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EXTREMUM STRATEGIES OF APPROACH OF CONTROLLED OBJECTS IN DYNAMIC GAME PROBLEMS WITH TERMINAL PAYOFF FUNCTION

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Abstract. *The authors propose a method for solving the problem of approach of controlled objects in dynamic game problems with a terminal payoff function. The method is reduced to the systematic use of the Fenchel–Moreau ideas on the general scheme of the method of resolving functions. The essence of the method is that the resolving function can be expressed in terms of the function conjugate to the payoff function and, using the involutivity of the connection operator for a convex closed function, it is possible to obtain a guaranteed estimate of the terminal value of the payoff function represented by the payoff value at the initial instant of time and integral of the resolving function. A feature of the method is the cumulative principle used in the current summation of the resolving function to assess the quality of the game before reaching a certain threshold. The notion of the upper and lower resolving functions of two types is introduced and sufficient conditions of a guaranteed result in the differential game with the terminal payoff function are obtained in the case where Pontryagin's principle does not hold. Two schemes of the method of resolving functions with extremum strategies of approach of controlled objects are constructed and the guaranteed times are compared.*

Keywords: *terminal payoff function, quasilinear differential game, multi-valued mapping, measurable selector, extremum strategy, resolving function.*

INTRODUCTION

We will consider the problem of approach of controlled objects in dynamic game problems with the terminal payoff function based on the method of resolving functions [1] and its modern version [2]. In contrast to the basic scheme of the method of resolving functions, we will examine the case where the Pontryagin's principle does not hold. We will consider special multi-valued mappings, which generate upper and lower resolving functions of two types, first introduced in [3]. With the help of these functions, we will obtain sufficient conditions for completing the game in some guaranteed time. We will propose two schemes of the method of resolving functions, generate extremum control strategies, and compare the guaranteed times.

The study continues the research [1–3], is related to the publications [4–14], expands the class of game problems of approach of controlled objects that have solutions, and outlines the new possibilities for applying convex analysis to the theory of conflict-controlled processes.

GENERAL SCHEME OF THE METHOD. RESOLVING FUNCTIONS OF THE FIRST KIND

Consider a conflict-controlled process whose evolution can be described by the equality

$$z(t) = g(t) + \int_0^t \Omega(t,\tau)\varphi(u(\tau), \upsilon(\tau))\, d\tau, \;\; t \ge 0\,. \tag{1}$$

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Here, $z(t) \in R^n$, function $g(t)$, $g: R_+ \to R^n$, is Lebesgue measurable [8] and bounded for $t > 0$, matrix function $\Omega(t, \tau)$, $t \geq \tau \geq 0$, is measurable with respect to t and summable with respect to τ for each $t \in R_+$. The control unit is defined by the function $\varphi(u, v)$, $\varphi: U \times V \to R^n$, which is considered continuous with respect to the set of variables on the direct product of non-empty compact sets U and V, and m, l, and n are natural numbers.

Controls of the players $u(\tau)$, $u: R_+ \to U$, and $v(\tau)$, $v: R_+ \to V$, are measurable functions of time. Along with process (1), a convex closed eigenfunction $\sigma(z)$, $\sigma: R^n \to R^1$, bounded below with respect to z is given, whose values on the trajectories of process (1) determine the time of the end of the game. If $z(t)$, $t \geq 0$, is the trajectory of system (1), then the game is considered terminated at time $t_1 > 0$ if

$$\sigma(z(t_1)) \leq 0. \tag{2}$$

The objectives of the first and second players are opposite. The first player (pursuer) tends to satisfy inequality (2) on the respective trajectory of process (1) in the shortest time, and the second player (evader) tends to postpone as much as possible the time of satisfying this inequality or to avoid the meeting at all.

Let us take the side with the first player and focus on the evader's choice of an arbitrary measurable function that takes the values from V as the control. If the game (1), (2) continues on the interval $[0, T]$, then we will choose the control of the first player at time t on the basis of information about $g(T)$ and $v_t(\cdot)$, i.e., in the form of a measurable function

$$u(t) = u(g(T), v_t(\cdot)), \quad t \in [0, T], \quad u(t) \in U, \tag{3}$$

where $v_t(\cdot) = \{v(s): s \in [0, t]\}$ is the previous history of control of the second player till the moment t, or in the form of counter-control

$$u(t) = u(g(T), v(\cdot)), \quad t \in [0, T], \quad u(t) \in U. \tag{4}$$

If, in particular, $g(t) = e^{At} z_0$, $\Omega(t, \tau) = e^{A(t-\tau)}$, $z(0) = z_0$, and e^{At} is a matrix exponent, then control $u(t) = u(z_0, v_t(\cdot))$ is assumed to implement a quasi-strategy [7], and counter-control [4] $u(t) = u(z_0, v(\cdot))$ is a manifestation of the Hajek stroboscopic strategy [8].

According to the definition of a conjugate function and taking into account the Fenchel–Moreau theorem [10], we get

$$\sigma(z) = \sup_{\psi \in R^n} [(\psi, z) - \sigma^*(\psi)],$$

where

$$\sigma^*(\psi) = \sup_{z \in R^n} [(\psi, z) - \sigma(z)]. \tag{5}$$

Function $\sigma^*(\psi)$ is a closed and convex eigenfunction [10]. An effective set of the function $\sigma^*(\psi)$ has the form $\operatorname{dom} \sigma^* = \{\psi \in R^n : \sigma^*(\psi) < +\infty\}$. Taking into account relation (5) and the fact that the eigenfunction $\sigma(z)$ is bounded from below, we obtain $\sigma^*(0) = -\inf_{z \in R^n} \sigma(z)$; thus, $0 \in \operatorname{dom} \sigma^*$.

We assume that L is the linear span of the set $\operatorname{dom} \sigma^*$ (intersection of all the linear subspaces that contain the set $\operatorname{dom} \sigma^*$). Then it is a linear subspace. Let $\Psi = \{\psi \in L: \|\psi\| = 1\}$ and π be the operator of orthogonal projection from R^n into L. If $\sigma^*(\psi)$ is continuous with respect to $\psi \in \Psi$, then the following relation is true:

$$\sigma(z) = \sigma(\pi z) = \max_{\psi \in \Psi} [(\psi, \pi z) - \sigma^*(\psi)], \quad z \in R^n.$$

If the function $\sigma^*(\psi)$ is continuous with respect to $\psi \in \Psi$, then, given the equality $\sigma(z(t)) = \sigma(\pi z(t))$, formula (1), and the definition of conjugate function, we obtain

$$\sigma(z(t)) = \max_{\psi \in \Psi} \left[(\psi, g(t)) + \int_0^t (\psi, \pi \Omega(t, \tau) \varphi(u(\tau), v(\tau))) d\tau - \sigma^*(\psi) \right], \quad t \geq 0. \tag{6}$$

Put $\varphi(U, \upsilon) = \{\varphi(u, \upsilon):\ u \in U\}$ and consider on the set $\Delta \times V$ the multi-valued mapping

$$W(t, \tau, \upsilon) = \operatorname{co} \pi\Omega(t, \tau)\varphi(U, \upsilon).$$

Here, $\operatorname{co} A$ is convexification of the set A [10], $\Delta = \{(t, \tau): 0 \le \tau \le t < \infty\}$.

