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OPTIMAL SPEED OF RESPONSE IN THE LOTKA–VOLTERRA CONTROLLED SYSTEM

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Abstract. *We consider a controlled system of Lotka–Volterra differential equations that describes the evolution of two interrelated populations of predators and prey. The system contains two control variables, which are chosen so as to minimize the time of transition to a stationary point. The control functions and the corresponding trajectories of motion in the state space are constructed and their optimality is substantiated.*

Keywords: *maximum principle, stationary point, minimum time.*

1. CONTROLLED LOTKA–VOLTERRA SYSTEM

Alfred James Lotka and Vito Volterra proposed a model of interaction of two populations of predators and prey. The model is a system of two ordinary nonlinear differential equations with two unknowns

$$\begin{aligned} \dot{x}_1 &= (\alpha - \beta x_2) x_1, \\ \dot{x}_2 &= (\delta x_1 - \gamma) x_2, \end{aligned} \tag{1}$$

where $x_1 = x_1(t)$ and $x_2 = x_2(t)$ denote respectively the numbers of prey and predators at instant of time t and positive numbers α, β, δ, and γ are parameters of the model. We assume that at some instant of time t_0 the inequalities $x_1(t_0) > 0$ and $x_2(t_0) > 0$ hold. Under these conditions, point $(x_1(t), x_2(t))$, where functions $x_1(t)$ and $x_2(t)$ satisfy system (1), moves counter-clockwise along a closed curve around a stationary point $(\gamma / \delta, \alpha / \beta)$ and the inequalities hold: $x_1(t) > 0$, $x_2(t) > 0$, $-\infty < t < +\infty$. Such models are applied in biology, economy, and in medicine.

Controlled dynamic prey–predator systems were analyzed in many publications. In [1], optimal controls are found for the Lotka–Volterra system with one variable of control, which determines the degree of separation of populations, and the cost of the ecosystem is maximized at certain instant of time. In [2–5], problems of optimal control by prey–predator systems with one control variable are solved. In [6], the Lotka–Volterra system with two control variables is used to minimize the expenses of an agricultural enterprise related to various methods of influence on the ecosystem, in particular, applying pesticides. In [7], a prey–predator system with two control variables is analyzed. The problem about optimal speed of response is solved, theorems on controllability and on the existence of optimal control are proved, however, under the conditions that are not satisfied for the Lotka–Volterra model.

In the present paper, we will analyze a controlled Lotka–Volterra system

$$\begin{aligned} \dot{x}_1 &= (\alpha - \beta x_2 + u_1) x_1, \\ \dot{x}_2 &= (\delta x_1 - \gamma + u_2) x_2, \\ \bar{u}_i &\le u_i(t) \le \bar{\bar{u}}_i, \quad i = 1, 2, \end{aligned} \tag{2}$$

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where $u_i = u_i(t)$ are control variables; $\bar{u}_i$ and $\bar{\bar{u}}_i$ are constants, $\bar{u}_i < 0 < \bar{\bar{u}}_i$, $i = 1, 2$. By means of the Pontryagin maximum principle, a control is generated that transfers a phase point from the given state $x_1(t_0) > 0$, $x_2(t_0) > 0$ to the stationary point $(\gamma / \delta, \alpha / \beta)$ of system (1) in minimum time. We will only consider piecewise continuous functions of control $u_i(t)$; at points of discontinuity, these functions are considered right-continuous.

