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FINDING COMPROMISE AND CONSENSUS IN MULTICRITERIA PROBLEMS

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Abstract. *Various approaches to solving multicriteria problems are considered depending on the role of constraints in the problem statement. If the constraints are fixed and specified, then the calculation algorithm includes the preferences of the decision maker and the corresponding multicriteria problem has a solution fundamentally based on compromise. If the values of the constraints can be varied, it becomes possible to obtain consensus decisions, and the calculation algorithm is free of heuristic elements.*

Keywords: *compromise, consensus, multicriteria problem, constraints, resources, calculation algorithm.*

INTRODUCTION

Each system has sufficient resources (financial, technical, etc.) to successfully achieve the set goals under the given operating conditions. These resources are usually constrained and this property of resources in optimization problems is taken into account in the constraints. The latter ones are of fundamental importance in finding solutions to multicriteria problems.

Constraints can be imposed both on the optimization arguments and on the values of the partial problem criteria. Constraints imposed on the characteristics of the system under certain circumstances can often be the reason for the introduction of a certain criterion. For example, under normal ground conditions, it is customary to evaluate the quality of the ergatic system by the amount (or by the loss rate) of oxygen consumed by a human operator during the performance of a given task. When a system functions without contact with the Earth's atmosphere (in space, underwater, etc.), it is a completely different situation. Here, oxygen resources are constrained and its consumption efficiency becomes a very important characteristic. The reflection of this requirement is the introduction of the corresponding criterion. In this case, it can be said that the constraint generates the criterion.

Even small changes in the constraints can significantly affect the solutions [1]. Very serious consequences can be obtained by canceling some constraints and adding others. In 1956, Brazilian entomologists recognized that bees were not producing enough honey. Hence, they crossed several European species of bees and added a variety of African ones. The hybrid bees did produce more honey, they were resistant to disease and tolerated the heat well, but at the same time they became incredibly aggressive and very venomous. More than 150 people and some animals, both domestic and wild, died from their stings in Brazil and the South of the United States.

Therefore, there is a great danger in the process of formal optimization of complex systems, which was considered by N. Wiener in the first publications on cybernetics. The fact is that without setting all the necessary constraints, it is possible to obtain unforeseen and undesirable side effects simultaneously with the optimization of the objective function.

In the theory of multicriteria decision-making, two main approaches to solving problems of vector optimization of complex systems are considered. One of them consists of the formalization of multicriteria problems under given (fixed)

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constraints and resource capabilities of the system. The second approach shows what new perspectives open up before the developers if they have the opportunity to manage the resource reserves (constraints) of multicriteria systems within certain limits. Some results in this field are considered in [2–4] for the system optimization theory.

COMPROMISE

To successfully achieve the set goals under the specified operating conditions, each system has certain reserves and resources (in terms of strength, heat resistance, amount of fuel, etc.) that are constrained. The concept of the first approach is characterized by the fact that the constraints and resource capabilities of the system are set and fixed.

Taking into account the fundamental importance of constraints in solving multicriteria problems, let us formulate, in contrast to the concept of Charnes–Cooper [5], the following principle of optimality: "away from the boundary values" [6]. In accordance with this principle, a nonlinear scheme of compromises is proposed that makes it possible to obtain a common compromise solution to the criteria, which distances partial criteria the most from their "red lines" (boundary values).

Requirements for the scalar convolution $Y[y(x)]$ of the criteria according to the nonlinear compromise scheme are as follows:

1) it should "penalize" partial criteria for approaching their boundary values;

2) it should be differentiable with respect to its arguments.

From the feasible functions that meet the above requirements, we choose the following simplest one:

$$Y[y(x)] = \sum_{k=1}^{s} A_k [A_K - y_k(x)]^{-1},$$

where A_k is an upper constraint on the values of criteria to be minimized, $y_k \le A_k$, $k \in [1, s]$.

Minimization of the objective function leads to a compromise solution that moves partial criteria away from their constraints to the greatest extent as follows:

$$x^* = \arg \min_{x \in X} \sum_{k=1}^{s} A_k [A_K - y_k(x)]^{-1}.$$

Scalar convolution by a nonlinear compromise scheme penalizes partial criteria for approaching their boundary values. Indeed, let some criterion $y_m(x)$, $m \in [1, s]$, dangerously approach its boundary value. This means that the difference $[A_m - y_m(x)]$ approaches zero; thus, the corresponding term of the minimized $\dfrac{A_m}{A_m - y_m(x)}$ is growing rapidly.

