power mean approximation, least squares method, variable weight function.*

PROBLEM STATEMENT

Let $f(X)$ be a function of *n* real variables, where X is a vector, $X=(x_1, x_2, \ldots, x_n)$, continuous in some bounded domain D ($D \subset R^n$, R^n is an *n*-dimensional vector space). Function $f(X)$, given on a set of points $\Omega = \{X_j\}_{j=1}^{s}$ from the domain D ($\Omega \subset D$), need to be approximated on the set of points Ω by the rational expression

$$R_{k,l}(a,b;X) = \frac{\sum_{i=0}^{k} a_i \varphi_i(X)}{\sum_{i=0}^{l-1} b_i \psi_i(X) + \psi_l(X)}, \tag{1}$$

where $\varphi_i(X)$, $i=\overline{0,k}$, and $\psi_i(X)$, $i=\overline{0,l}$, are systems of linearly independent real functions D, continuous on a_i, $i=\overline{0,k}$, and b_i, $i=\overline{0,l-1}$, are unknown parameters: $\{a_i\}_{i=0}^{k} \in A$, $A \subseteq R^{k+1}$, and $\{b_i\}_{i=0}^{l-1} \in B$, $B \subseteq R^l$. To construct the Chebyshev approximation of the function $f(X)$ by the rational expression (1) with the condition at a point U ($U \in \Omega$) is to calculate the values of the parameter a^* and b^* that minimize the value of the approximation error

$$\max_{X \in \Omega} |f(X) - R_{k,l}(a^*, b^*; X)| = \min_{a \in A,\, b \in B} \max_{X \in \Omega} |f(X) - R_{k,l}(a,b;X)|, \tag{2}$$

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 1060-0396/23/5901-0146

and at the approximation point U, $R_{k,l}(a^*, b^*; X)$ returns the value of the function $f(U)$:

$$f(U) = R_{k,l}(a^*, b^*; U) = \nu. \tag{3}$$

Approximation by the rational expression $R_{k,l}(a^*, b^*; X)$ that satisfies condition (2), (3) is called Chebyshev approximation with a constraint or with interpolation [1, 2].

Chebyshev approximation by a rational expression with interpolation is used in solving many applied problems [1–4], in particular, for linearization of thermometric characteristics of thermodiode sensors [5, 6], design of manometers [7], etc.

The problem of constructing a Chebyshev approximation of functions by a constrained rational expression is formulated in [8]. The properties of a constrained Chebyshev approximation were first described in [9]. In the Chebyshev approximation by a rational expression, reduction to the sequential solution of a linear programming problem [10, 11] or a nonlinear optimization method [2, 10] are mostly used. The articles [12, 13] describe algorithms for calculating the parameters of the Chebyshev approximation of functions of one variable based on the Remez scheme with the use of differential adjustment.

We will propose a method for constructing the Chebyshev approximation of multivariable functions by a rational expression as a limiting approximation in the norm of the space L^p as $p \to \infty$ [11]. It implies sequential construction of the power mean approximations with interpolation condition [14, 15]. The values of the parameters of power mean approximations by a rational expression with interpolation can be calculated using an iterative scheme based on the least squares method using two variable weight functions whose values are specified with regard for all the intermediate power mean approximations [11, 16]. The parameters of the rational approximation by the least squares method can be found using linearization [17].

CALCULATING THE PARAMETERS OF THE CHEBYSHEV APPROXIMATION BY A RATIONAL EXPRESSION WITH A CONDITION

If for a function $f(X)$ on a set of points Ω there exists a continuous Chebyshev approximation by a rational expression $R_{k,l}(a, b; X)$ with an interpolation condition, then it can be constructed by successive calculation of the power mean approximation with a condition in the space E^p for $p = 2, 3, 4, \ldots$ [11, 15]. To estimate the approximation error in the space E^p, we use the norm

$$\|\Delta\|_{E^p} = \left(\sum_{X \in \Omega} |f(X) - R_{k,l}(a, b; X)|^p \right)^{1/p}.$$