2. CONSTRUCTING OPTIMAL CONTROLS AND TRAJECTORIES

Changing variables $x_1 = \delta y_1 - \gamma$ and $x_2 = \beta y_2 - \alpha$ in the system

$$\begin{gathered} \dot{x}_1 = -(x_2 - u_1)(x_1 + \gamma), \\ \dot{x}_2 = (x_1 + u_2)(x_2 + \alpha), \\ \bar{u}_i \le u_i(t) \le \bar{\bar{u}}_i, \ i = 1, 2, \end{gathered} \tag{3}$$

yields system (2); therefore, in what follows, we solve the problem of minimizing the time of attaining the origin of coordinates for system (3). Let on the time interval $[t_0, t_1]$ a phase point subject to some control pass from the given position into the origin of coordinates. It is required to select the control so as to minimize the transition time $t_1 - t_0$. The initial values should satisfy the conditions $x_1(t_0) > -\gamma$ and $x_2(t_0) > -\alpha$. We assume that the inequalities hold:

$$\alpha > -\bar{u}_1, \ \gamma > \bar{\bar{u}}_2. \tag{4}$$

Integrating the system of differential equations (3) provided that u_1 and u_2 take constant values, we obtain the relation

$$e^{-x_1}(x_1 + \gamma)^{\gamma - u_2} e^{-x_2}(x_2 + \alpha)^{\alpha + u_1} = c, \tag{5}$$

where $c > 0$ does not depend on time. From the analysis of this relation, it follows that phase curves of system (3) are closed [8]. If control values are constant, point $X(t) = (x_1(t), x_2(t))$, where functions $x_1(t)$ and $x_2(t)$ satisfy system (3), moves counter-clockwise along a closed curve around a stationary point $(-u_2, u_1)$ and the inequalities $x_1(t) > -\gamma$ and $x_2(t) > -\alpha$ hold. An example of the phase trajectories of system (3) is presented in Fig. 1.

To generate optimal controls, we will use the Pontryagin maximum principle [9]. Consider functions $\psi_1(t)$, $\psi_2(t)$, and H

$$H = -\psi_1(x_2 - u_1)(x_1 + \gamma) + \psi_2(x_1 + u_2)(x_2 + \alpha)$$

and the system of differential equations

$$\begin{gathered} \dot{\psi}_1 = (x_2 - u_1)\psi_1 - (x_2 + \alpha)\psi_2, \\ \dot{\psi}_2 = (x_1 + \gamma)\psi_1 - (x_1 + u_2)\psi_2. \end{gathered} \tag{6}$$

We only consider nontrivial solutions $\Psi(t) = (\psi_1(t), \psi_2(t))$ of this system of equations. At each instant of time $t \in [t_0, t_1]$, optimal vector of control $U(t) = (u_1(t), u_2(t))$ should satisfy the condition

$$H(\Psi(t), X(t), U(t)) = \max_V H(\Psi(t), X(t), V), \tag{7}$$

where vector $V = (v_1, v_2)$ belongs to the set of feasible controls, i.e., $\bar{u}_1 \le v_1 \le \bar{\bar{u}}_1$ and $\bar{u}_2 \le v_2 \le \bar{\bar{u}}_2$. From relation (7) it follows that

$$\begin{gathered} u_1(t) = \bar{\bar{u}}_1 \ \text{if } \psi_1(t) > 0, \\ u_1(t) = \bar{u}_1 \ \text{if } \psi_1(t) < 0, \\ u_2(t) = \bar{\bar{u}}_2 \ \text{if } \psi_2(t) > 0, \\ u_2(t) = \bar{u}_2 \ \text{if } \psi_2(t) < 0. \end{gathered} \tag{8}$$