Condition 1. Function $\sigma^*(\psi)$ is continuous with respect to $\psi \in \Psi$, mapping $\pi\Omega(t, \tau)\varphi(U, \upsilon)$ is closed-valued, and boundaries of the sets $W(t, \tau, \upsilon)$ and $\pi\Omega(t, \tau)\varphi(U, \upsilon)$ coincide on the set $\Delta \times V$.

Taking into account the assumptions about the matrix function $\Omega(t, \tau)$, we conclude that for any fixed $t > 0$, the vector function $\pi\Omega(t, \tau)\varphi(u, \upsilon)$ is $\mathfrak{L} \otimes \mathfrak{B}$-measurable with respect to $(\tau, \upsilon) \in [0, t] \times V$ and continuous with respect to $u \in U$. Therefore, for any fixed $t > 0$, multi-valued mappings $\pi\Omega(t, \tau)\varphi(U, \upsilon)$ and $W(t, \tau, \upsilon)$ have closed values and are $\mathfrak{L} \otimes \mathfrak{B}$-measurable with respect to $(\tau, \upsilon) \in [0, t] \times V$ [9].

Condition 2 (Pontryagin's Principle). On the set Δ, there exists a Pontryagin selector $\gamma_0(t, \tau)$, for which the inclusion holds:

$$0 \in \bigcap_{\upsilon \in V} [\pi\Omega(t, \tau)\varphi(U, \upsilon) - \gamma_0(t, \tau)].$$

In convex analysis [10], support functions $C^*(X, \psi) = \sup_{x \in X} (\psi, x)$, where X is from R^n, play a key role in describing sets. If the set X is convex and closed, then function $C^*(X, \psi)$ is convex and positive homogeneous [10]. We will call such functions upper support functions, and in parallel introduce lower support functions $C_*(X, \psi) = \inf_{x \in X} (\psi, x)$.

If set X is convex and closed, then there exists a one-to-one correspondence between it and its upper and lower support functions [10], and

$$X = \{x: (x, \psi) \le C^*(X, \psi)\ \ \forall \psi \in R^n\} = \{x: (x, \psi) \ge C_*(X, \psi)\ \ \forall \psi \in R^n\}.$$

Let $\gamma(t, \tau)$, $\gamma: \Delta \to L$, $\Delta = \{(t, \tau):\ 0 \le \tau \le t < \infty\}$, be some almost everywhere bounded, measurable with respect to t, and summable with respect to τ, $\tau \in [0, t]$, for each $t > 0$ function, which, following [3], will be called shift function.

Condition 3. Condition 1 is satisfied and for some shift function $\gamma(t, \tau)$, $\gamma: \Delta \to L$, on the set $\Delta \times V$ the inequality holds $\max_{\psi \in \Psi} C_*(W(t, \tau, \upsilon) - \gamma(t, \tau), \psi) \le 0$.

Remark 1. Condition 3 is equivalent to the inclusion $0 \in [W(t, \tau, \upsilon) - \gamma(t, \tau)]$ for all $\tau \in [0, t]$, $\upsilon \in V$, which, generally speaking, does not guarantee that Condition 2 is satisfied. In this case, if $W(t, \tau, \upsilon) = \pi\Omega(t, \tau)\varphi(U, \upsilon)$, then Condition 3 guarantees that Condition 2 is true.

Let us fix some shift function $\gamma(t, \tau)$ and put

$$\xi(t) = \xi(t, g(t), \gamma(t, \cdot)) = \pi g(t) + \int_0^t \gamma(t, \tau)\, d\tau.$$

Consider the set

$$\mathrm{P}(g(\cdot), \gamma(\cdot, \cdot)) = \{t \ge 0: \sigma(\xi(t, g(t), \gamma(t, \cdot)) \le 0\}.$$

If the inequality in curly brackets does not hold for any $t \ge 0$, then put $\mathrm{P}(g(\cdot), \gamma(\cdot, \cdot)) = \varnothing$.

THEOREM 1. Suppose that for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$ Condition 3 is satisfied, for the corresponding shift function $\gamma(t, \tau)$ set $\mathrm{P}(g(\cdot), \gamma(\cdot, \cdot))$ is not empty, and $\mathrm{P} \in \mathrm{P}(g(\cdot), \gamma(\cdot, \cdot))$. Then, if the maximum in (6) is attained on some vector $\psi(\mathrm{P})$, the game can be terminated at time P using control (4).

Proof. Let $\upsilon(\tau)$ be an arbitrary measurable selector of the compact set V, $\tau \in [0, \mathrm{P}]$, and $\psi(\mathrm{P})$ be the vector specified in the condition of the theorem. Let us present the way of choosing the control by the pursuer.

Consider for $\tau \in [0, \mathrm{P}]$, $\upsilon \in V$, a compact multi-valued mapping

$$U_0(\tau, \upsilon) = \{u \in U: (\pi\Omega(\mathrm{P}, \tau)\varphi(u, \upsilon) - \gamma(\mathrm{P}, \tau), \psi(\mathrm{P})) = C_*(W(\mathrm{P}, \tau, \upsilon) - \gamma(\mathrm{P}, \tau), \psi(\mathrm{P}))\}. \tag{7}$$

With regard for the properties of the parameters of process (1) and of the lower support function $C_*(W(\mathrm{P}, \tau, \upsilon) - \gamma(\mathrm{P}, \tau), \psi(\mathrm{P}))$, the compact multi-valued mapping $U_0(\tau, \upsilon)$ is $\mathfrak{L} \otimes \mathfrak{B}$-measurable [2] for $\tau \in [0, \mathrm{P}]$, $\upsilon \in V$. Therefore, by the theorem on measurable choice of selector [9], the multi-valued mapping $U_0(\tau, \upsilon)$ contains $\mathfrak{L} \otimes \mathfrak{B}$-measurable selector $u_0(\tau, \upsilon)$, which is a superpositionally measurable function [2].

Let the control of the first player be $u_0(\tau) = u_0(\tau, \upsilon(\tau))$, $\tau \in [0, \mathrm{P}]$. Taking into account relation (6), we obtain

$$\sigma(z(\mathrm{P})) = (\psi(\mathrm{P}), \xi(\mathrm{P})) + \int_0^{\mathrm{P}} (\psi(\mathrm{P}), \pi\Omega(\mathrm{P}, \tau)\varphi(u_0(\tau), \upsilon(\tau)) - \gamma(\mathrm{P}, \tau)) d\tau - \sigma^*(\psi(\mathrm{P})). \tag{8}$$

Then, with regard for Condition 3, relations (7) and (8) determine

$$\sigma(z(\mathrm{P})) \le (\psi(\mathrm{P}), \xi(\mathrm{P})) - \sigma^*(\psi(\mathrm{P})) \le \sigma(\xi(\mathrm{P}, g(\mathrm{P}), \gamma(\mathrm{P}, \cdot))) \le 0.$$

Hence inequality (2) follows at time P.

Remark 2. From Condition 2 it follows that there exists a measurable selector $\gamma_0(t, \tau)$, $\gamma_0(t, \tau) \in \bigcap_{\upsilon \in V} \pi\Omega(t, \tau)\varphi(U, \upsilon)$ for which Condition 3 is satisfied.

On the set $\Delta \times V$, consider the multi-valued mapping

$$\mathfrak{A}(t, \tau, \upsilon) = \left\{ \alpha \ge 0 : \max_{\psi \in \Psi} [C_*(W(t, \tau, \upsilon) - \gamma(t, \tau), \psi) + \alpha[(\psi, \xi(t)) - \sigma^*(\psi)]] \le 0 \right\}.$$

Condition 4. Condition 1 is satisfied and the multi-valued mapping $\mathfrak{A}(t, \tau, \upsilon)$ is not empty on the set $\Delta \times V$.