The limiting value of the norm $\|\Delta\|_{E^p}$ as $p \to \infty$, similarly to the norm of the space L^p as $p \to \infty$, coincides with the norm of the space of continuous functions $\|\Delta\|_C$ [18]:

$$\|\Delta\|_C = \max_{X \in \Omega} |f(X) - R_{k,l}(a, b; X)|.$$

In [18], Remez used numerical examples to theoretically substantiate and illustrate the convergence of the computing schemes of constructing the Chebyshev approximation on the basis of the power mean approximation.

Let for the function $f(X)$ given on a set of points Ω, there exist a Chebyshev approximation by a rational expression $R_{k,l}(a, b; X)$ with a condition at point U. To construct such a Chebyshev approximation of the function $f(X)$ means to successively calculate the power mean approximations $f(X)$ on the set of points $\overline{\Omega} = \Omega \setminus \{U\}$ by the expression

$$\overline{R}_{k,l}(a, b; X) = \frac{a_0 \varphi_0(X) + \sum_{i=1}^{k} a_i \varphi_i(X)}{\sum_{i=0}^{l-1} b_i \psi_i(X) + \psi_l(X)}, \tag{4}$$

where

$$a_0 = \left(\nu \left(\sum_{i=0}^{l-1} b_i \psi_i(U) + \psi_l(U) \right) - \sum_{i=1}^{k} a_i \varphi_i(U) \right) / \varphi_0(U),$$

a_i $(i=\overline{1,k})$ and b_i $(i=\overline{0,l-1})$ are unknown parameters. Expression $\overline{R}_{k,l}(a,b;X)$ is obtained from the rational expression (1) with regard for condition (3). When obtaining the values of the parameter a_0, it was assumed that $\varphi_0(U)$ is not equal to zero. To calculate the values of the parameters of the power mean approximations by the expression $\overline{R}_{k,l}(a,b;X)$ in the space E^p for $p=2,3,4,\dots$, we use an iterative scheme based on the least squares method [11, 16]

$$\sum_{X\in\overline{\Omega}} \rho_r(X)(f(X)) - \overline{R}_{k,l}(a,b;X)^2 \xrightarrow[a\in A, b\in B]{} \min, \quad r=0,\dots,p-2, \; p=2,3,\dots, \tag{5}$$

and then specify the values of the weight function

$$\rho_0(X)\equiv 1, \; \rho_r(X)=\rho_{r-1}(X)\,|\Delta_r(X)| = \prod_{i=1}^{r} |\Delta_i(X)|, \; r=1,\dots,p-2, \tag{6}$$

where $\Delta_s(X) = f(X) - \overline{R}_{k,l,s-1}(a,b;X)$, $k=\overline{1,r}$, $\overline{R}_{k,l,s}(a,b;X)$ is an approximation of the function $f(X)$ by expression (4) by the least squares method with the weight function $\rho_s(X)$ on the set of points $\overline{\Omega}$ with interpolation at point U. Approximation $\overline{R}_{k,l,s}(a,b;X)$ corresponds to the power mean approximation of the function $f(X)$ of degree $p=s+2$.

Constructing an approximation by a rational expression using the least squares method is a nonlinear problem [19]. Calculation of a rational approximation by the least squares method is often reduced to a nonlinear optimization problem [20], where the objective function can have many local minima. In practice, other approaches are used to obtain such an approximation [19–21], in particular, in [20] the calculation of approximation parameters by a rational expression is reduced to a quadratic programming problem with a strictly convex objective function.

