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ON MINIMAX INTERPOLATION OF STATIONARY SEQUENCES

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Abstract. *The problem of the optimal linear estimation of functionals that depend on the unknown values of the stochastic stationary sequence of observations of a sequence with missing values is considered. Formulas for calculating the root-mean-square error and the spectral characteristic of the optimal linear estimate of the functionals are derived under the spectral determinacy, where the spectral density of the sequence is known exactly. The minimax (robust) method of estimation is applied in the case where the spectral density of the sequence is not known exactly while some classes of feasible spectral densities are given. Formulas that determine the least favourable spectral densities and minimax spectral characteristics are derived for optimal linear estimation of functionals for some special classes of spectral densities.*

Keywords: *stationary sequence, minimax-robust estimate, least favorable spectral density, minimax spectral characteristic.*

INTRODUCTION

Problems of estimating unknown values of random processes are important in the theory of random processes. Kolmogorov [1] formulated interpolation, extrapolation, and filtering problems for stationary processes and reduced them to problems of function theory. Wiener [2] and Yaglom [3] developed efficient methods for finding estimates of unknown values of stationary sequences and processes. Later, these methods were developed in [4, 5]. The classical estimation theory is based on the assumption that spectral densities of sequences and processes are known. However, complete information on spectral densities is mostly absent in practice. In this case, to avoid difficulties, it is necessary to search for parametric or non-parametric estimates of spectral densities or to add density based on other considerations. According to [6], such an approach may significantly increase the estimate error. Therefore, it is expedient to search for estimates that are optimal for all the densities from a certain class of possible spectral densities. Such estimates are called minimax ones because they minimize the maximum value of the error. In [7], such an approach was applied for the first time to the problem of extrapolation of stationary processes. An overview of the results of minimax (robust) data analysis methods can be found in [8]. The latest results in minimax estimation for stationary processes, periodically correlated processes, and processes with stationary increments are described in [9–14].

We will analyze the problem of optimal root-mean-square (RMS) estimation of functionals

$$A_{S_1}\xi = \sum_{j=0}^{N} a(j)\xi(j) + \sum_{j=N+M_2+1}^{\infty} a(j)\xi(j), \quad A_{S_2}\xi = \sum_{j=-\infty}^{-M_1-1} a(j)\xi(j) + A_{S_1}\xi$$

of unknown values of the stationary sequence $\xi(j)$, $j \in \mathbb{Z}$, based on observations of the sequence at points $j \in S_k$, where

$$S_1 = \{\ldots, -2, -1\} \cup \{N+1, \ldots, N+M_2\}, \quad S_2 = \{-M_1, \ldots, -1\} \cup \{N+1, \ldots, N+M_2\}.$$

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The problem is analyzed for the case of spectral determinacy, where the spectral density of the sequence is known, and for the case of spectral uncertainty, where only the set of feasible spectral densities is given.

CLASSICAL INTERPOLATION METHOD

Let $\xi(j),\ j\in\mathbb{Z}$, be a stationary (in a broad sense) stochastic sequence that has the covariance function $r(n)=E\xi(j+n)\overline{\xi(j)}$ that admits spectral decomposition [15]

$$r(n)=\int_{-\pi}^{\pi} e^{in\lambda}F(d\lambda)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\lambda}f(\lambda)d\lambda,$$

where $F(d\lambda)$ is the spectral measure of the sequence, and $f(\lambda)$ is the spectral density of the sequence, which satisfies the minimality condition

$$\int_{-\pi}^{\pi}(f(\lambda))^{-1}d\lambda<\infty. \tag{1}$$

Such a condition is necessary and sufficient to ensure that error-free interpolation of unknown values of the sequence is impossible [4].

Each stationary sequence $\xi(j),\ j\in\mathbb{Z}$, admits spectral decomposition [15]

$$\xi(j)=\int_{-\pi}^{\pi} e^{ij\lambda}Z(d\lambda), \tag{2}$$

where $Z(\Delta)$ is the orthogonal stochastic measure of the sequence that satisfies the relation

$$EZ(\Delta_1)\overline{Z(\Delta_2)}=F(\Delta_1\cap\Delta_2)=\frac{1}{2\pi}\int_{\Delta_1\cap\Delta_2}f(\lambda)d\lambda.$$

Consider the problem of RMS optimal estimation of the functionals

$$A_{S_1}\xi=\sum_{j=0}^{N}a(j)\xi(j)+\sum_{j=N+M_2+1}^{\infty}a(j)\xi(j),\ A_{S_2}\xi=\sum_{j=-\infty}^{-M_1-1}a(j)\xi(j)+A_{S_1}\xi$$

of unknown values of the stationary sequence $\xi(j),\ j\in\mathbb{Z}$, based on observations of the sequence at points $j\in S_k$, where

$$S_1=\{\ldots,-2,-1\}\cup\{N+1,\ldots,N+M_2\},\ S_2=\{-M_1,\ldots,-1\}\cup\{N+1,\ldots,N+M_2\}.$$

From the spectral decomposition (2) of the sequence $\xi(j)$ it follows that the functionals $A_{S_1}\xi$ and $A_{S_2}\xi$ can be represented as

$$A_{S_1}\xi=\int_{-\pi}^{\pi}A_{S_1}(e^{i\lambda})Z(d\lambda),\ A_{S_2}\xi=\int_{-\pi}^{\pi}A_{S_2}(e^{i\lambda})Z(d\lambda), \tag{3}$$

where

$$A_{S_1}(e^{i\lambda})=\sum_{j=0}^{N}a(j)e^{ij\lambda}+\sum_{j=N+M_2+1}^{\infty}a(j)e^{ij\lambda},\ A_{S_2}(e^{i\lambda})=\sum_{j=-\infty}^{-M_1-1}a(j)e^{ij\lambda}+A_{S_1}(e^{i\lambda}).$$