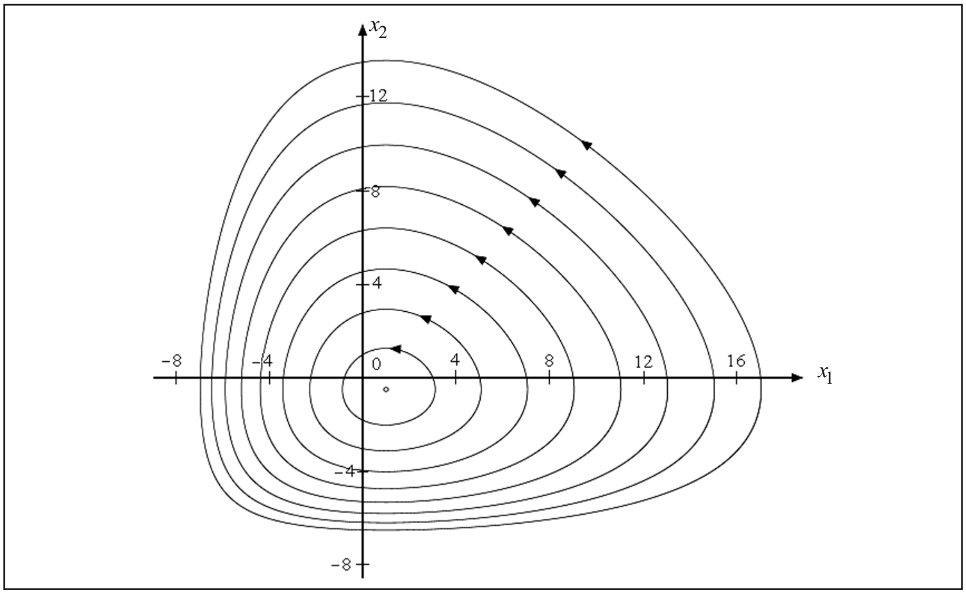

Fig. 1. Phase trajectories of system (3) under the conditions $\alpha = 8,\ \gamma = 10,$ $u_1(t) \equiv -0.5,\ u_2(t) \equiv -1$.

At instants of time such that $\psi_1(t) = 0$ ($\psi_2(t) = 0$), the value of $u_1(t)$ ($u_2(t)$) is selected from the right-continuity condition. Optimal controls $u_1(t)$ and $u_2(t)$ are piecewise constant functions and can take the values $\overline{u}_1$, $\overline{\overline{u}}_1$, $\overline{u}_2$, and $\overline{\overline{u}}_2$.

The part of the phase trajectory corresponding to the time interval on which u_1 and u_2 are constant is a part of the curve defined by (5), and the stationary point of system (3) is equal to $(-u_2, u_1)$. Under the optimality condition, vector $(-u_2, u_1)$ can take one of the four values: $A = (-\overline{u}_2, \overline{\overline{u}}_1)$, $B = (-\overline{\overline{u}}_2, \overline{\overline{u}}_1)$, $C = (-\overline{\overline{u}}_2, \overline{u}_1)$ or $D = (-\overline{u}_2, \overline{u}_1)$.

Solution $\Psi(t)$ of system (6) intersects all the four quadrants counter-clockwise. Indeed, if at the moment t point $\Psi(t)$ is on the positive abscissa semiaxis, then from (6) it follows that $\dot{\psi}_2 = (x_1 + \gamma)\psi_1 > 0$, i.e., $\Psi(t)$ passes from the fourth quadrant into the first one. Similarly, from the first quadrant point $\Psi(t)$ passes into the second one, from the second into the third one, and from the third into the fourth one. Moreover, from Lemma 1 proved below it follows that point $\Psi(t)$ stays a limited time in each quadrant. Therefore, vectors A, B, C, and D eventually follow in the following order: $\ldots, A, B, C, D, A, B, C, D, \ldots$.

In Fig. 2, thin lines represent motion trajectories, bold lines represent the set of control switching points, dashed lines represent arcs that simultaneously are both trajectories and switching lines. Phase points move counter-clockwise along spiral lines. Before the phase point hits the origin of coordinates, it moves along one of four dashed arcs which we denote by $\breve{A}, \breve{B}, \breve{C}$, and $\breve{D}$. These arcs satisfy relation (5) and are related to the stationary points A, B, C, and D, respectively.

Initial points of the arcs $\breve{A}, \breve{B}, \breve{C}$, and $\breve{D}$ can be easily calculated by a numerical method. For example, to find the initial point $\breve{A}$, it is necessary to solve the system of differential equations (3), (6) with the initial values t_1, $X(t_1) = (0, 0)$, $\Psi(t_1) = (1, 0)$. Let t_1' be an instant of time such that $t_1' < t_1$, $\psi_1(t_1') = 0$ and $\psi_1(t) > 0$, $t_1' < t < t_1$. Point $X(t_1')$ is the initial point of the arc $\breve{A}$.