To construct the approximation by a rational expression using the least squares method, we will apply linearization and use a variable weight function [11, 17], which consists in iterative refinement of the approximation by the rational expression (4). According to this linearization method, for each fixed value of p $(p=2,3,4,\dots)$, we calculate an approximation of the function $f(X)$ by the expression $\overline{R}_{k,l}(a,b;X)$ (4) by the least squares method

$$\sum_{X\in\overline{\Omega}} \rho_r(X) \upsilon_{r,t}(X) (\Phi_{r,t}(a,b;X))^2 \xrightarrow[a\in A,\, b\in B]{} \min, \; r=p-2, \; t=0,1,\dots, \tag{7}$$

where

$$\Phi_{r,t}(a,b;X) = f(X)\left(\sum_{i=0}^{l-1} b_{i,r,t} \psi_i(X) + \psi_l(X) \right) - \sum_{i=1}^{k} a_{i,r,t} \varphi_i(X) - a_0 \varphi_0(X). \tag{8}$$

We calculate the value of the weight function $\rho_r(X)$ by formula (6), and the value of the weight function $\upsilon_{r,t}(X)$ by the formula

$$\upsilon_{r,t}(X) = \begin{cases} 1 & \text{if } r=0, \; t=0, \\ \left(\sum_{i=0}^{l-1} b_{i,r,t-1} \psi_i(X) + \psi_l(X) \right)^{-2} & \text{if } t>0. \end{cases} \tag{9}$$

We can control the refinement of the approximation by the expression $\overline{R}_{k,l}(a,b;X)$ by the least squares method (7) with the weight functions (6) and (9) by the accuracy ε_1 of satisfying the condition

$$|\eta_{r,t-1} - \eta_{r,t}| \le \varepsilon_1 \eta_{r,t}, \tag{10}$$

where

$$\eta_{r,t} = \sum_{X\in\overline{\Omega}} \rho_r(X) \upsilon_{r,t}(X) (\Phi_{r,t}(a,b;X))^2 .$$

Satisfying condition (10) means that the power mean approximation of the degree $p = r + 2$ by the rational expression $\bar{R}_{k,l,r}(a,b;X)$ is calculated with the accuracy ε_1. The values of the approximation parameters $\bar{R}_{k,l,r}(a,b;X)$ are as follows:

$$a_{j,r} = a_{j,r,t} \quad (j = \overline{1,k}), \;\; b_{j,r} = b_{j,r,t} \quad (j = \overline{0,l-1}).$$

Therefore, to construct the Chebyshev approximation by the rational expression (4) means to apply two iterative processes: inner (7), (9) and external (5), (6) iterations. Completion of iterations (5), (6) can be controlled by reaching a given accuracy ε

$$\mu_{r-1} - \mu_r \le \varepsilon \mu_r, \tag{11}$$

where

$$\mu_r = \max_{(x,y)\in\bar{\Omega}} |f(X) - \bar{R}_{k,l,r}(a,b;X)|.$$

In solving test examples for functions of one, two, and three variables, the accuracy $\varepsilon = 0.003$ was achieved when five to 22 iterations (5), (6) were used. This accuracy ensured the convergence of two or three significant digits of the error of the Chebyshev approximation with the rational expression. Here, the accuracy $\varepsilon_1 = 0.003$ of determining intermediate approximations with a rational expression was achieved in three to four inner iterations (7), (9). If for $r \ge 1$ the values of the weight function $\upsilon_{r,0}(X)$ are not changed (remain the same as the previous ones $\upsilon_{r-1,t}(X)$), then only two iterations (7), (9) were sufficient to refine the rational expression.

As a result, the Chebyshev approximation of the continuous function $f(X)$ by the rational expression (1) on the set of points Ω with interpolation at point U is determined by the values of the parameters calculated for the approximation by expression (4)

$$R_{k,l}(a,b;X) = \bar{R}_{k,l,r}(a,b;X). \tag{12}$$

If $k \ge l$, then to obtain approximation (12), we carry out another correction using the additive adjustmment

$$\bar{\alpha}_0 = (\mu_{\max} + \mu_{\min})/2,$$

where

$$\mu_{\max} = \max_{X\in\Omega_u} (f(X) - R_{k,l}(a,b;X)), \;\; \mu_{\min} = \min_{X\in\Omega_u} (f(X) - R_{k,l}(a,b;X)).$$