If Condition 4 is satisfied, then we consider the upper and lower scalar resolving functions of the first kind

$$\alpha^*(t, \tau, \upsilon) = \sup \{\alpha : \alpha \in \mathfrak{A}(t, \tau, \upsilon)\},$$

$$\alpha_*(t, \tau, \upsilon) = \inf \{\alpha : \alpha \in \mathfrak{A}(t, \tau, \upsilon)\}, \ \tau \in [0, t], \ \upsilon \in V.$$

By analogy with [2], we can show that the multi-valued mapping $\mathfrak{A}(t, \tau, \upsilon)$ is closed-valued, $\mathfrak{L} \otimes \mathfrak{B}$-measurable on the strength of (τ, υ), $\tau \in [0, t]$, $\upsilon \in V$, and the upper and lower resolving functions, being respectively the upper and lower support function of the multi-valued mapping $\mathfrak{A}(t, \tau, \upsilon)$ in the direction +1, are $\mathfrak{L} \otimes \mathfrak{B}$-measurable on the strength of (τ, υ), $\tau \in [0, t]$, $\upsilon \in V$. Therefore, they are superpositionally measurable [2], i.e., $\alpha^*(t, \tau, \upsilon(\tau))$ and $\alpha_*(t, \tau, \upsilon(\tau))$ are measurable with respect to τ, $\tau \in [0, t]$, for any measurable function $\upsilon(\cdot) \in V(\cdot)$, where $V(\cdot)$ is a set of measurable functions $\upsilon(\tau)$, $\tau \in [0, +\infty]$, with values from the set V. Note also that the upper resolving function is upper semi-continuous and the lower one is lower semi-continuous with respect to the variable υ and functions $\inf_{\upsilon \in V} \alpha^*(t, \tau, \upsilon)$ and $\sup_{\upsilon \in V} \alpha_*(t, \tau, \upsilon)$ are measurable with respect to τ, $\tau \in [0, t]$.

Consider the set

$$\mathrm{P}_*^1(g(\cdot), \gamma(\cdot, \cdot)) = \left\{ t \ge 0 : \sigma(\xi(t, g(t), \gamma(t, \cdot))) \le 0, \int_0^t \sup_{\upsilon \in V} \alpha_*(t, \tau, \upsilon) d\tau < 1 \right\}.$$

If the inequalities in curly brackets do not hold for any $t \ge 0$, then put $\mathrm{P}_*^1(g(\cdot), \gamma(\cdot, \cdot)) = \varnothing$.

THEOREM 2. Let for a conflict-controlled process (1), (2) with terminal functional $\sigma(z)$ Condition 4 be satisfied, for the corresponding shift function $\gamma(t, \tau)$ set $\mathrm{P}_*^1(g(\cdot), \gamma(\cdot, \cdot))$ be non-empty, and $\mathrm{P}_*^1 \in \mathrm{P}_*^1(g(\cdot), \gamma(\cdot, \cdot))$. Then, if the maximum in relation (6) is attained on some vector $\psi(\mathrm{P}_*^1)$, the game can be terminated at time P_*^1 with the use of control (4).

Proof. Let $\upsilon(\tau)$ be an arbitrary measurable selector of the compact set V, $\tau \in [0, \mathrm{P}_*^1]$, and $\psi(\mathrm{P}_*^1)$ be the vector specified in the condition of the theorem. Let us present the way of choosing the control by the pursuer.

For $\tau \in [0, \mathrm{P}_*^1]$, $\upsilon \in V$, consider the compact multi-valued mapping

$$U_*^1(\tau, \upsilon) = \{u \in U : (\pi\Omega(\mathrm{P}_*^1, \tau)\varphi(u, \upsilon) - \gamma(\mathrm{P}_*^1, \tau), \psi(\mathrm{P}_*^1))$$
$$= C_*(W(\mathrm{P}_*^1, \tau, \upsilon) - \gamma(\mathrm{P}_*^1, \tau), \psi(\mathrm{P}_*^1))\}. \quad (9)$$

With regard for the properties of the parameters of process (1) and of the lower support function $C_*(W(\mathrm{P}_*^1, \tau, \upsilon) - \gamma(\mathrm{P}_*^1, \tau), \psi(\mathrm{P}_*^1))$, the compact-valued multi-valued mapping $U_*^1(\tau, \upsilon)$ is $\mathfrak{L} \otimes \mathfrak{B}$-measurable [2] for $\tau \in [0, \mathrm{P}_*^1]$, $\upsilon \in V$. Therefore, by the theorem on the measurable choice of selector [9], the multi-valued mapping $U_*^1(\tau, \upsilon)$ contains $\mathfrak{L} \otimes \mathfrak{B}$-measurable selector $u_*^1(\tau, \upsilon)$, which is a superpositionally measurable function [2].

Let the control of the first player be $u_*^1(\tau) = u_*^1(\tau, \upsilon(\tau))$, $\tau \in [0, \mathrm{P}_*^1]$. Then given Condition 4 for $\tau \in [0, \mathrm{P}_*^1]$ the inequality holds:

$$(\psi(\mathrm{P}_*^1), \pi\Omega(\mathrm{P}_*^1, \tau)\varphi(u_*^1(\tau), \upsilon(\tau)) - \gamma(\mathrm{P}_*^1, \tau))$$
$$+ \alpha_*(\mathrm{P}_*^1, \tau, \upsilon(\tau))[(\psi(\mathrm{P}_*^1), \xi(\mathrm{P}_*^1)) - \sigma^*(\psi(\mathrm{P}_*^1))] \le 0. \quad (10)$$

Given relation (6), we obtain

$$\sigma(z(\mathrm{P}_*^1)) = (\psi(\mathrm{P}_*^1), \xi(\mathrm{P}_*^1)) + \int_0^{\mathrm{P}_*^1} (\psi(\mathrm{P}_*^1), \pi\Omega(\mathrm{P}_*^1, \tau)\varphi(u_*(\tau), \upsilon(\tau)) - \gamma(\mathrm{P}_*^1, \tau))d\tau - \sigma^*(\psi(\mathrm{P}_*^1)).$$

Adding and subtracting in this equality the expression

$$[(\psi(\mathrm{P}_*^1), \xi(\mathrm{P}_*^1)) - \sigma^*(\psi(\mathrm{P}_*^1))] \int_0^{\mathrm{P}_*^1} \alpha_*(\mathrm{P}_*^1, \tau, \upsilon(\tau))d\tau,$$

we obtain

$$\sigma(z(\mathrm{P}_*^1)) = [(\psi(\mathrm{P}_*^1), \xi(\mathrm{P}_*^1)) - \sigma^*(\psi(\mathrm{P}_*^1))]\left(1 - \int_0^{\mathrm{P}_*^1} \alpha_*(\mathrm{P}_*^1, \tau, \upsilon(\tau))d\tau\right)$$
$$+ \int_0^{\mathrm{P}_*^1} [(\psi(\mathrm{P}_*^1), \pi\Omega(\mathrm{P}_*^1, \tau)\varphi(u_*^1(\tau), \upsilon(\tau)) - \gamma(\mathrm{P}_*^1, \tau))$$
$$+ \alpha_*(\mathrm{P}_*^1, \tau, \upsilon(\tau))[(\psi(\mathrm{P}_*^1), \xi(\mathrm{P}_*^1)) - \sigma^*(\psi(\mathrm{P}_*^1))]]d\tau. \quad (11)$$