We will consider $\xi(j)$ as elements of the Hilbert space $H=L_2(\Omega,\mathcal{F},P)$ of random variables γ with zero mean value $\mathrm{E}\gamma$, finite variance $\mathrm{E}|\gamma|^2<\infty$, and scalar product $(\gamma_1,\gamma_2)=\mathrm{E}\gamma_1\overline{\gamma_2}$. Denote by $H^{S_k}(\xi)$ a closed linear subspace of space $H=L_2(\Omega,\mathcal{F},P)$, generated by the quantities $\{\xi(j):j\in S_k\}$ and by $L_2(f)$ a Hilbert space of functions $a(\lambda)$ such that

$\int_{-\pi}^{\pi} |a(\lambda)|^2 f(\lambda) d\lambda < \infty$. Also, denote by $L_2^{S_k}(f)$ a subspace in $L_2(f)$, generated by the functions $\{e^{in\lambda}, n \in S_k\}$. The RMS optimal estimate $\hat{A}_{S_k}\xi$ of the functional $A_{S_k}\xi$ of the observations of sequence $\xi(j)$ at points $j \in S_k$ belongs to the space $H^{S_k}(\xi)$ and can be presented as

$$\hat{A}_{S_k}\xi = \int_{-\pi}^{\pi} h_k(e^{i\lambda}) Z(d\lambda), \tag{4}$$

where $h_k(e^{i\lambda}) \in L_2^{S_k}(f)$ is the spectral characteristic of the estimate $\hat{A}_{S_k}\xi$. The RMS error $\Delta(h_k; f) = E|A_{S_k}\xi - \hat{A}_{S_k}\xi|^2$ of the estimate $\hat{A}_{S_k}\xi$ can be calculated by the formula

$$\Delta(h_k; f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |A_{S_k}(e^{i\lambda}) - h_k(e^{i\lambda})|^2 f(\lambda) d\lambda.$$

The classical method of orthogonal projection in Hilbert space, proposed by Kolmogorov [1], makes it possible to find the spectral characteristic $h_k(e^{i\lambda})$ and the RMS error $\Delta(h_k; f)$ of the optimal linear estimate of the functional $A_{S_k}\xi$ in the case where the density $f(\lambda)$ is known and condition (1) is satisfied. The spectral characteristic of the estimate are determined under the following conditions:

(i) $h_k(e^{i\lambda}) \in L_2^{S_k}(f)$;

(ii) $A_{S_k}(e^{i\lambda}) - h_k(e^{i\lambda}) \perp L_2^{S_k}(f)$.

From condition (ii) it follows that for all $\eta \in H^{S_k}(\xi)$ the following equality should hold:

$$(A_{S_k}\xi - \hat{A}_{S_k}\xi, \eta) = E[(A_{S_k}\xi - \hat{A}_{S_k}\xi)\overline{\eta}] = 0,$$

which is equivalent to the equalities $E[(A_{S_k}\xi - \hat{A}_{S_k}\xi)\overline{\xi(j)}] = 0$, $j \in S_k$. We obtain the following relations from (3) and (4) and the definition of scalar product in $H = L_2(\Omega, \mathcal{F}, P)$:

$$\int_{-\pi}^{\pi} (A_{S_k}(e^{i\lambda}) - h_k(e^{i\lambda})) f(\lambda) e^{-ij\lambda} d\lambda = 0, \ j \in S_k.$$

It follows herefrom that functions $(A_{S_k}(e^{i\lambda}) - h_k(e^{i\lambda})) f(\lambda)$ can be presented as

$$(A_{S_k}(e^{i\lambda}) - h_k(e^{i\lambda})) f(\lambda) = C_{S_k}(e^{i\lambda}),$$

where

$$C_{S_1}(e^{i\lambda}) = \sum_{j=0}^{N} c(j) e^{ij\lambda} + \sum_{j=N+M_2+1}^{\infty} c(j) e^{ij\lambda}, \ C_{S_2}(e^{i\lambda}) = \sum_{j=-\infty}^{-M_1-1} c(j) e^{ij\lambda} + C_{S_1}(e^{i\lambda}),$$

and $c(j)$ are unknown coefficients to be determined. From the last relation it follows that the spectral characteristic $h_k(e^{i\lambda})$ of the optimal linear estimate of the functional $A_{S_k}\xi$ has the form

$$h_k(e^{i\lambda}) = A_{S_k}(e^{i\lambda}) - C_{S_k}(e^{i\lambda})(f(\lambda))^{-1}. \tag{5}$$

To find the equation for the unknown coefficients $c(j)$, we will use the Fourier decomposition of the function $(f(\lambda))^{-1}$

$$\frac{1}{f(\lambda)} = \sum_{m=-\infty}^{\infty} b(m) e^{im\lambda}, \tag{6}$$

where $b(m)$ are the Fourier coefficients of the function $(f(\lambda))^{-1}$. Substituting (6) into (5), we obtain the following formulas for calculating the spectral characteristics:

$$h_1(e^{i\lambda}) = \sum_{j=N+M_2+1}^{\infty} a(j)e^{ij\lambda} + \sum_{j=0}^{N} a(j)e^{ij\lambda}$$

$$-\left(\sum_{m=-\infty}^{\infty} b(m)e^{im\lambda}\right)\left(\sum_{j=0}^{N} c(j)e^{ij\lambda} + \sum_{j=N+M_2+1}^{\infty} c(j)e^{ij\lambda}\right), \quad (7)$$

$$h_2(e^{i\lambda}) = \sum_{j=-\infty}^{-M_1-1} a(j)e^{ij\lambda} + \sum_{j=0}^{N} a(j)e^{ij\lambda} + \sum_{j=N+M_2+1}^{\infty} a(j)e^{ij\lambda}$$

$$-\left(\sum_{m=-\infty}^{\infty} b(m)e^{im\lambda}\right)\left(\sum_{j=-\infty}^{-M_1-1} c(j)e^{ij\lambda} + \sum_{j=0}^{N} c(j)e^{ij\lambda} + \sum_{j=N+M_2+1}^{\infty} c(j)e^{ij\lambda}\right). \quad (8)$$