The adjusted approximation of the continuous function $f(X)$, given on the set of points $X \in \Omega$ by the rational expression (1) for $k \ge l$ with the condition at point U, is

$$R_{k,l}(a,b;X) = R_{k,l}(a,b;X) + \bar{a}_0. \tag{13}$$

Note that the adjusted approximation by the rational expression (13) of the function $f(X)$ given on the set of points $X \in \Omega$ with interpolation at point U decreases the absolute error of approximation by $\bar{a}_0$ but at the same time causes an error of the same value $\bar{a}_0$ when estimating the value of the function at point U.

When constructing the Chebyshev approximation of tabular functions by a rational expression with a condition for some values of the powers of the numerator k and of the denominator l, a discontinuous approximation can be obtained. This means that for the given function there is no continuous Chebyshev approximation by a rational expression with a condition for these power values k and l. If it is necessary to construct a Chebyshev approximation with a rational expression for this function, the calculation can be continued with other values of powers k and l.

CALCULATING THE PARAMETERS OF THE CHEBYSHEV APPROXIMATION BY A RATIONAL EXPRESSION WITH RELATIVE ERROR

If the continuous function $f(X)$ does not take values equal to zero at the set of points Ω, and there exists Chebyshev approximation $f(X)$ by the generalized rational expression (1) with interpolation at point U, then such an approximation with a relative error can be calculated by the method described above. The scheme for calculating the Chebyshev approximation by a rational expression with a relative error and condition at point U provides the following changes.

Let the initial values of the weight function $\rho_0(X)$ in (6) be

$$\rho_0(X) = \frac{1}{f(X)^2}. \quad (14)$$

We will calculate the error μ_r of the Chebyshev approximation by the rational expression (4) in condition (11) by the formula

$$\mu_r = \max_{X \in \Omega} \left| 1 - \frac{\bar{R}_{k,l,r-1}(a,b;X)}{f(X)} \right| \quad (15)$$

and calculate the refined values of the weight function $\rho_r(X)$ by formula (6), where

$$\Delta_s(X) = 1 - \frac{\bar{R}_{k,l,s}(a,b;X)}{f(X)}. \quad (16)$$

If condition (11) is satisfied, taking into account (15), we assume that the Chebyshev approximation of the continuous function $f(X)$ by the rational expression (1) on the set of points Ω with a relative error and condition at point U is calculated with accuracy ε. The values of the parameters of the Chebyshev approximation by the rational expression (1) will be equal to the corresponding parameters of the obtained approximation by expression (4)

$$R_{k,l}(a,b;X) = \bar{R}_{k,l,r}(a,b;X). \quad (17)$$

For approximation (17), we can also perform refinement using the multiplicative adjustment d

$$R_{k,l}(a,b;X) = d\bar{R}_{k,l}(a,b;X),$$

where

$$d = \frac{2f(X_{\max})f(X_{\min})}{R_{k,l}(a,b;X_{\min})f(X_{\max}) + R_{k,l}(a,b;X_{\max})f(X_{\min})},$$

$X_{\max}$ and $X_{\min}$ are respectively the points at which the relative error μ_r of approximation (17) reaches its maximum and minimum values at the set of points Ω. The value of the adjustment d can be found as the solution of the one-parameter problem of Chebyshev approximation of the function $f(X)$ by the expression $dR_{k,l}(a,b;X)$ on the set of points Ω with the relative error

$$\max_{X \in \Omega} \left| \frac{f(X) - dR_{k,l}(a,b;X)}{f(X)} \right| \xrightarrow[d]{} \min.$$

Example 1. Let us find the Chebyshev approximation of the function $y(x) = e^x$ defined on the interval $[-1, 2]$ at points x_i, $i = \overline{0, 30}$, $x_i = -1 + 0.1i$, by the rational expression $R_{2,1}(a,b;x)$, in which the numerator and denominator are polynomials of the second and first degrees, respectively, with respect to the variable x, with the condition at the point $\bar{x} = x_6 = -0.4$.