Taking into account (9)–(11), the pursuer can guarantee at time P_*^1 that the inequality holds:

$$\sigma(z(\mathrm{P}_*^1)) \le [(\psi(\mathrm{P}_*^1), \xi(\mathrm{P}_*^1)) - \sigma^*(\psi(\mathrm{P}_*^1))]\left(1 - \int_0^{\mathrm{P}_*^1} \alpha_*(\mathrm{P}_*^1, \tau, \upsilon(\tau))\, d\tau\right).$$

In this case, by definition P_*^1 the relations hold:

$$(\psi(\mathrm{P}_*^1), \xi(\mathrm{P}_*^1)) - \sigma^*(\psi(\mathrm{P}_*^1)) \le \sigma(\xi(\mathrm{P}_*^1, g(\mathrm{P}_*^1), \gamma(\mathrm{P}_*^1, \cdot))) \le 0,$$
$$1 - \int_0^{\mathrm{P}_*^1} \alpha_*(\mathrm{P}_*^1, \tau, \upsilon(\tau))d\tau \ge 1 - \int_0^{\mathrm{P}_*^1} \sup_{\upsilon \in V} \alpha_*(\mathrm{P}_*^1, \tau, \upsilon)\, d\tau > 0.$$

Therefore, we get

$$\sigma(z(\mathrm{P}_*^1)) \le \sigma(\xi(\mathrm{P}_*^1, g(\mathrm{P}_*^1), \gamma(\mathrm{P}_*^1, \cdot)))\left(1 - \int_0^{\mathrm{P}_*^1} \alpha_*(\mathrm{P}_*^1, \tau, \upsilon(\tau))d\tau\right) \le 0,$$

which completes the proof of the theorem.

Remark 3. If for some shift function $\gamma(t,\tau), \gamma:\Delta\to L$, on the set $\Delta\times V$ Condition 3 is satisfied, then $0\in\mathfrak{A}(t,\tau,\upsilon)$, $\tau\in[0,t]$, $\upsilon\in V$. Therefore, Condition 4 is satisfied and $\alpha_*(t,\tau,\upsilon)=\inf\{\alpha:\alpha\in\mathfrak{A}(t,\tau,\upsilon)\}=0$ on the set $\Delta\times V$.

Condition 5. Condition 4 is satisfied and on the set Δ the inequality holds:

$$\sup_{\upsilon\in V}\max_{\psi\in\Psi}\left[C_*(W(t,\tau,\upsilon)-\gamma(t,\tau),\psi)+\sup_{\upsilon\in V}\alpha_*(t,\tau,\upsilon)[(\psi,\xi(t))-\sigma^*(\psi)]\right]\le 0.$$

Remark 4. If for some shift function $\gamma(t,\tau), \gamma:\Delta\to L$, on the set $\Delta\times V$ Condition 3 is satisfied, then Condition 5 holds and $\sup_{\upsilon\in V}\alpha_*(t,\tau,\upsilon)=0$ on the set Δ.

Consider the set

$$T(g(t),\gamma(\cdot,\cdot))=\left\{t\ge 0:\int_0^t\inf_{\upsilon\in V}\sigma^*(t,\tau,\upsilon)d\tau\ge 1,\ \int_0^t\sup_{\upsilon\in V}\alpha_*(t,\tau,\upsilon)d\tau<1\right\}. \tag{12}$$

If for some $t>0$ we get $\alpha^*(t,\tau,\upsilon)\equiv+\infty$ for $\tau\in[0,t]$, $\upsilon\in V$, then in this case it is natural to set the value of the corresponding integral in curly brackets in (12) equal to $+\infty$ and $t\in T(g(t),\gamma(\cdot,\cdot))$ if for this t the other inequality in the curly brackets in (12) holds. If the inequalities in (12) do not hold for all $t>0$, then we put $T(g(t),\gamma(\cdot,\cdot))=\varnothing$.

THEOREM 3. Let Condition 5 be satisfied for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$, for the corresponding shift function $\gamma(\cdot,\cdot)$ set $T(g(\cdot),\gamma(\cdot,\cdot))$ be non-empty, and $T\in T(g(\cdot),\gamma(\cdot,\cdot))$. Then, if the maximum in (6) is attained for some vector $\psi(T)$, the game can be terminated at time T using control (3).

Proof. Let $\upsilon(\tau)$ be an arbitrary measurable selector of the compact set V, $\tau\in[0,T]$, and $\psi(T)$ be the vector specified in the condition of the theorem.

First, consider the case $\sigma(\xi(T,g(T),\gamma(T,\cdot)))>0$ and introduce a control function

$$h(t)=1-\int_0^t\alpha^*(T,\tau,\upsilon(\tau))d\tau-\int_t^T\sup_{\upsilon\in V}\alpha_*(T,\tau,\upsilon)d\tau,\ t\in[0,T].$$

By the definition of T, we get

$$h(0)=1-\int_0^T\sup_{\upsilon\in V}\alpha_*(T,\tau,\upsilon)\,d\tau>0,$$

$$h(T)=1-\int_0^T\sigma^*(T,\tau,\upsilon(\tau))\,d\tau\le 1-\int_0^T\inf_{\upsilon\in V}\alpha^*(t,\tau,\upsilon)\,d\tau\le 0.$$

Since the function $h(t)$ is continuous, there exists time t_*, $t_*\in(0,T]$, such that $h(t_*)=0$. Note that the switching time t_* depends on the previous history of control of the second player $\upsilon_{t_*}(\cdot)=\{\upsilon(s):s\in[0,t_*]\}$.

Consider for $\tau\in[0,T]$ and $\upsilon\in V$ the compact multi-valued mapping

$$U_1(\tau,\upsilon)=\{u\in U:(\pi\Omega(T,\tau)\varphi(u,\upsilon)-\gamma(T,\tau),\psi(T))$$

$$=C_*(W(T,\tau,\upsilon)-\gamma(T,\tau),\psi(T))\}. \tag{13}$$

With regard for the properties of the parameters of process (1) and of the lower support function $C_*(W(T,\tau,\upsilon)-\gamma(T,\tau),\psi(T))$, the compact-valued multi-valued mapping $U_1(\tau,\upsilon)$ is $\mathfrak{L}\otimes\mathfrak{B}$-measurable [2] for $\tau\in[0,T]$, $\upsilon\in V$. Therefore, by the theorem on the measurable choice of selector [9], the multi-valued mapping $U_1(\tau,\upsilon)$ contains $\mathfrak{L}\otimes\mathfrak{B}$-measurable selector $u_1(\tau,\upsilon)$, which is a superpositionally measurable function [2].

Put the control of the first player $u_1(\tau)=u_1(\tau,\upsilon(\tau))$, $\tau\in[0,T]$.