With the significant growth of $y_m(x)$ and of the term $\dfrac{A_m}{A_m - y_m(x)}$ as well, the minimization of the entire sum is reduced to the minimization of only this most "disadvantageous" term. This means that the non-linear scheme of compromises under the dangerous growth of one or more partial criteria being minimized acts as a minimax (Chebyshev) optimization model

$$x^* = \arg \min_{x \in X} \max_{k \in [1, s]} \frac{y_k}{A_k}(x).$$

A situation where the partial criteria are dangerously close to their boundary values is considered to be stressed. On the contrary, the situation is calm if the criteria are far from their boundary values. In a calm situation, the nonlinear scheme of compromises acts as an integral optimization model

$$x^* = \arg \min_{x \in X} \sum_{k=1}^{s} \frac{y_k(x)}{A_k}.$$

Thus, the nonlinear scheme of compromises adapts to the decision-making situation. The optimization model varies from integral in calm situations to egalitarian (Chebyshev) in tense situations. In intermediate situations, schemes

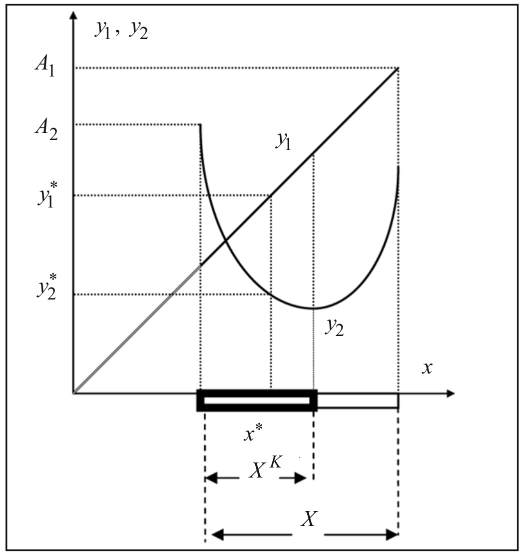

Fig. 1

of compromises are formed that satisfy partial criteria to the extent that they are far from their boundary values. This means that instead of choosing a scheme of compromises in different situations, a single universal non-linear scheme of compromises can be applied that automatically provied a scheme adequate to that particular situation.

Model example. Let a certain system be evaluated according to two criteria that are minimized as follows:

$$y_1 = x;$$

$$y_2 = (x-5)^2 + 2.$$

Restrictions on criteria values are set as follows: $y_1 \le A_1 = 6.5$ and $y_2 \le A_2 = 6$. It is necessary to find a compromise-optimal solution to $x^* \in X^K \subset X$ according to the nonlinear scheme of compromises.

In accordance with the concept of a nonlinear scheme of compromises for this example,

$$x^* = \arg\min_x \left[\frac{A_1 - y_1}{A_1} + \frac{A_2 - y_2}{A_2}\right] = \arg\min_x \left[\frac{6.5 - x}{6.5} + \frac{6 - (x-5)^2 - 2}{6}\right].$$

The necessary condition for the minimum of the function has the form $\frac{\partial}{\partial x} Y[y(x)] = 0$, i.e.,

$$-\frac{1}{6.5} + \frac{-2(x-5)}{6} = 0,$$

where $x^* = 4.08$; $y_1^* = 4.08$; $y_2^* = 2.85$.

A graphic illustration of the example is presented in Fig. 1. A characteristic feature of the first approach is the presence of the area of compromises $X^K \subset X$ (the Pareto set, the negotiation set) where a compromise-optimal solution x^* is obtained.

CONSENSUS

The second approach is used when the constraints on partial criteria are not fixed. Let us consider how this affects the choice of a scheme of compromises.

Each scheme of compromises is a reflection of a certain useful property that the designed multicriteria system should have in a given situation according to the developers [7]. Thus, the egalitarian principle of uniformity provides the implementation of the most uniform change idea at the level of each of the partial criteria. As an example, let us note that a well-designed mechanism wears out evenly and fails simultaneously in all its links after the estimated operation time.

On the other hand, the application of the utilitarian principle of integral optimality shows that the developers focus on saving the total consumption of stocks and resources of the designed system. For example, the above mechanism has to work as long as possible.

The traditional problem statement forces one to choose a single adequate to the given conditions scheme of compromises, ignoring the useful qualities that are present in other optimality principles.

Meanwhile, the practice of solving applied multicriteria problems shows that the solutions obtained by different schemes of compromises are not always significantly different. In the best samples of multicriteria systems, they are close and they almost coincide sometimes. What does it depend on and what does it indicate?