Using the proposed method for $\varepsilon = 0.003$, in five iterations (5) with the weight function (6) for the function $y(x)$, we obtained the rational expression

$$R_{2,1}(a,b;x) = \frac{0.9560543189x^2 + 2.921771312x + 3.848025101}{3.823924539 - x}, \quad (18)$$

which, with regard for the additive adjustment $\bar{a}_0 = 0.0002140135$, ensures the absolute error of the approximation of 0.022702257. During the calculation of approximation (18), the error in iterations (5) took the following values:

$$0.0320089585,\ 0.0240330006,\ 0.023045826,\ 0.022958886,\ 0.022916264. \quad (19)$$

The curve in Fig. 1 shows the error of approximation of function $y(x)$ by the rational expression (18). It corresponds to the characteristic property of the Chebyshev approximation with interpolation [7]: has four extremum points at which the absolute value of the deviation is the maximum. The values of the absolute values of these deviations

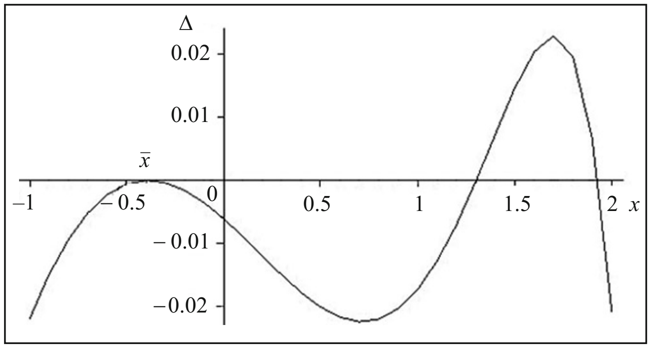

Fig. 1. Curve of the error of approximation of function $y(x)$ by the rational expression (18) with the condition at the point $\bar{x} = -0.4$.

coincide (or are equal) within the accuracy of the calculated approximation, and the sign of the deviations at these points alternates, except for the points adjacent to the points of the condition $\bar{x} = -0.4$:

$$(-1, -0.0223232025),\ (0.7, -0.022702257),\ (1.7, 0.022702257),\ (2, -0.021220319). \tag{20}$$

According to the characteristic property of the Chebyshev approximation with the condition, the signs of the error of the deviation of the approximation at the extremum points adjacent to the point of the condition should coincide [7].

At the end extremum points (20), the absolute value of the approximation error is somewhat smaller. The spread of the absolute values of the approximation error at the extremum points (20) is equal to 0.001481938, which is 6.66% of the average absolute value of the approximation error at the extremum points 0.02223700888. For a better alignment of the absolute values of the approximation error at the extremum points (20), it is possible to increase the calculation accuracy of the Chebyshev approximation by reducing the value ε in condition (11). Chebyshev approximation of the function $y(x)$ by the rational expression with interpolation at point $\bar{x} = -0.4$ for $\varepsilon = 0.00003$ was obtained in 115 iterations (5). The rational expression

$$\hat{R}_{2,1}(a, b; x) = \frac{0.9566342954x^2 + 2.921051278x + 3.846359357}{3.82355946 - x} \tag{21}$$

with regard for the adjustment $\bar{a}_0 = -0.0000322565$ ensures the absolute approximation error of 0.02236887.

For the approximation of function $y(x)$ by the rational expression (21), the following values of the error were observed at the extremum points:

$$(-1, -0.0222769132),\ (0.7, -0.022336613),\ (1.7, 0.022336614),\ (2, -0.022272896). \tag{22}$$

It follows from (22) that an increase in the accuracy of the calculated approximation did not change the extremum points. At the end extremum points, we also observe a slightly smaller absolute value of the approximation error; however, the spread of the absolute values of the approximation error at the extremum points has significantly decreased by 0.000063708 and is only 0.286% of the average absolute value of the approximation error at these points 0.02230575905.