From the condition $h_k(e^{i\lambda}) \in L_2^{S_k}(f)$ it follows that the Fourier coefficients of the function $h_k(e^{i\lambda})$ are equal to zero for $j \in Z \setminus S_k$, i.e.,

$$\int_{-\pi}^{\pi} h_k(e^{i\lambda})e^{-ij\lambda}d\lambda = 0, \; j \in Z \setminus S_k.$$

Using relations (7) and (8), we obtain systems for finding the unknown coefficients $c(j), \; j \in Z \setminus S_1$,

$$a(k) = \sum_{j \in Z\setminus S_1} c(j)\, b(-j+k), \; k = \overline{0, N},$$

$$a(N+M_2+k) = \sum_{j \in Z\setminus S_1} c(j) b(N+M_2+k-j), \quad k \geq 1, \quad (9)$$

and coefficients $c(j), \; j \in Z \setminus S_2$,

$$a(-M_1-k) = \sum_{j \in Z\setminus S_2} c(j) b(-M_1-k-j), \; k \geq 1,$$

$$a(k) = \sum_{j \in Z\setminus S_2} c(j) b(-j+k), \; k = \overline{0, N},$$

$$a(N+M_2+k) = \sum_{j \in Z\setminus S_2} c(j) b(N+M_2+k-j), \; k \geq 1. \quad (10)$$

Denote by $\vec{a}_1^{\,\mathrm{T}} = (\vec{a}_N^{\,\mathrm{T}}, \vec{a}_{N+M_2+1}^{\,\mathrm{T}})$, $\vec{a}_{-M_1-1}^{\,\mathrm{T}}$ vectors formed from the coefficients $a(j)$

$$\vec{a}_N^{\,\mathrm{T}} = (a(0), a(1), \ldots, a(N)), \; \vec{a}_{-M_1-1}^{\,\mathrm{T}} = (a(-M_1-1), a(-M_1-2), \ldots),$$

$$a_{N+M_2+1}^{\mathrm{T}} = (a(N+M_2+1), \; a(N+M_2+2), \ldots),$$

and by B_{S_1} matrix $B_{s_1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, where B_k are matrices formed by the Fourier coefficients of the function $(f(\lambda))^{-1}$

$$B_1(i,j) = b(i-j), \; 0 \leq i, j \leq N, \; B_4(i,j) = b(i-j), \; 0 \leq i, j < \infty,$$

$$B_2(i,j) = b(-N-M_2-1+i-j), \; 0 \leq i \leq N, \; 0 \leq j < \infty,$$

$$B_3(i,j) = b(N+M_2+1+i-j), \; 0 \leq j \leq N, \; 0 \leq i < \infty.$$

Then the system of equations (9) can be presented as

$$\vec{a}_1 = B_{S_1}\vec{c}_1, \tag{11}$$

and the system of equations (10) as

$$\vec{a}_N = B_1\vec{c}_N + B_5\vec{c}_{-M_1-1} + B_2\vec{c}_{N+M_2+1},$$

$$\vec{a}_{-M_1-1} = B_6\vec{c}_N + B_7\vec{c}_{-M_1-1} + B_8\vec{c}_{N+M_2+1},$$

$$\vec{a}_{N+M_2+1} = B_3\vec{c}_N + B_9\vec{c}_{-M_1-1} + B_4\vec{c}_{N+M_2+1}, \tag{12}$$

where $\vec{c}_1^{\,\mathrm{T}} = (\vec{c}_N^{\,\mathrm{T}}, \vec{c}_{N+M_2+1}^{\,\mathrm{T}})$, $\vec{c}_{-M_1-1}^{\,\mathrm{T}}$ are vectors formed from the unknown coefficients $c(j)$

$$\vec{c}_N^{\,\mathrm{T}} = (c(0), c(1), \dots, c(N)),\ \vec{c}_{-M_1-1}^{\,\mathrm{T}} = (c(-M_1-1),\ c(-M_1-2), \dots),$$

$$\vec{c}_{N+M_2+1}^{\,\mathrm{T}} = (c(N+M_2+1),\ c(N+M_2+2), \dots),$$

and B_k are matrices formed by the Fourier coefficients of the function $(f(\lambda))^{-1}$

$$B_5(i,j) = b(M_1+1+i+j),\ 0 \le i \le N,\ 0 \le j < \infty,$$

$$B_6(i,j) = b(-M_1-1-i-j),\ 0 \le j \le N,\ 0 \le i < \infty,\ B_7(i,j) = b(j-i),\ 0 \le i, j < \infty,$$

$$B_8(i,j) = b(-N-M_2-1-j-M_1-1-i),\ 0 \le i, j < \infty,$$

$$B_9(i,j) = b(N+M_2+1+i+M_1+1+j),\ 0 \le i, j < \infty.$$

Since the matrix B_{S_1} has the inverse one [16], we obtain from (11) that $\vec{c}_1 = B_{S_1}^{-1}\vec{a}_1$. Therefore, the unknown coefficients $c(j)$, $j \in Z \setminus S_1$, can be calculated by the formula

$$c(j) = (B_{S_1}^{-1}\vec{a}_1)(j),\ j \in Z \setminus S_1,$$

where $(B_{S_1}^{-1}\vec{a}_1)(j)$ is the jth element of the vector $B_{S_1}^{-1}\vec{a}_1$. Thus, the spectral characteristic of the estimate $\hat{A}_{S_1}\xi$ can be calculated by the formula

$$h_1(e^{i\lambda}) = \sum_{j=N+M_2+1}^{\infty} a(j)e^{ij\lambda} + \sum_{j=0}^{N} a(j)e^{ij\lambda}$$

$$-\left(\sum_{m=-\infty}^{\infty} b(m)e^{im\lambda}\right)\left(\sum_{j=0}^{N}(B_{S_1}^{-1}\vec{a}_1)(j)e^{ij\lambda} + \sum_{j=N+M_2+1}^{\infty}(B_{S_1}^{-1}\vec{a}_1)(j)e^{ij\lambda}\right), \tag{13}$$