According to Condition 5, with regard for Condition 4 and relation (13) for $\tau\in[0,t_*]$ the inequality holds:

$$(\psi(T),\pi\Omega(T,\tau)\varphi(u_1(\tau),\upsilon(\tau))-\gamma(T,\tau))$$

$$+\alpha^*(T,\tau,\upsilon(\tau))[(\psi(T),\xi(T))-\sigma^*(\psi(T))]\le 0, \tag{14}$$

and for $\tau \in [t_*, T]$ the inequality holds:

$$(\psi(T), \pi\Omega(T,\tau)\varphi(u_1(\tau), \upsilon(\tau)) - \gamma(T,\tau))$$

$$+ \sup_{\upsilon \in V} \alpha_*(T,\tau,\upsilon)[(\psi(T), \xi(T)) - \sigma^*(\psi(T))] \le 0. \qquad (15)$$

Given the relation (6), we obtain

$$\sigma(z(T)) = (\psi(T), \xi(T)) + \int_0^T (\psi(T), \pi\Omega(T,\tau)\varphi(u_1(\tau), \upsilon(\tau)) - \gamma(T,\tau))d\tau - \sigma^*(\psi(T)).$$

In this equality, we add and subtract the expression

$$[(\psi(T), \xi(T)) - \sigma^*(T)]\left[\int_0^{t_*} \alpha^*(T,\tau,\upsilon(\tau))\,d\tau + \int_{t_*}^T \sup_{\upsilon \in V} \alpha_*(T,\tau,\upsilon)\,d\tau\right].$$

Then we get

$$\sigma(z(T)) = [(\psi(T), \xi(T)) - \sigma^*(\psi(T))]\,h(t_*) + \int_0^{t_*} [(\psi(T), \pi\Omega(T,\tau)\varphi(u_1(\tau), \upsilon(\tau)) - \gamma(T,\tau))$$

$$+ \alpha^*(T,\tau,\upsilon(\tau))[(\psi(T), \xi(T)) - \sigma^*(\psi(T))]]d\tau + \int_{t_*}^T [(\psi(T), \pi\Omega(T,\tau)\varphi(u_1(\tau), \upsilon(\tau)) - \gamma(T,\tau))$$

$$+ \sup_{\upsilon \in V} \alpha_*(T,\tau,\upsilon)[(\psi(T), \xi(T)) - \sigma^*(\psi(T))]]d\tau.$$

With regard for (14) and (15), this means that the pursuer can guarantee at time T that the inequality holds

$$\sigma(z(T)) \le [(\psi(T), \xi(T)) - \sigma^*(\psi(T))]\,h(t_*) \le \sigma(\xi(T))h(t_*) = 0.$$

For the case $\sigma(\xi(T, g(T), \gamma(T,\cdot))) \le 0$ it will suffice to apply Theorem 2, which completes the proof of the theorem.

Condition 6. Condition 4 is satisfied and on the set Δ the inequality holds:

$$\sup_{\upsilon \in V} \max_{\psi \in \Psi}\left[C_*(W(t,\tau,\upsilon) - \gamma(t,\tau), \psi) + \inf_{\upsilon \in V} \alpha^*(t,\tau,\upsilon)[(\psi, \xi(t)) - \sigma^*(\psi)]\right] \le 0.$$

THEOREM 4. Let for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$ Conditions 5 and 6 be satisfied, for the corresponding shift function $\gamma(\cdot,\cdot)$ set $T(g(\cdot), \gamma(\cdot,\cdot))$ be non-empty, and $T \in T(g(\cdot), \gamma(\cdot,\cdot))$. Then, if the maximum in (6) is attained on some vector $\psi(T)$, the game can be terminated at time T using control (4).

Proof. Let $\upsilon(\tau)$ be an arbitrary measurable selector of the compact set V, $\tau \in [0, T]$, and $\psi(T)$ be the vector specified in the condition of the theorem.

First, consider the case $\sigma(\xi(T, g(T), \gamma(T,\cdot))) > 0$ and introduce the control function

$$h(t) = 1 - \int_0^t \inf_{\upsilon \in V} \alpha^*(T,\tau,\upsilon)d\tau - \int_t^T \sup_{\upsilon \in V} \alpha_*(T,\tau,\upsilon)d\tau, \ \ t \in [0, T].$$

By definition of T, we get

$$h(0) = 1 - \int_0^T \sup_{\upsilon \in V} \alpha_*(T,\tau,\upsilon)\,d\tau > 0, \ \ h(T) = 1 - \int_0^T \inf_{\upsilon \in V} \alpha^*(T,\tau,\upsilon)\,d\tau \le 0.$$

Due to the continuity of the function $h(t)$ there exists time t_*, $t_* \in (0, T]$, such that $h(t_*) = 0$. Note that the time of switching t_* does not depend on the previous history of control of the second player $v_{t_*}(\cdot) = \{v(s): s \in [0, t_*]\}$.

For $\tau \in [0, T]$, $v \in V$, consider the compact-valued multi-valued mapping

$$\widetilde{U}_1(\tau, v) = \{u \in U : (\pi\Omega(T, \tau)\varphi(u, v) - \gamma(T, \tau), \psi(T))$$

$$= C_*(W(T, \tau, v) - \gamma(T, \tau), \psi(T))\}. \tag{16}$$

With regard for the properties of the parameters of process (1) and of the lower support function $C_*(W(T, \tau, v) - \gamma(T, \tau), \psi(T))$, the compact-valued multi-valued mapping $\widetilde{U}_1(\tau, v)$ is $\mathfrak{L} \otimes \mathfrak{B}$-measurable [1] for $\tau \in [0, T]$, $v \in V$. Therefore, by the theorem on the measurable choice of selector [9], the multi-valued mapping $\widetilde{U}_1(\tau, v)$ contains $\mathfrak{L} \otimes \mathfrak{B}$-measurable selector $\widetilde{u}_1(\tau, v)$, which is a superpositionally measurable function [2].

Let the control of the first player be $\widetilde{u}_1(\tau) = \widetilde{u}_1(\tau, v(\tau))$, $\tau \in [0, T]$.

According to Condition 6, with regard for Condition 4 and relation (16), for $\tau \in [0, t_*]$ the inequality holds:

$$(\psi(T), \pi\Omega(T, \tau)\varphi(\widetilde{u}_1(\tau), v(\tau)) - \gamma(T, \tau))$$

$$+ \inf_{v \in V} \alpha^*(T, \tau, v)[(\psi(T), \xi(T)) - \sigma^*(\psi(T))] \le 0. \tag{17}$$

According to Condition 5 for $\tau \in [t_*, T]$ the inequality holds:

$$(\psi(T), \pi\Omega(T, \tau)\varphi(\widetilde{u}_1(\tau), v(\tau)) - \gamma(T, \tau))$$

$$+ \sup_{v \in V} \alpha_*(T, \tau, v)[(\psi(T), \xi(T)) - \sigma^*(\psi(T))] \le 0. \tag{18}$$

Given the relation (6), we obtain

$$\sigma(z(T)) = (\psi(T), \xi(T)) + \int_0^T (\psi(T), \pi\Omega(T, \tau)\varphi(\widetilde{u}_1(\tau), v(\tau)) - \gamma(T, \tau))d\tau - \sigma^*(\psi(T)).$$

Adding and subtracting in this equality the expression

$$[(\psi(T), \xi(T)) - \sigma^*(T)]\left[\int_0^{t_*} \inf_{v \in V} \alpha^*(T, \tau, v)\, d\tau + \int_{t_*}^T \sup_{v \in V} \alpha_*(T, \tau, v)\, d\tau\right]$$

yields

$$\sigma(z(T)) = [(\psi(T), \xi(T)) - \sigma^*(\psi(T))]\, h(t_*) + \int_0^{t_*} [(\psi(T), \pi\Omega(T, \tau)\varphi(\widetilde{u}_1(\tau), v(\tau)) - \gamma(T, \tau))$$

$$+ \inf_{v \in V} \alpha^*(T, \tau, v)[(\psi(T), \xi(T)) - \sigma^*(\psi(T))]]d\tau + \int_{t_*}^T [(\psi(T), \pi\Omega(T, \tau)\varphi(\widetilde{u}_1(\tau), v(\tau)) - \gamma(T, \tau))$$

$$+ \sup_{v \in V} \alpha_*(T, \tau, v)[(\psi(T), \xi(T)) - \sigma^*(\psi(T))]]d\tau.$$

Given the relations (17) and (18), this means that the pursuer can guarantee at time T that the inequality holds

$$\sigma(z(T)) \le [(\psi(T), \xi(T)) - \sigma^*(\psi(T))]\, h(t_*) \le \sigma(\xi(T))h(t_*) = 0.$$

For the case $\sigma(\xi(T, g(T), \gamma(T, \cdot))) \le 0$ it will suffice to apply Theorem 2, which completes the proof of the theorem.