Chebyshev approximation of the function $y(x)$ by the rational expression $R_{2,1}(a, b; x)$ with the condition at the point $\bar{x} = -0.4$ and relative error for $\varepsilon = 0.003$ was calculated in ten iterations (5). The rational expression

$$\bar{R}_{2,1}(a, b; x) = \frac{0.7972125368x^2 + 2.746838211x + 3.678106394}{3.659074535 - x} \tag{23}$$

with regard for the adjustment $d = 0.999968438$ ensures the relative approximation error of 1.021717265%. During the calculation of the approximation (23), the error according to iterations (5) took the following values:

$$\begin{aligned}&0.02904823484,\ 0.01270885344,\ 0.01084661589,\ 0.01065015905,\ 0.01051006218,\\&0.01041509521,\ 0.01034981727,\ 0.01030468772,\ 0.01027323863,\ 0.01024905742.\end{aligned} \tag{24}$$

Analyzing the change in the values of the approximation error by iterations (5), given in (19) and (24), we can see that the value of the approximation error decreases rather quickly at the initial iterations. On average, two to four iterations are sufficient to refine the first two significant digits in the value of the approximation error.

The graph of the relative error of approximation (23) also corresponds to the characteristic features of the Chebyshev approximation with a condition at one point [7, 15], namely, at the extremum points adjacent to the point of the condition, the sign of the relative error coincides, and at the rest of the extremum points, it alternates. The relative error at the extremum points took the following values (in percent):

$$(-1, -0.9778652453),\ (0.2, -1.021717195),\ (1.5, 1.021717265), (2, -0.9682237628).$$

To attain a better alignment of the absolute values of the approximation error at the extremum points, the Chebyshev approximation was obtained by the rational expression $R_{2,1}(a, b; x)$ with the accuracy $\varepsilon = 0.00003$ in 84 iterations (5). With regard for the adjustment $d = 1.000003503$, the expression

$$\widehat{R}_{2,1}(a, b; x) = \frac{0.7574484034x^2 + 2.687510612x + 3.644765167}{3.614415586 - x}$$

ensures the relative approximation error of 1.012%. For this approximation, the following values of the error (in percent) were observed at the extremum points:

$$(-1, -1.010535459),\ (0.2, -1.011625929),$$

$$(1.5, 1.011625981),\ (2, -1.010751901).$$

These results confirm the convergence of the proposed method. By increasing the calculation accuracy of the Chebyshev approximation in condition (11), it is possible to obtain an approximation with the required accuracy. It is clear that the number of iterations usually increases during the calculation of the approximation parameters. A practically suitable accuracy of the calculation of the Chebyshev approximation is provided for $\varepsilon = 0.003$. For both absolute and relative errors, this accuracy provides an approximation with two to three correct significant digits of the approximation error using five to ten iterations to approximate the function of one variable by the rational expression with four parameters. In the case of increasing the approximation calculation accuracy, beginning with the value $\varepsilon = 0.003$, changes in points with extremum approximation errors were not observed.

Example 2. Let us find the Chebyshev approximation of function $z(x, y) = e^{-(x^2+y^2)}$ defined at points (x_i, y_j), $i = \overline{0, 10}$, $j = \overline{0, 10}$, where $x_i = -1 + 0.2i$ and $y_j = -1 + 0.2j$, by the rational expression $R_{2,2}(a, b; x, y)$, in which the numerator and denominator are polynomials of second degree in the variables x and y, with interpolation at point $(\bar{x}, \bar{y}) = (-0.8, -0.8)$.