and the RMS errors of estimates of the functionals $\hat{A}_{S_1}\xi$ and $\hat{A}_{S_2}\xi$ can be calculated by the formulas

$$\Delta(h_1; f) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{j=0}^{N} c(j)e^{ij\lambda} + \sum_{j=N+M_2+1}^{\infty} c(j)e^{ij\lambda}\right)\left(\sum_{j=0}^{N}\overline{c(j)}e^{-ij\lambda} + \sum_{j=N+M_2+1}^{\infty}\overline{c(j)}\,e^{-ij\lambda}\right)$$

$$\times\left(\sum_{m=-\infty}^{\infty} b(m)e^{im\lambda}\right)d\lambda = \langle \vec{c}_1, B_{S_1}\vec{c}_1\rangle = \langle B_{S_1}^{-1}\vec{a}_1, \vec{a}_1\rangle, \tag{14}$$

$$\Delta(h_2; f) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{j=-\infty}^{-M_1-1} c(j)e^{ij\lambda} + \sum_{j=0}^{N} c(j)e^{ij\lambda} + \sum_{j=N+M_2+1}^{\infty} c(j)e^{ij\lambda}\right)$$

$$\times\left(\sum_{j=-\infty}^{-M_1-1}\overline{c(j)}e^{-ij\lambda}+\sum_{j=0}^{N}\overline{c(j)}e^{-ij\lambda}+\sum_{j=N+M_2+1}^{\infty}\overline{c(j)}e^{-ij\lambda}\right)\left(\sum_{m=-\infty}^{\infty}b(m)e^{im\lambda}\right)d\lambda$$

$$=\langle\vec{c}_N,\vec{a}_N\rangle+\langle\vec{c}_{-M_1-1},\vec{a}_{-M_1-1}\rangle+\langle\vec{c}_{N+M_2+1},\vec{a}_{N+M_2+1}\rangle, \tag{15}$$

where $\langle\cdot,\cdot\rangle$ denotes scalar product in the space l_2.

Let us formulate the obtained result as a theorem.

THEOREM 1. Let $\xi(j)$ be a stationary stochastic sequence that has spectral density $f(\lambda)$, which satisfies the minimality condition (1). The RMS errors $\Delta(h_k,f)$ and spectral characteristics $h_k(e^{i\lambda})$ of the optimal linear estimates $\hat{A}_{S_k}\xi$ of the functionals $A_{S_k}\xi$ can be calculated by formulas (8), (13)–(15).

Consider the problem of RMS optimal estimation of the functionals

$$A_{S_3}\xi=\sum_{j=0}^{N}a(j)\xi(j)+\sum_{j=N+M_2+1}^{N+M_2+N_2}a(j)\xi(j),$$

$$A_{S_4}\xi=\sum_{j=-M_1-N_1}^{-M_1-1}a(j)\xi(j)+\sum_{j=0}^{N}a(j)\xi(j)+\sum_{j=N+M_2+1}^{N+M_2+N_2}a(j)\xi(j),$$

of unknown values of the stationary sequence $\xi(j)$, $j\in\mathbb{Z}$, based on the observations of the sequence at points $j\in S_k$, where

$$S_3=Z\setminus(\{0,\dots,N\}\cup\{N+M_2+1,\dots,N+M_2+N_1\}),$$

$$S_4=Z\setminus(\{-M_1-N_1,\dots,-M_1-1\}\cup\{0,\dots,N\}\cup\{N+M_2+1,\dots,N+M_2+N_1\}).$$

Denote by $\vec{a}_3^{\,\mathrm{T}}=(\vec{a}_N^{\,\mathrm{T}},\vec{a}_{N+M_2+1}^{\,N_2\mathrm{T}})$, $\vec{a}_4^{\,\mathrm{T}}=(\vec{a}_N^{\,\mathrm{T}},\vec{a}_{-M_1-1}^{\,N_1\mathrm{T}},\vec{a}_{N+M_2+1}^{\,N_2\mathrm{T}})$ vectors generated from coefficients $a(j)$:

$$\vec{a}_N^{\,\mathrm{T}}=(a(0),a(1),\dots,a(N)),\ \vec{a}_{-M_1-1}^{\,N_1\mathrm{T}}=(a(-M_1-1),a(-M_1-2),\dots,a(-M_1-N_1)),$$

$$\vec{a}_{N+M_2+1}^{\,N_2\mathrm{T}}=(a(N+M_2+1),a(N+M_2+2),\dots,a(N+M_2+N_2)).$$

Let $B_{s_3}=\begin{pmatrix}B_1 & B_5\\ B_6 & B_7\end{pmatrix}$ and $B_{s_4}=\begin{pmatrix}B_1 & B_2 & B_5\\ B_3 & B_4 & B_8\\ B_6 & B_9 & B_7\end{pmatrix}$ be matrices generated by the Fourier coefficients of the function $(f(\lambda))^{-1}$:

$$B_1(i,j)=b(i-j),\ 0\le i,j\le N,\ B_4(i,j)=b(j-i),\ 0\le i,j<N_1,$$

$$B_2(i,j)=b(M_1+1+i+j),\ 0\le i\le N,\ 0\le j<N_1,$$

$$B_3(i,j)=b(-M_1-1-i-j),\ 0\le j\le N,\ 0\le i<N_1,$$

$$B_5(i,j)=b(-N-M_2-1+i-j),\ 0\le i\le N,\ 0\le j<N_2,$$

$$B_6(i,j)=b(N+M_2+1+i-j),\ 0\le j\le N,\ 0\le i<N_2,$$

$$B_7(i,j)=b(i-j),0\le i,j<N_2,$$

$$B_8(i,j)=b(-N-M_2-1-j-M_1-1-i),\ 0\le i<N_1,\ 0\le j<N_2,$$

$$B_9(i,j)=b(N+M_2+1+i+M_1+1+j),\ 0\le i<N_1,\ 0\le j<N_2.$$

COROLLARY 1. Let $\xi(j)$ be a stationary stochastic sequence that has spectral density $f(\lambda)$, which satisfies the minimality condition (1). The RMS errors $\Delta(h_k, f)$ and spectral characteristics $h_k(e^{i\lambda})$ of the optimal linear estimates $\hat{A}_{S_k}\xi$ of the functionals $A_{S_k}\xi$ based on the observations of the sequence $\xi(j)$ for $j \in S_k$, where $k = 3, 4$, can be calculated as follows:

$$h_3(e^{i\lambda}) = \sum_{j=0}^{N} a(j)e^{ij\lambda} + \sum_{j=N+M_2+1}^{N+M_2+N_2} a(j)e^{ij\lambda}$$

$$-\left(\sum_{m=-\infty}^{\infty} b(m)e^{im\lambda}\right)\left(\sum_{j=0}^{N}(B_{S_3}^{-1}\vec{a}_3)(j)e^{ij\lambda} + \sum_{j=N+M_2+1}^{N+M_2+N_2}(B_{S_3}^{-1}\vec{a}_3)(j)e^{ij\lambda}\right), \tag{16}$$

$$\Delta(h_3; f) = \langle B_{S_3}^{-1}\vec{a}_3, \vec{a}_3\rangle, \tag{17}$$

$$h_4(e^{i\lambda}) = \sum_{j=-M_1-N_1}^{-M_1-1} a(j)e^{ij\lambda} + \sum_{j=0}^{N} a(j)e^{ij\lambda} + \sum_{j=N+M_2+1}^{N+M_2+N_2} a(j)e^{ij\lambda} - \left(\sum_{m=-\infty}^{\infty} b(m)e^{im\lambda}\right)$$

$$\times\left(\sum_{j=0}^{N}(B_{S_4}^{-1}\vec{a}_4)(j)e^{ij\lambda} + \sum_{j=-M_1-N-1}^{-M_1-1}(B_{S_4}^{-1}\vec{a}_4)(j)e^{ij\lambda} + \sum_{j=N+M_2+1}^{N+M_2+N_2}(B_{S_4}^{-1}\vec{a}_4)(j)e^{ij\lambda}\right), \tag{18}$$

$$\Delta(h_4; f) = \langle B_{S_4}^{-1}\vec{a}_4, \vec{a}_4\rangle. \tag{19}$$

THE MINIMAX (ROBUST) INTERPOLATION METHOD

The classical method of interpolation is used when the spectral density $f(\lambda)$ of the sequence is known exactly. However, as we have already mentioned, complete information on the spectral density is mostly absent in practice. However, if the density $f(\lambda)$ is known to belong to a certain class of spectral densities D, it is expedient to apply the minimax approach. Then, instead of finding an estimate that would be optimal for a certain spectral density, an estimate that would minimize the error at the same time for all the spectral densities in this class D is sought for.

Definition 1. For the given class of spectral densities D, spectral density $f_k^0(\lambda) \in D$ is called the least favorable in D for the optimal linear interpolation of the functional $A_{S_k}\xi$ if

$$\Delta(f_k^0) = \Delta(h_k(f_k^0); f_k^0) = \max_{f \in D} \Delta(h_k(f); f).$$

Definition 2. For the given class of spectral densities D, spectral characteristic $h_k^0(e^{i\lambda})$ of the optimal estimate of functional $A_{S_k}\xi$ is called minimax (robust) if

$$h_k^0(e^{i\lambda}) \in H_D^k = \bigcap_{f \in D} L_2^{S_k}(f), \quad \min_{h \in H_D^k} \max_{f \in D} \Delta(h; f) = \max_{f \in D} \Delta(h_k^0; f).$$

LEMMA 1. Spectral density $f_k^0(\lambda) \in D$ is least favorable in class D for the optimal linear interpolation of the functional $A_{S_k}\xi$, $k = 1, 3, 4$, if the Fourier coefficients of the function $(f_k^0(\lambda))^{-1}$ specify the matrix $B_{S_k}^0$, which determines the solution of the extremum problem

$$\max_{f \in D} \langle B_{S_k}^{-1}\vec{a}_k, \vec{a}_k\rangle = \langle B_{S_k}^{0-1}\vec{a}_k, \vec{a}_k\rangle. \tag{20}$$

The minimax spectral characteristic $h_k^0 = h_k(f_k^0)$ can be calculated by formulas (13), (16), and (18) provided that $h_k(f_k^0) \in H_D^k$.

LEMMA 2. Spectral density $f_2^0(\lambda)\in D$ is least favorable in the class D for the optimal linear interpolation of the functional $A_{S_2}\xi$ if the Fourier coefficients of the function $(f_2^0(\lambda))^{-1}$ specify the matrices B_k^0 that determine the solution of the extremum problem

$$\max_{f\in D}(\langle \vec{c}_N,\vec{a}_N\rangle+\langle \vec{c}_{-M_1-1},\vec{a}_{-M_1-1}\rangle+\langle \vec{c}_{N+M_2+1},\vec{a}_{N+M_2+1}\rangle)$$
$$=\langle \vec{c}_N^{\,0},\vec{a}_N\rangle+\langle \vec{c}_{-M_1-1}^{\,0},\vec{a}_{-M_1-1}\rangle+\langle \vec{c}_{N+M_2+1}^{\,0},\vec{a}_{N+M_2+1}\rangle . \tag{21}$$

The minimax spectral characteristic $h_2^0=h_2(f_2^0)$ can be calculated by formula (8) provided that $h_2(f_2^0)\in H_D^2$.