Assume that at the instant of time t' phase point X hits the arc $\breve{D}$, i.e., the arc OP (see Fig. 2). It means that beginning with the instant of time t', the control vector $(\overline{u}_1, \overline{u}_2)$ that corresponds to the stationary point D of system (3) is involved and till the instant of time t' the control vector $(\overline{u}_1, \overline{\overline{u}}_2)$ that corresponds to the stationary point C was used. Solving the system of differential equations (3), (6) with the initial values t', $X(t')$, $\Psi(t') = (-1, 0)$ by a numerical method, we find the instant of time t'' such that $t'' < t'$, $\psi_1(t'') = 0$ and $\psi_1(t) < 0$, $t'' < t < t'$. At time t'', the control vector changes its value from $(\overline{\overline{u}}_1, \overline{\overline{u}}_2)$ to $(\overline{u}_1, \overline{\overline{u}}_2)$. If point $X(t')$ runs the arc OP, then point $X(t'')$ runs the arc P_1P_2, which is the control switching line. The curvilinear tetragon OPP_2P_1 is filled with the arcs satisfying (5) provided that $(u_1, u_2) = (\overline{u}_1, \overline{\overline{u}}_2)$, along which points move; OP_1 and PP_2 belong to such arcs.

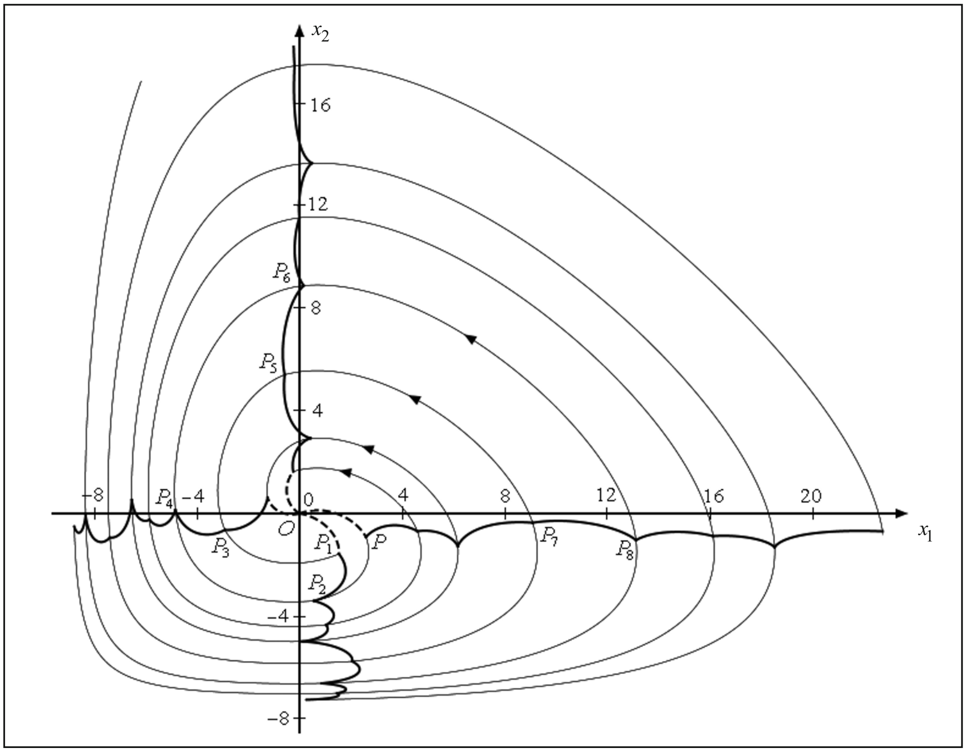

Fig. 2. Optimal trajectories and switching lines of system (3) under the conditions $\alpha = 8,\ \gamma = 10,\ \bar{u}_1 = -1.5,\ \bar{\bar{u}}_1 = 0.9,\ \bar{u}_2 = -0.7,\ \bar{\bar{u}}_2 = 0.3.$