MODIFICATION OF THE METHOD. RESOLVING FUNCTIONS OF THE SECOND KIND

Consider the multi-valued mapping

$$\mathfrak{A}(t,\tau)=\bigcap_{v\in V}\mathfrak{A}(t,\tau,v),\ (t,\tau)\in\Delta .$$

Condition 7. Condition 4 is satisfied and the multi-valued mapping $\mathfrak{A}(t,\tau)$ is not empty on the set Δ.

If Condition 7 is satisfied, consider the upper and lower scalar resolving functions of the second kind

$$\alpha^*(t,\tau)=\sup\{\alpha:\alpha\in\mathfrak{A}(t,\tau)\},\ \alpha_*(t,\tau)=\inf\{\alpha:\alpha\in\mathfrak{A}(t,\tau)\},\ \tau\in[0,t],\ v\in V.$$

It can be shown [2] that the multi-valued mapping $\mathfrak{A}(t,\tau)$ is closed-valued, $\mathfrak{L}$-measurable with respect to τ, $\tau\in[0,t]$, $v\in V$, and the upper and lower resolving functions, being respectively the upper and lower support functions of the multi-valued mapping $\mathfrak{A}(t,\tau)$ in the direction +1, are $\mathfrak{L}$-measurable with respect to τ, $\tau\in[0,t]$.

Remark 5. If for some shift function $\gamma(t,\tau)$ on the set Δ Condition 5 is satisfied, then $\sup_{v\in V}\alpha_*(t,\tau,v)\in\mathfrak{A}(t,\tau)$, $\tau\in[0,t]$. Then Condition 7 is satisfied and the equality holds: $\sup_{v\in V}\alpha_*(t,\tau,v)=\alpha_*(t,\tau)$, $\tau\in[0,t]$. If for some shift function $\gamma(t,\tau)$ on the set Δ Condition 6 is satisfied, then $\inf_{v\in V}\alpha^*(t,\tau,v)\in\mathfrak{A}(t,\tau)$, $\tau\in[0,t]$. Then Condition 7 is satisfied and the equality holds: $\inf_{v\in V}\alpha^*(t,\tau,v)=\alpha^*(t,\tau)$, $\tau\in[0,t]$.

Consider the set

$$\mathrm{P}_*^2(g(\cdot),\gamma(\cdot,\cdot))=\left\{t\ge 0:\sigma(\xi(t,g(t),\gamma(t,\cdot)))\le 0,\int_0^t\alpha_*(t,\tau)d\tau<1\right\}.$$

If the inequalities in curly brackets do not hold for any $t\ge 0$, then put $\mathrm{P}_*^2(g(\cdot),\gamma(\cdot,\cdot))=\varnothing$.

THEOREM 5. Let for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$ Condition 7 be satisfied, for the corresponding shift function $\gamma(t,\tau)$ the set $\mathrm{P}_*^2(g(\cdot),\gamma(\cdot,\cdot))$ be non-empty, and $\mathrm{P}_*^2\in\mathrm{P}_*^2(g(\cdot),\gamma(\cdot,\cdot))$. Then, if the maximum in (6) is attained for some vector $\psi(\mathrm{P}_*^2)$, the game can be terminated at time P_*^2 with the use of control (4).

Proof. Let $v(\tau)$ be an arbitrary measurable selector of the compact set V, $\tau\in[0,\mathrm{P}_*^2]$, and $\psi(\mathrm{P}_*^2)$ be the vector specified in the condition of the theorem. Let us present the way of choosing the control by the pursuer.

For $\tau\in[0,\mathrm{P}_*^2]$, $v\in V$, consider the compact-valued multi-valued mapping

$$U_*^2(\tau,v)=\{u\in U:(\pi\Omega(\mathrm{P}_*^2,\tau)\varphi(u,v)-\gamma(\mathrm{P}_*^2,\tau),\psi(\mathrm{P}_*^2))$$
$$=C_*(W(\mathrm{P}_*^2,\tau,v)-\gamma(\mathrm{P}_*^2,\tau),\psi(\mathrm{P}_*^2))\}. \tag{19}$$

With regard for the properties of the parameters of process (1) and of the lower support function $C_*(W(\mathrm{P}_*^2,\tau,v)-\gamma(\mathrm{P}_*^2,\tau),\psi(\mathrm{P}_*^2))$, the compact-valued multi-valued mapping $U_*^2(\tau,v)$ is $\mathfrak{L}\otimes\mathfrak{B}$-measurable [2] for $\tau\in[0,\mathrm{P}_*^2]$, $v\in V$. Therefore, by the theorem on the measurable choice of selector [9], the multi-valued mapping $U_*^2(\tau,v)$ contains a $\mathfrak{L}\otimes\mathfrak{B}$-measurable selector $u_*^2(\tau,v)$, which is a superpositionally measurable function [2].

Let the control of the first player be $u_*^2(\tau)=u_*^2(\tau,v(\tau))$, $\tau\in[0,\mathrm{P}_*^2]$. Then given Condition 7 for $\tau\in[0,\mathrm{P}_*^2]$ the inequality holds:

$$(\psi(\mathrm{P}_*^2),\pi\Omega(\mathrm{P}_*^2,\tau)\varphi(u_*^2(\tau),v(\tau))-\gamma(\mathrm{P}_*^2,\tau))$$
$$+\alpha_*(\mathrm{P}_*^2,\tau)[(\psi(\mathrm{P}_*^2),\xi(\mathrm{P}_*^2))-\sigma^*(\psi(\mathrm{P}_*^2))]\le 0. \tag{20}$$

Taking into account relation (6), we obtain

$$\sigma(z(\mathrm{P}_*^2)) = (\psi(\mathrm{P}_*^2), \xi(\mathrm{P}_*^2))$$

$$+\int_0^{\mathrm{P}_*^2} (\psi(\mathrm{P}_*^2), \pi\Omega(\mathrm{P}_*^2, \tau)\varphi(u_*(\tau), \upsilon(\tau)) - \gamma(\mathrm{P}_*^2, \tau))d\tau - \sigma^*(\psi(\mathrm{P}_*^2)).$$