The least favorable spectral density f_k^0 and the minimax spectral characteristic h_k^0 generate the saddle point of the function $\Delta(h;f)$ on the set $H_D^k\times D$. The inequalities of the saddle point

$$\Delta(h;f_k^0)\ge\Delta(h_k^0;f_k^0)\ge\Delta(h_k^0;f)\ \ \forall f\in D,\ \forall h\in H_D^k$$

hold if $h_k^0=h_k(f_k^0)$ and $h_k(f_k^0)\in H_D^k$, where f_k^0 is the solution of the conditional extremum problem

$$\widetilde{\Delta}_k(f)=-\Delta(h_k^0;f)=-\frac{1}{2\pi}\int_{-\pi}^{\pi}|C_k^0(e^{i\lambda})|^2\frac{f(\lambda)}{(f_k^0(\lambda))^2}d\lambda\to\inf,\ f(\lambda)\in D, \tag{22}$$

where

$$C_1^0(e^{i\lambda})=\sum_{j=0}^{N}(B_{S_1}^{0-1}\vec{a}_1)(j)e^{ij\lambda}+\sum_{j=N+M_2+1}^{\infty}(B_{S_1}^{0-1}\vec{a}_1)(j)e^{ij\lambda},$$
$$C_2^0(e^{i\lambda})=\sum_{j=-\infty}^{M_1-1}c^0(j)e^{ij\lambda}+\sum_{j=0}^{N}c^0(j)e^{ij\lambda}+\sum_{j=N+M_2+1}^{\infty}c^0(j)e^{ij\lambda},$$
$$C_3^0(e^{i\lambda})=\sum_{j=0}^{N}(B_{S_3}^{0-1}\vec{a}_3)(j)e^{ij\lambda}+\sum_{j=N+M_2+1}^{N+M_2+N_2}(B_{S_3}^{0-1}\vec{a}_3)(j)e^{ij\lambda},$$
$$C_4^0(e^{i\lambda})=\sum_{j=0}^{N}(B_{S_4}^{0-1}\vec{a}_4)(j)e^{ij\lambda}+\sum_{j=-M_1-N-1}^{-M_1-1}(B_{S_4}^{0-1}\vec{a}_4)(j)e^{ij\lambda}+\sum_{j=N+M_2+1}^{N+M_2+N_2}(B_{S_4}^{0-1}\vec{a}_4)(j)e^{ij\lambda}.$$

The conditional extremum problem (22) is equivalent to the unconditional extremum problem

$$\Delta_D^k(f)=\widetilde{\Delta}_k(f)+\delta(f\,|\,D)\to\inf,$$

where $\delta(f\,|\,D)$ is an indicator function of the set D. The solution f_k^0 of this problem is characterized by the condition $0\in\partial\Delta_D^k(f_k^0)$, where $\partial\Delta_D^k(f_k^0)$ is the subdifferential of the convex functional $\Delta_D^k(f)$. This condition makes it possible to find the least favorable spectral densities for specific classes D [17, 18]. Note that the form of the functional $\Delta(h_k^0;f)$ is convenient for applying the method of indeterminate Lagrange factors to finding the solution of problem (22). Using the method of Lagrange multipliers and the form of the subdifferential of the indicator function, we can describe the relations that determine the least favorable spectral densities in some classes of spectral densities.

LEAST FAVORABLE SPECTRAL DENSITIES IN THE CLASS D_0^-

Consider the problem of minimum estimation of functionals $A_{S_1}\xi$ and $A_{S_2}\xi$ of unknown values of the stationary sequence $\xi(j)$, which has the spectral density from the class

$$D_0^-=\left\{f(\lambda)\,\middle|\,\frac{1}{2\pi}\int_{-\pi}^{\pi}f^{-1}(\lambda)d\lambda\ge p\right\},$$

where p is a given number, and the sequence $a(k)$, $k \in Z \setminus S_k$, which defines the functional $A_{S_k}\xi$, is strictly positive [19]. To find the solution of problem (22), we will use the method of indeterminate Lagrange factors. We obtain the following equations:

$$|C_k^0(e^{i\lambda})|^2((f_k^0(\lambda))^2)^{-1} = \alpha_k^2((f_k^0(\lambda))^2)^{-1},$$

where α_k^2 are unknown Lagrange multipliers. Hence, we obtain that the Fourier coefficients of the functions $(f_k^0(\lambda))^{-1}$ satisfy the equations

$$\left|\sum_{j=0}^{N} c(j)e^{ij\lambda} + \sum_{j=N+M_2+1}^{\infty} c(j)e^{ij\lambda}\right|^2 = \alpha_1^2, \tag{23}$$

$$\left|\sum_{j=-\infty}^{M_1-1} c(j)e^{ij\lambda} + \sum_{j=0}^{N} c(j)e^{ij\lambda} + \sum_{j=N+M_2+1}^{\infty} c(j)e^{ij\lambda}\right|^2 = \alpha_2^2, \tag{24}$$

where $c(j)$, $j \in Z \setminus S_1$, are coordinates of the vector $\vec{c}_1$ that satisfies the equation $B_{S_1}^0 \vec{c}_1 = \vec{a}_1$, matrix $B_{S_1}^0$ is generated from the Fourier coefficients of the function $(f_1^0(\lambda))^{-1} = \sum_{k=-\infty}^{\infty} b_1^0(k)e^{ij\lambda}$, the values $c(j)$, $j \in Z \setminus S_2$, are the coordinates of vectors $\vec{c}_N$, $\vec{c}_{-M_1-1}$, $\vec{c}_{N+M_2+1}$, which satisfy the equations

$$\vec{a}_N = B_1^0 \vec{c}_N + B_2^0 \vec{c}_{-M_1-1} + B_5^0 \vec{c}_{N+M_2+1},$$

$$\vec{a}_{-M_1-1} = B_3^0 \vec{c}_N + B_4^0 \vec{c}_{-M_1-1} + B_8^0 \vec{c}_{N+M_2+1},$$

$$\vec{a}_{N+M_2+1} = B_6^0 \vec{c}_N + B_9^0 \vec{c}_{-M_1-1} + B_7^0 \vec{c}_{N+M_2+1},$$

and matrices B_k^0 are generated from the Fourier coefficients of the function $(f_2^0(\lambda))^{-1} = \sum_{k=-\infty}^{\infty} b_2^0(k)e^{ij\lambda}$.