If the solutions obtained by different schemes of compromises coincide (or are quite close), it means that the resources and reserves of the designed system are selected and used so successfully that the system simultaneously meets all the requirements laid down in various optimality principles under the given operation conditions. For example, a rationally constructed mechanism works for a long time, and at the end of its service life it turns out that its parts are worn out equally. If the solutions are significantly different, then the reserves and resources of the system are selected incorrectly and they are not balanced for the given operation conditions; thus, the system is organized irrationally.

Thus, an indication of a rationally organized multicriteria system is the coincidence (closeness) of solutions obtained according to various optimality principles, while the means of rational organization is the selection of stocks and resources dependent upon by the system constraints. It is expedient not only to passively state that the system is designed irrationally (or rationally), but when and how it is feasible to include a purposeful selection (organization) of stocks and resources of the designed multicriteria system into the problem statement.

Let us consider such a design procedure when developers within certain limits can change the values of all or some reserves and resources of the system, harmoniously selecting a set of constraints adequate to the given conditions and, thereby, determining an adequate feasible range of solutions.

Let us formulate the principle of rational organization in multicriteria management problems. In a rationally organized multicriteria system, constrained resources and stocks are selected in such a way that the optimization of the efficiency vector according to various schemes of compromises leads to convergent (or close) solutions under given operational conditions.

The principle of rational organization is universal and is a logical basis for formalizing the solution to multicriteria problems of various natures. In the case of matching solutions, there is no problem of choosing a compromise scheme and the corresponding heuristic element from the method of solving the multicriteria problem is not used. Then, the vector optimization problem is completely and objectively reduced to a scalar one.

In the new formulation of the multicriteria problem, constraints A are not given constants, but are independent variables with arguments x being independent variables. In this case, the result of solving the vector problem consists in determining such vectors $x^o \in X$ and A^o, for whom the principle of rational organization is satisfied.

It can be argued that if the principle of rational organization is strictly followed, then the optimal solution x^o has the following features:

- is belongs to the Pareto set;
- it simultaneously satisfies all schemes of compromises that lead to Pareto-optimal solutions;
- it is unique.

From here it follows that the mathematically strict implementation of the rational organization principle is nothing more than a rational selection of a vector A^o of all Pareto solutions to a single point x^o that is the desired optimal solution to the multicriteria problem.

In view of the above, it can be asserted that the implementation of the rational organization principle simultaneously satisfies the polar schemes of compromises corresponding to the egalitarian principle of uniformity and the utilitarian principle of efficiency. The principle of uniformity is the equality of relative criteria

$$\frac{y_1(x)}{A_1} = \frac{y_2(x)}{A_2} = \ldots = \frac{y_s(x)}{A_s}, \tag{1}$$

while the efficiency principle requires the minimum of function

$$Y\,[y(x)] = \sum_{k=1}^{s} \frac{y_k(x)}{A_k} = \min_{x}. \tag{2}$$

By expanding expression (1), we obtain

$$\frac{y_j(x)}{A_j} - \frac{y_{j+1}(x)}{A_{j+1}} = 0, \;\; j \in [1, s-1]. \tag{3}$$

Under the assumption that the solution is reached within the feasible region of the arguments, condition (2) gives rise to the equation system

$$\frac{\partial Y}{\partial x_i} = \frac{\partial}{\partial x_i} \sum_{k=1}^{s} \frac{y_k(x)}{A_k} = 0, \;\; i \in [1, n], \tag{4}$$

where n is the dimensionality of the argument vector.

The principle of rational organization requires that equations (3) and (4) form a compatible and complete equation system. However, it is clear that this system is uncertain (one equation is missing). This is explained by the fact that its solution can be obtained with an infinite number of combinations of the absolute values of the constraints according to the rational organization principle. Therefore, it is necessary to determine the problem with an additional condition. Such a condition can be, for example, equality of the lth relative criterion (and, automatically, of all relative criteria) given for physical reasons to a specific value

$$\frac{y_l(x)}{A_l} = \mu, \;\; l \in [1, s], \;\; 0 < \mu \le 1. \tag{5}$$

Thus, to solve the multicriteria problem according to the rational organization principle, it is necessary to solve system of equations (3)–(5). As a result, we obtain n of the sought-after components of the solution vector x^o and s of optimal components of the vector of the boundary values of the criteria A^o. The described simple and objective technique can be applied if the rational organization problem has an exact solution in the given area of arguments.

Among the solutions obtained according to the rational organization principle, a special place is occupied by the one where the limited reserves and resources of the system are spent completely. Such a solution can be obtained by crossing conditions (3) and (4) if $\mu = 1$. The obtained solution x^O can be logically called a global consensus as it is balanced by all criteria and provided with fully expendable system resource capabilities, and it is non-contradictory.