Adding and subtracting in this equality the expression

$$[(\psi(\mathrm{P}_*^2), \xi(\mathrm{P}_*^2)) - \sigma^*(\psi(\mathrm{P}_*^1))] \int_0^{\mathrm{P}_*^2} \alpha_*(\mathrm{P}_*^2, \tau)d\tau,$$

we get

$$\sigma(z(\mathrm{P}_*^2)) = [(\psi(\mathrm{P}_*^2), \xi(\mathrm{P}_*^2)) - \sigma^*(\psi(\mathrm{P}_*^2))]\left(1 - \int_0^{\mathrm{P}_*^2} \alpha_*(\mathrm{P}_*^2, \tau)d\tau\right)$$

$$+\int_0^{\mathrm{P}_*^2} [(\psi(\mathrm{P}_*^2), \pi\Omega(\mathrm{P}_*^2, \tau)\varphi(u_*^2(\tau), \upsilon(\tau)) - \gamma(\mathrm{P}_*^2, \tau))$$

$$+\alpha_*(\mathrm{P}_*^2, \tau, \upsilon(\tau))[(\psi(\mathrm{P}_*^2), \xi(\mathrm{P}_*^2)) - \sigma^*(\psi(\mathrm{P}_*^2))]]d\tau. \tag{21}$$

With regard for (19)–(21), the pursuer can guarantee at time P_*^2 that the inequality holds:

$$\sigma(z(\mathrm{P}_*^2)) \le [(\psi(\mathrm{P}_*^2), \xi(\mathrm{P}_*^2)) - \sigma^*(\psi(\mathrm{P}_*^2))]\left(1 - \int_0^{\mathrm{P}_*^2} \alpha_*(\mathrm{P}_*^2, \tau)\, d\tau\right).$$

By the definition of P_*^2 the following relations are true:

$$(\psi(\mathrm{P}_*^2), \xi(\mathrm{P}_*^2)) - \sigma^*(\psi(\mathrm{P}_*^2)) \le \sigma(\xi(\mathrm{P}_*^2, g(\mathrm{P}_*^2), \gamma(\mathrm{P}_*^2, \cdot))) \le 0,$$

$$1 - \int_0^{\mathrm{P}_*^2} \alpha_*(\mathrm{P}_*^2, \tau)\, d\tau > 0.$$

Therefore, we get

$$\sigma(z(\mathrm{P}_*^2)) \le \sigma(\xi(\mathrm{P}_*^2, g(\mathrm{P}_*^2), \gamma(\mathrm{P}_*^2, \cdot)))\left(1 - \int_0^{\mathrm{P}_*^2} \alpha_*(\mathrm{P}_*^2, \tau)d\tau\right) \le 0,$$

which completes the proof of the theorem.

Remark 6. If for some shift function $\gamma(t, \tau)$, $\gamma : \Delta \to L$, on the set $\Delta \times V$ Condition 3 is satisfied, then $0 \in \mathfrak{A}(t, \tau)$, $\tau \in [0, t]$. Therefore, Conditions 5 and 7 are satisfied and on the set Δ the equality is true: $\sup_{\upsilon \in V} \alpha_*(t, \tau, \upsilon) = \alpha_*(t, \tau) = 0$.

Consider the set

$$\Theta(g(t), \gamma(\cdot, \cdot)) = \left\{ t \ge 0 : \int_0^t \alpha^*(t, \tau)\, d\tau \ge 1, \int_0^t \alpha_*(t, \tau)\, d\tau < 1 \right\}. \tag{22}$$

If for some $t > 0$ we get $\alpha^*(t, \tau) \equiv +\infty$ for $\tau \in [0, t]$, $\upsilon \in V$, then in this case it is natural to set the value of the corresponding integral in curly brackets in (22) equal to $+\infty$ and $t \in \Theta(g(t), \gamma(\cdot, \cdot))$ if for this t the other inequality in the curly brackets in (22) is true. If the inequalities in (22) do not hold for all $t > 0$, we put $\Theta(g(t), \gamma(\cdot, \cdot)) = \varnothing$.

THEOREM 6. Let for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$ Condition 7 be satisfied, for the corresponding shift function $\gamma(\cdot, \cdot)$ set $\Theta(g(\cdot), \gamma(\cdot, \cdot))$ be non-empty, and $\Theta \in \Theta(g(\cdot), \gamma(\cdot, \cdot))$. Then, if the maximum in (6) is attained for some vector $\psi(\Theta)$, the game can be terminated at time Θ with the use of control (4).

Proof. Let $\upsilon(\tau)$ be an arbitrary measurable selector of the compact set V, $\tau \in [0, \Theta]$, and $\psi(\Theta)$ be the vector specified in the condition of the theorem.

First, consider the case $\sigma(\xi(\Theta, g(\Theta), \gamma(\Theta, \cdot))) > 0$ and introduce the control function

$$h(t) = 1 - \int_0^t \alpha^*(\Theta, \tau) d\tau - \int_t^\Theta \alpha_*(\Theta, \tau) d\tau, \ t \in [0, \Theta].$$

By definition of Θ, we get

$$h(0) = 1 - \int_0^\Theta \alpha_*(\Theta, \tau)\, d\tau > 0, \ h(\Theta) = 1 - \int_0^\Theta \inf_{\upsilon \in V} \alpha^*(\Theta, \tau)\, d\tau \le 0.$$

Due to the continuity of the function $h(t)$ there exists a time t_*, $t_* \in (0, \Theta]$, such that $h(t_*) = 0$. Note that the switching time t_* does not depend on the previous history of the control of the second player $\upsilon_{t_*}(\cdot) = \{\upsilon(s) : s \in [0, t_*]\}$.

For $\tau \in [0, \Theta]$, $\upsilon \in V$, consider the compact-valued multi-valued mapping

$$\widetilde{U}_2(\tau, \upsilon) = \{u \in U : (\pi\Omega(\Theta, \tau)\varphi(u, \upsilon) - \gamma(\Theta, \tau), \psi(\Theta))$$

$$= C_*(W(\Theta, \tau, \upsilon) - \gamma(\Theta, \tau), \psi(\Theta))\}. \tag{23}$$

With regard for the properties of the parameters of process (1) and of the lower support function $C_*(W(\Theta, \tau, \upsilon) - \gamma(\Theta, \tau), \psi(\Theta))$, the compact-valued multi-valued mapping $\widetilde{U}_2(\tau, \upsilon)$ is $\mathfrak{L} \otimes \mathfrak{B}$-measurable [2] for $\tau \in [0, T]$, $\upsilon \in V$. Therefore, by the theorem on the measurable choice of selector [9], the multi-valued mapping $\widetilde{U}_2(\tau, \upsilon)$ contains $\mathfrak{L} \otimes \mathfrak{B}$-measurable selector $\widetilde{u}_2(\tau, \upsilon)$, which is a superpositionally measurable function [2].

Let the control of the first player be $\widetilde{u}_2(\tau) = \widetilde{u}_2(\tau, \upsilon(\tau))$, $\tau \in [0, \Theta]$.