Equation (23) and equation $B_{S_1}^0 \vec{c}_1 = \vec{a}_1$ are satisfied by the Fourier coefficients $b_1^0(n) = b_1^0(-n)$, $n \in Z \setminus S_1$, which can be found from the equation $B_{S_1}^0 \vec{\alpha}_1 = \vec{a}_1$, where $\vec{\alpha}_1 = (\alpha_1, 0, 0, \dots)$. The last equation can be presented as a system of equations $\alpha_1 b_1^0(n) = a(n)$, $n \in Z \setminus S_1$. From the first equation of the system ($n = 0$), we get the unknown $\alpha_1 = a(0)(b_1^0(0))^{-1}$. Due to the extremum condition (20) and the constraint for the spectral density in the class D_0^-, we obtain

$$b_1^0(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f_1^0(\lambda))^{-1}d\lambda = p.$$

Thus,

$$b_1^0(n) = b_1^0(-n) = \begin{cases} p\,a(n)(a(0))^{-1}, & n \in Z \setminus S_1; \\ 0, & n \in S_1. \end{cases}$$

Function $(f_1^0(\lambda))^{-1}$ can be represented as

$$(f_1^0(\lambda))^{-1} = \sum_{n=-\infty}^{-N-M_2-1} b_1^0(n)e^{in\lambda} + \sum_{n=-N}^{N} b_1^0(n)e^{in\lambda} + \sum_{j=N+M_2+1}^{\infty} b_1^0(n)e^{in\lambda}. \tag{25}$$

Based on the result [19], we can represent the function $(f_1^0(\lambda))^{-1}$ as follows:

$$(f_1^0(\lambda))^{-1} = \left|\sum_{j=0}^{\infty} \gamma_{1n} e^{in\lambda}\right|^2, \; \lambda \in [\pi, -\pi], \tag{26}$$

where $\gamma_{1n}=0$, $n\in\{N+1,\dots,N+M_2\}$. The minimax spectral characteristic $h_1(f_1^0)$ of the optimal linear estimate of the functional $A_{S_1}\xi$ can be calculated by formula (7), where

$$\sum_{j=0}^{N}c(j)e^{ij\lambda}+\sum_{j=N+M_2+1}^{\infty}c(j)\,e^{ij\lambda}=\alpha_1=a(0)p^{-1},$$

i.e.,

$$h_1(f_2^0)=\sum_{j=0}^{N}a(j)e^{ij\lambda}+\sum_{j=N+M_2+1}^{\infty}a(j)\,e^{ij\lambda}-\left(\sum_{k=-\infty}^{\infty}b_1^0(k)\,e^{ij\lambda}\right)a(0)p^{-1}=-\sum_{j=0}^{N}a(j)e^{-ij\lambda}-\sum_{j=N+M_2+1}^{\infty}a(j)\,e^{-ij\lambda}. \quad (27)$$

Let us formulate the obtained result as a theorem.

THEOREM 2. The spectral density (25) with the Fourier coefficients

$$b_1^0(n)=b_1^0(-n)=pa(n)(a(0))^{-1},\ n\in Z\setminus S_1,$$

is the least favorable spectral density in the class D_0^- for the optimal linear interpolation of the functional $A_{S_1}\xi$, which is defined by a strictly positive sequence $a(k)$, $k\in Z\setminus S_1$. The minimax spectral characteristic $h_1(f_1^0)$ can be calculated by formula (27). The least favorable spectral density in the class D_0^- for the optimal linear interpolation of the functional $A_{S_2}\xi$, which is defined by a strictly positive sequence $a(k)$, $k\in Z\setminus S_2$, satisfies relation (24) and defines the solution of the extremum problem (21). The minimax spectral characteristic $h_2(f_2^0)$ can be calculated by formula (8).

Consider the problem of minimum estimation of functionals $A_{S_k}\xi$, where $k=3,4$. To obtain the solution of problem (22), we will use the method of indeterminate Lagrange factors. We obtain the following equations:

$$\left|\sum_{j=0}^{N}c(j)e^{ij\lambda}+\sum_{j=N+M_2+1}^{N+M_2+N_2}c(j)\,e^{ij\lambda}\right|^2=\alpha_3^2, \quad (28)$$

$$\left|\sum_{j=-M_1-N_1}^{M_1-1}c(j)e^{ij\lambda}+\sum_{j=0}^{N}c(j)e^{ij\lambda}+\sum_{j=N+M_2+1}^{N+M_2+N_2}c(j)\,e^{ij\lambda}\right|^2=\alpha_4^2, \quad (29)$$

where α_k^2 are unknown Lagrange multipliers, $c(j)$, $j\in Z\setminus S_k$, are coordinates of vectors $\vec{c}_k$:

$$\vec{c}_3^{\,\mathrm{T}}=(\vec{c}_N^{\,\mathrm{T}}\vec{c}_{N+M_2+1}^{\,N_2\mathrm{T}}),\ \ \vec{c}_4^{\,\mathrm{T}}=(\vec{c}_N^{\,\mathrm{T}},\vec{c}_{-M_1-1}^{\,N_1\mathrm{T}},\vec{c}_{N+M_2+1}^{\,N_2\mathrm{T}}),$$

$$\vec{c}_N^{\,\mathrm{T}}=(c(0),c(1),\dots,c(N)),\vec{c}_{-M_1-1}^{\,N_1\mathrm{T}}=(c(-M_1-1),c(-M_1-2),\dots,c(-M_1-N_1)),$$

$$\vec{c}_{N+M_2+1}^{\,N_2\mathrm{T}}=(c(N+M_2+1),c(N+M_2+2),\dots,c(N+M_2+N_2)),$$

that satisfy the equation $B_{S_k}^0\vec{c}_k=\vec{a}_k$, and matrices $B_{S_k}^0$ are generated from the Fourier coefficients of the function $(f_k^0(\lambda))^{-1}$. Equation (23) and the equation $B_{S_k}^0\vec{c}_k=\vec{a}_k$ satisfy the Fourier coefficients

$$b_k^0(n)=b_k^0(-n),\ n\in Z\setminus S_k,\ k=3,4,$$

which can be found from the equation $B_{S_k}^0\vec{\alpha}_k=\vec{a}_k$, where $\vec{\alpha}_3=(\alpha_3,0,0,\dots,0)$, $\vec{\alpha}_4=(0,0,\dots,\alpha_4)$. The last equality can be represented as a system of equations

$$\alpha_3 b_3^0(n)=a(n),\ n\in Z\setminus S_3,\ \alpha_4 b_4^0(n-N-M_2-N_2)=a(n),n\in Z\setminus S_4.$$

Hence, for $n=0$ and $n=N+M_2+N_2$ we find the unknowns $\alpha_3=a(0)(b_3^0(0))^{-1}$ and $\alpha_4=a(N+M_2+N_2)(b_4^0(0))^{-1}$, respectively.