Example 1. Let the system be evaluated by the same two criteria being minimized

$$y_1 = x,$$

$$y_2 = (x-5)^2 + 2.$$

Now, let us change the problem in such a way that the constraints A_1 and A_2 are not fixed and can be selected from an open domain. The parameter $\mu = 0.6$ is chosen for physical reasons.

The values of x^o, A_1^o, and A_2^o where the rational organization principle is satisfied has to be found. To do this, let us construct system of equations (3)–(5) as follows:

$$\frac{x}{A_1} = \frac{(x-5)^2 + 2}{A_2},$$

$$\frac{\partial}{\partial x}\left[\frac{x}{A_1} + \frac{(x-5)^2 + 2}{A_2}\right] = 0,$$

$$\frac{x}{A_1} = \mu = 0.6.$$

Solving this system leads to the quadratic equation $1.8x^2 - 12x + 16.2 = 0$ with two roots $x_1^o = 4.79$ where $A_{11}^o = 7.98$; $y_{11}^o = 4.79$; $A_{21}^o = 3.13$; and $y_{21}^o = 2.05$ and $x_2^o = 1.88$ where $A_{21}^o = 3.41$; $y_{21}^o = 1.88$ $A_{22}^o = 19.56$; and $y_{22}^o = 11.73$. The first option seems preferable.

Example 2. Let us select the parameter $\mu = 1$ under the same conditions, for which the solution to system of equations (3)–(5) gives the values x^O, A_1^O, and A_2^O that correspond to the global consensus.

During the solving process, we obtain a quadratic equation $3x^2 - 20x + 27 = 0$ that has two roots $x_1^O = 4.79$ where $A_{11}^O = 4.79$; $y_{11}^O = 4.79$; $A_{21}^O = 2.05$; and $y_{21}^O = 2.05$ and $x_2^O = 1.88$ where $A_{21}^O = 1.88$; $y_{21}^O = 1.88$; $A_{22}^O = 11.73$; and $y_{22}^O = 11.73$. Both options are feasible.

Example 3. For the system that is evaluated according to three minimized criteria

$$y_1 = x,$$

$$y_2 = (x-5)^2 + 2,$$

$$y_3 = 2 - 0.5x,$$

it is necessary to determine the conditions of the global consensus x^O, A_1^O, A_2^O, and A_3^O.

Let us construct system of equations (3)–(5) as follows:

$$\frac{x}{A_1} = \frac{(x-5)^2 + 2}{A_2}; \ \frac{(x-5)^2 + 2}{A_2} = \frac{2 - 0.5x}{A_3},$$

$$\frac{\partial}{\partial x}\left[\frac{x}{A_1} + \frac{(x-5)^2 + 2}{A_2} + \frac{2 - 0.5x}{A_3}\right] = 0,$$

$$\frac{x}{A_1} = \mu = 1.$$

During the solving process, we obtain the cubic equation

$$2x^3 - 21x^2 + 67x - 54 = 0.$$

Solving this equation using Vieta’s formula gives us two complex roots and one real root that is the search for argument. Thus, $x^O = 1.22$, where $A_1^O = 1.22$; $y_1^O = 1.22$; $A_2^O = 14.3$; $y_2^O = 14.3$; $A_3^O = 1.39$; and $y_3^O = 1.39$.

CONCLUSIONS

Two approaches to the formalization of solving multicriteria problems are considered. One of them is applied under fixed constraint values of partial criteria and is the fact that a desirable compromise-optimal solution x^* is obtained as the minimization argument of the scalar convolution of the criteria according to the nonlinear compromise scheme that depends on given constraints A [8].

The second approach is used when the constraints A are not specified and it is possible to consider them as independent variable problems. In this case, the solution x^o, A^o can be obtained according to the rational organization principle; if the resource capabilities are spent completely, the solution x^O, A^O can be obtained that expresses the global consensus of partial criteria.

Both approaches were used when solving specific vector optimization problems of space control systems. Thus, the concept of a non-linear compromise scheme was used during the development of the control law for the glide descent of the Buran spacecraft during the process of creating a regression model of “reliability-cost” expert assessments for the insurance problems of space activity objects, as well as during the development of energy systems for aerospace purposes, etc.

The rational organization principle makes it possible to most efficiently solve the technical and economic problems of the implementation of complex technical and social projects [9] as it makes it possible to harmoniously coordinate the (financial, technical, ergonomic, etc.) resource capabilities of systems in the process of designing and implementing projects under conditions of strict requirements.

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