According to Condition 7, with regard for relation (23), for $\tau \in [0, t_*]$ the inequality holds:

$$(\psi(\Theta), \pi\Omega(\Theta, \tau)\varphi(\widetilde{u}_2(\tau), \upsilon(\tau)) - \gamma(\Theta, \tau))$$

$$+ \alpha^*(\Theta, \tau)[(\psi(\Theta), \xi(\Theta)) - \sigma^*(\psi(\Theta))] \le 0. \tag{24}$$

According to Condition 7 for $\tau \in [t_*, T]$ the following inequality holds:

$$(\psi(\Theta), \pi\Omega(\Theta, \tau)\varphi(\widetilde{u}_2(\tau), \upsilon(\tau)) - \gamma(\Theta, \tau))$$

$$+ \alpha_*(\Theta, \tau)[(\psi(\Theta), \xi(\Theta)) - \sigma^*(\psi(\Theta))] \le 0. \tag{25}$$

Taking into account relation (6), we obtain

$$\sigma(z(\Theta)) = (\psi(\Theta), \xi(\Theta))$$

$$+ \int_0^\Theta (\psi(\Theta), \pi\Omega(\Theta, \tau)\varphi(\widetilde{u}_2(\tau), \upsilon(\tau)) - \gamma(\Theta, \tau)) d\tau - \sigma^*(\psi(\Theta)).$$

Adding and subtracting in this equality the expression

$$[(\psi(\Theta), \xi(\Theta)) - \sigma^*(\Theta)] \left[\int_0^{t_*} \alpha^*(\Theta, \tau)\, d\tau + \int_{t_*}^\Theta \alpha_*(\Theta, \tau)\, d\tau \right]$$

yields

$$\sigma(z(\Theta)) = [(\psi(\Theta), \xi(\Theta)) - \sigma^*(\psi(\Theta))]\,h(t_*) + \int_0^{t_*} [(\psi(\Theta), \pi\Omega(\Theta,\tau)\varphi(\tilde{u}_2(\tau), \upsilon(\tau)) - \gamma(\Theta,\tau))$$

$$+\alpha^*(\Theta,\tau)[(\psi(\Theta), \xi(\Theta)) - \sigma^*(\psi(\Theta))]]\,d\tau + \int_{t_*}^{\Theta} [(\psi(\Theta), \pi\Omega(\Theta,\tau)\varphi(\tilde{u}_2(\tau), \upsilon(\tau)) - \gamma(\Theta,\tau))$$

$$+\alpha_*(\Theta,\tau)[(\psi(\Theta), \xi(\Theta)) - \sigma^*(\psi(\Theta))]]\,d\tau.$$

With regard for (24) and (25), this means that the pursuer can guarantee at time Θ that the following inequality holds:

$$\sigma(z(\Theta)) \le [(\psi(\Theta), \xi(\Theta)) - \sigma^*(\psi(\Theta))]\,h(t_*) \le \sigma(\xi(\Theta))h(t_*) = 0.$$

For the case $\sigma(\xi(\Theta, g(\Theta), \gamma(\Theta,\cdot))) \le 0$ it will suffice to apply Theorem 5, which completes the proof of the theorem.

COMPARING THE GUARANTEED TIMES

LEMMA 1. Let for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$ Condition 7 be satisfied and $\sigma(\xi(t, g(t), \gamma(t,\cdot))) > 0$. Then the following inequalities hold:

$$\sup_{\upsilon\in V} \alpha_*(t,\tau,\upsilon) \le \alpha_*(t,\tau), \quad (t,\tau)\in\Delta, \tag{26}$$

$$\inf_{\upsilon\in V} \alpha^*(t,\tau,\upsilon) \ge \alpha^*(t,\tau), \quad (t,\tau)\in\Delta. \tag{27}$$

If Condition 5 is also satisfied, then inequality (26) turns into an equality. If Condition 6 is satisfied, then inequality (27) turns into an equality.

THEOREM 7. Let Condition 7 be satisfied for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$. Then the following inclusions take place:

$$T(g(\cdot), \gamma(\cdot,\cdot)) \supset \Theta(g(\cdot), \gamma(\cdot,\cdot)), \quad \mathrm{P}(g(\cdot), \gamma(\cdot,\cdot)) \supset \mathrm{P}_*^1(g(\cdot), \gamma(\cdot,\cdot)) \supset \mathrm{P}_*^2(g(\cdot), \gamma(\cdot,\cdot)).$$

If Conditions 5 and 6 are satisfied as well, then the following equalities are true:

$$T(g(\cdot), \gamma(\cdot,\cdot)) = \Theta(g(\cdot), \gamma(\cdot,\cdot)), \quad \mathrm{P}_*^1(g(\cdot), \gamma(\cdot,\cdot)) = \mathrm{P}_*^2(g(\cdot), \gamma(\cdot,\cdot)).$$

If Condition 3 is satisfied, then we get

$$\mathrm{P}(g(\cdot), \gamma(\cdot,\cdot)) = \mathrm{P}_*^1(g(\cdot), \gamma(\cdot,\cdot)) = \mathrm{P}_*^2(g(\cdot), \gamma(\cdot,\cdot)),$$

and if Condition 2 is satisfied, a certain Pontryagin selector can be determined as $\gamma(\cdot,\cdot)$ [1].

The proof of Lemma 1 and Theorem 7 follows immediately from the constructions of the corresponding definitions, remarks, and theorems.

CONCLUSIONS

We have considered quasilinear conflict-controlled processes of general form with the terminal payoff function and have formulated sufficient conditions for game termination in a finite guaranteed time in case where Pontryagin's principle does not hold. We have proposed two schemes of the method of resolving functions that ensure the completion of conflict-controlled process with a terminal payoff function in the class of extremum quasi-strategies and counter-controls. We have also compared the guaranteed times.

REFERENCES

1. A. A. Chikrii, Conflict Controlled Processes, Springer Sci. and Business Media, Dordrecht–Boston–London (2013).
2. A. A. Chikrii and I. S. Rappoport, "Method of resolving functions in the theory of conflict-controlled processes," Cybern. Syst. Analysis, Vol. 48, No. 4, 512–531 (2012).
3. A. A. Chikrii and V. K. Chikrii, "Image structure of multialued mappings in game problems of motion control," J. Autom. Inform. Sci., Vol. 48, No. 3, 20–35 (2016).
4. N. N. Krasovskii and A. I. Subbotin, Positional Differential Games [in Russian], Nauka, Moscow (1974).
5. L. S. Pontryagin, Selected Scientific Works, Vol. 2 [in Russian], Nauka, Moscow (1988).
6. M. S. Nikol'skii, The First Direct Pontryagin Method in Differential Games [in Russian], Izd. MGU, Moscow (1984).
7. A. I. Subbotin and A. G. Chentsov, Guarantee Optimization in Control Problems [in Russian], Nauka, Moscow (1981).
8. O. Hajek, Pursuit Games, Vol. 12, Academic Press, New York (1975).
9. J.-P., Aubin and H. Frankowska, Set-Valued Analysis, Birkhauser, Boston–Basel–Berlin (1990).
10. R. T. Rockafellar, Convex Analysis, Princeton Univ. Press (1970).
11. A. D. Ioffe and V. M. Tikhomirov, Theory of Extremum Problems [in Russian], Nauka, Moscow (1974).
12. I. S. Rappoport, "Resolving-functions method in the theory of conflict-controlled processes with a terminal payoff function," J. Autom. Inform. Sci., Vol. 48, No. 5, 74–84 (2016). https://doi.org/10.1615/JAutomatInfScien.v48.i5.70.
13. I. S. Rappoport, "Sstroboscopic strategy in the method of resolving functions for game control problems with terminal payoff function," Cybern. Syst. Analysis, Vol. 52, No. 4, 577–587 (2016). https://doi.org/10.1007/s10559-016-9860-z.
14. I. S. Rappoport, "Sufficient conditions of the guaranteed result in differential game with a terminal payoff function," J. Autom. Inform. Sci., Vol. 50, No. 2, 14–27 (2018). https://doi.org/10.1615/JAutomatInfScien.v50.i2.20.