Using the proposed method for $\varepsilon = 0.003$, the following rational expression was obtained in 16 iterations (5), (6) for the function $z(x, y)$:

$$R_{2,2}(a, b; x, y) = \frac{P_2(a; x, y)}{Q_2(b; x, y)}, \tag{25}$$

where

$$P_2(a; x, y) = 1.01168691 + 0.01271841065x + 0.01271832247y$$

$$+ 0.01883247505xy - 0.3389069836x^2 - 0.3389072952y^2,$$

$$Q_2(b; x, y) = 1 + 0.01749531648x + 0.01749495916y$$

$$+ 0.0798051921xy + 0.801369663x^2 + 0.8013684061y^2.$$

Rational expression (25) with regard for the adjustment $\bar{a}_0 = -0.0000309595$ provides the absolute approximation error of 0.0122648252. Figure 2 shows the surface of the error of approximation of function $z(x, y)$ by the rational expression (25).

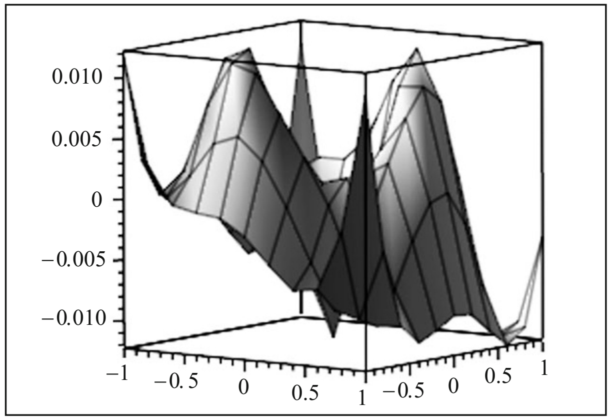

Fig. 2. The surface of the error of approximation of function $z(x, y)$ by the rational expression (25) with interpolation at point $(\bar{x}, \bar{y}) = (-0.8, -0.8)$.

Chebyshev approximation of function $z(x, y)$ by the rational expression with interpolation at point $(\bar{x}, \bar{y}) = (-0.8, -0.8)$ for $\varepsilon = 0.00003$ was obtained in 186 iterations (5)

$$\hat{R}_{2,2}(a, b; x, y) = \frac{\hat{P}_2(a; x, y)}{\hat{Q}_2(b; x, y)}, \tag{26}$$

$$\hat{P}_2(a; x, y) = 1.011896416 + 0.01275075547\,y - 0.01280516368\,x$$

$$+0.01467104117\,xy - 0.3371140452\,x^2 - 0.337412009\,y^2,$$

$$\hat{Q}_2(b; x, y) = 1 + 0.01764260288\,y + 0.01813318269\,x$$

$$+0.3314116329\,xy + 0.8242332228\,x^2 + 0.8232947458\,y^2.$$

The rational expression (26) provides the approximation error of 0.0119322935. The adjustment for this approximation is $\bar{a}_0 = 0.1483705\text{e}-4$. Taking into account this adjustment does not make practical sense since it is only 0.124% of the approximation error (26).

Chebyshev approximation of the function $z(x, y)$ by the rational expression $R_{2,2}(a, b; x, y)$ with interpolation at point $(\bar{x}, \bar{y}) = (-0.8, -0.8)$ with the relative error for $\varepsilon = 0.003$ was calculated in three iterations (5). The rational expression

$$R_{2,2}(a, b; x, y) = \frac{P_2(a; x, y)}{Q_2(b; x, y)},$$

in which

$$P_2(a; x, y) = 1.029409171 + 0.0009985053314\,y + 0.0009985051201\,x$$

$$+0.005933912472\,xy - 0.3263794034\,x^2 - 0.3263794041\,y^2,$$

$$Q_2(b; x, y) = 1 - 0.002981207706\,y - 0.00298120858\,x$$

$$+0.04299484992\,xy + 0.9178425704\,x^2 + 0.9178425682\,y^2,$$

with regard for the adjustment $d = 1.000294165$ provides the relative approximation error of 2.94%.