Due to the extremum condition (20) and the constraint for the spectral densities in the class D_0^-, we obtain

$$b_k^0(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(f_k^0(\lambda))^{-1}d\lambda=p.$$

Thus,

$$b_3^0(n)=b_3^0(-n)=\begin{cases} pa(n)(a(0))^{-1}, & n\in Z\setminus S_3;\\ 0, & n\in S_3,\end{cases}$$

$$b_4^0(n-N-M_2-N_2)=b_4^0(N+M_2+N_2-n)=\begin{cases} pa(n)(a(N+M_2+N_2))^{-1}, & n\in Z\setminus S_4;\\ 0, & n\in S_4.\end{cases}$$

Functions $(f_k^0(\lambda))^{-1}$, $k=3,4$, can be represented as

$$(f_3^0(\lambda))^{-1}=\sum_{k=N-M_2-N_2}^{-N-M_2-1}b_3^0(k)e^{ij\lambda}+\sum_{k=-N}^{N}b_3^0(k)e^{ij\lambda}+\sum_{k=N+M_2+1}^{N+M_2+N_2}b_3^0(k)e^{ij\lambda}, \tag{30}$$

$$(f_4^0(\lambda))^{-1}=\sum_{k=-N-M_2-N_2-M_1-N_1}^{-N-M_2-N_2-M_1-1}b_4^0(k)e^{ik\lambda}+\sum_{k=-N-M_2-N_2}^{-M_2-N_2}b_4^0(k)e^{ik\lambda}$$

$$+\sum_{k=-(N_2-1)}^{N_2-1}b_4^0(k)e^{ik\lambda}+\sum_{k=M_2+N_2}^{N+M_2+N_2}b_4^0(k)e^{ik\lambda}\sum_{k=N+M_2+N_2+M_1+1}^{N+M_2+N_2+M_1+N_1}b_4^0(k)e^{ik\lambda}. \tag{31}$$

Based on the result [19], the functions $(f_k^0(\lambda))^{-1}$, $k=3,4$, can be represented as follows:

$$(f_k^0(\lambda))^{-1}=\left|\sum_{n=0}^{\infty}\gamma_{kn}e^{-in\lambda}\right|^2,\ \lambda\in[\pi,-\pi], \tag{32}$$

where

$$\gamma_{3n}=0: N+1\le n\le N+M_2,\, n>N+M_2+N_2,\ \gamma_{4n}=0: N_2\le n<M_2+N_2,$$

$$N+M_2+N_2<n\le N+M_2+N_2+M_1,\ \ n>N+M_2+N_2+M_1+N_1.$$

The minimax spectral characteristics $h_k(f_k^0)$ of the optimal linear estimate of the functionals $A_{S_k}\xi$, $k=3,4$, can be calculated by the formulas

$$h_3(f_3^0)=-\sum_{j=1}^{N}a(j)e^{-ij\lambda}-\sum_{j=N+M_2+1}^{N+M_2+N_2}a(j)e^{-ij\lambda}, \tag{33}$$

$$h_4(f_4^0)=-\sum_{j=1}^{N_2-1}a(N+M_2+N_2-j)e^{i(N+M_2+N_2+j)\lambda}$$

$$-\sum_{j=M_2+N_2}^{N+M_2+N_2}a(N+M_2+N_2-j)e^{i(N+M_2+N_2+j)\lambda}-\sum_{j=N+M_2+N_2+M_1+1}^{N+M_2+N_2+M_1+N_1}a(N+M_2+N_2-j)e^{i(N+M_2+N_2+j)\lambda}. \tag{34}$$

COROLLARY 1. Spectral densities (30) and (31) with the Fourier coefficients

$$b_3^0(n)=b_3^0(-n)=pa(n)(a(0))^{-1},\ n\in Z\setminus S_3,$$

$$b_4^0(n-N-M_2-N_2)=b_4^0(N+M_2+N_2-n)=pa(n)(a(N+M_2+N_2))^{-1},\ n\in Z\setminus S_4,$$

are the least favorable spectral densities in the class D_0^- for the optimal linear interpolation of the functionals $A_k\xi$, $k=3,4$, defined by the strictly positive sequence $a(n)$, $n\in Z\setminus S_k$.

The minimax spectral characteristics $h_k(f_k^0)$ can be calculated by formulas (33) and (34).

CONCLUSIONS

We have presented the methods for solving the problem of optimal linear estimation of functionals of unknown values of the stochastic stationary sequence.

We have analyzed the problem for the case of spectral determinacy, where the spectral densities of the sequences are known exactly. Therefore, we have proposed an approach based on the method of orthogonal projections in the Hilbert space. We have derived formulas for calculating the spectral characteristics and root mean square errors of optimal estimates of the functionals. In the case of spectral uncertainty, where the spectral densities are not known exactly, but some classes of feasible spectral densities are specified, the minimax-robust method is used. We have obtained the image of the root mean square error in the form of a linear functional with respect to the spectral densities, which makes it possible to solve the corresponding conditional optimization problem and describe the minimax (robust) estimates of the functionals. We have derived the formulas that determine the least favorable spectral densities and the minimax (robust) spectral characteristic of the optimal linear estimates of functionals for certain classes of feasible spectral densities.

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