Chebyshev approximation of function $z(x, y)$ by the rational expression $R_{2,2}(a, b; x, y)$ with interpolation at point $(\bar{x}, \bar{y}) = (-0.8, -0.8)$ with the relative error for $\varepsilon = 0.00003$ was obtained in 24 iterations (5). With regard for the adjustment $d = 0.9999926746$, the obtained Chebyshev approximation provided the relative error of 2.77%.

Example 3. Let us find the Chebyshev approximation of function $z_3(x, y, t) = e^{-(x+y+t)}$ defined at points (x_i, y_j, t_r), $i = \overline{0, 20}$, $j = \overline{0, 20}$, $r = \overline{0, 20}$, where $x_i = -1 + 0.1i$, $y_j = -1 + 0.1j$, and $t_r = -1 + 0.1r$, by the rational expression $R_{1,1}(a, b; x, y, t)$, where the numerator and denominator are first-degree polynomials in the variables x, y, and t, with interpolation at point $(\bar{x}, \bar{y}, \bar{t}) = (-0.8, -0.8, -0.8)$.

Using the proposed method (5) for $\varepsilon = 0.003$ in 22 iterations for the function $z_3(x, y, t)$, we obtained an approximation by the rational expression

$$R_{1,1}(a, b; x, y, t) = \frac{P_1(a; x, y, t)}{Q_1(b; x, y, t)}, \tag{27}$$

in which

$$P_1(a; x, y, t) = 1.65548594111 - 1.09662621859x - 1.09662693548y - 1.09662646991t,$$

$$Q_1(a; x, y, t) = 1 + 0.254993735029x + 0.254993459307y + 0.254993922881t.$$

The rational expression (27) provides the absolute approximation error of 0.97579906188 with the adjustment $\bar{a}_0 = 0.026366720935$.

Chebyshev approximation of function $z_3(x, y, t)$ by the rational expression $R_{1,1}(a, b; x, y, t)$ with a relative error was not obtained. A discontinuous rational expression was obtained in calculating the approximation with the relative error on iterations (6), (8).

Chebyshev approximation of function $z_3(x, y, t)$ by the rational expression in which the numerator and denominator are second-degree polynomials in the variables x, y, and t, using method (5) for $\varepsilon = 0.003$, was obtained in 11 iterations. The absolute error of this approximation, with regard for the adjustment $\bar{a}_0 = 0.0002501987786$, was equal to 0.0233863597314. Chebyshev approximation of the function $z_3(x, y, t)$ by the rational expression in which the numerator and denominator are second-degree polynomials in the variables x, y, and t, with the relative error using iterations (5) with the weight function (14) for $\varepsilon = 0.003$, was obtained in 19 iterations. With regard for the adjustment $d = 1.00006973092$ it provided the relative approximation error of 2.16%.

CONCLUSIONS

The proposed method of constructing the Chebyshev approximation of tabulated multivariable functions by a rational expression with a condition is based on the idea of constructing a limiting power mean approximation with a condition. It implies sequential calculation of the power mean approximations by a rational expression with a condition implemented by the least-squares method with two variable weight functions. One weight function ensures the construction of the power mean approximation by the rational expression with a condition, and the second function ensures the adjustment of the approximation parameters by a rational expression using the least-squares method according to the linearization scheme. The convergence of the method provides an original way of sequentially specifying the values of the weight functions according to formulas (6) and (9) for the absolute error and taking into account formulas (14), (16) for the relative error. The method is easy to implement and provides the possibility of calculating the parameters of the Chebyshev approximation using a rational expression with an interpolation condition for the absolute and relative errors with the required accuracy. The results of solving test examples confirm the fast convergence of the proposed method for approximation by a rational expression with the condition of functions of one, two, and three variables. When solving the test examples by this method, the coincidence of two to three significant digits of the error of the Chebyshev approximation by the rational expression with the condition was achieved with the use of 5 to 22 iterations.

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