After constructing the curvilinear tetragon OPP_2P_1, we similarly construct the curvilinear tetragon $P_1P_2P_4P_3$ by using the switching arc P_1P_2 as the set of already calculated points $X(t'')$ of phase trajectories. Solving the system of differential equations (3), (6) with the initial values t'', $X(t'')$, $\Psi(t'') = (0, 1)$, we find the instant of time t''' such that $t''' < t''$, $\psi_2(t''') = 0$ and $\psi_2(t) > 0$, $t''' < t < t''$. If point $X(t'')$ runs the arc P_1P_2, then point $X(t''')$ runs the arc P_3P_4, which is a switching line. Similarly, we construct the curvilinear tetragons $P_3P_4P_6P_5$, $P_5P_6P_8P_7, \ldots$ and perform the described construction for the arcs $\breve{A}, \breve{B}, \breve{C}$, and $\breve{D}$.

Figure 2 shows an example of synthesis of optimal trajectories. Inside each curvilinear tetragon, control vector takes a constant value. The control vector takes the same value on its sides, except for the side through which the trajectories abandon the tetragon. To determine the optimal control at the given phase point, it is necessary to find a curvilinear tetragon to which this point belongs. If the phase point lies on the switching line, control is selected from the condition of right-continuity.

LEMMA 1. If on the time interval (t', t'') point $\Psi(t)$ belongs to one of the quadrants and conditions (3), (4), (6), (8) are satisfied, then $t'' - t' < \infty$.

Proof. From the conditions of the lemma it follows that the constant control vector (u_1, u_2) is used on the interval (t', t''). Let $X(t) = (-u_2, u_1)$, $t \in (t', t'')$. In this case, system (6) becomes

$$\dot{\psi}_1 = -(u_1 + \alpha)\psi_2,$$

$$\dot{\psi}_2 = (-u_2 + \gamma)\psi_1.$$

Taking into account (4), it is easy to prove that the solution $\Psi(t)$ of this system of equations intersects each quadrant in a finite time; therefore, the statement of the lemma is true.

Assume that $X(t) \neq (-u_2, u_1)$, $t \in (t', t'')$. Let us prove the lemma for the case $(u_1, u_2) = (\bar{\bar{u}}_1, \bar{u}_2)$. For the three other cases: $(u_1, u_2) = (\bar{\bar{u}}_1, \bar{\bar{u}}_2)$, $(u_1, u_2) = (\bar{u}_1, \bar{\bar{u}}_2)$, and $(u_1, u_2) = (\bar{u}_1, \bar{u}_2)$, the proof is similar. From condition (8) it follows that on the interval (t', t'') point $\Psi(t)$ belongs to the fourth quadrant. Denote by L a closed curve satisfying (5), which contains points $X(t)$, $t \in (t', t'')$, corresponds to the control $(\bar{\bar{u}}_1, \bar{u}_2)$ and to the stationary point $(-\bar{u}_2, \bar{\bar{u}}_1)$ of system (3). Let T be the time it takes for the phase point to transits once the curve L under the control $(\bar{\bar{u}}_1, \bar{u}_2)$. It is obvious that $T < \infty$.

Assume that $t'' - t' > T$. In the interval (t', t''), there exist two instants of time τ_1 and τ_2 such that $X(\tau_1) = (-\bar{u}_2 + \sigma_1, \bar{\bar{u}}_1)$ and $X(\tau_2) = (-\bar{u}_2 - \sigma_2, \bar{\bar{u}}_1)$, and $\sigma_i > 0$, $X(\tau_i) \in L$, $i = 1, 2$. Taking into account (4), we obtain

$$H(t) = -\psi_1(x_2 - \bar{\bar{u}}_1)(x_1 + \gamma) + \psi_2(x_1 + \bar{u}_2)(x_2 + \alpha),$$

$$H(\tau_1) = \psi_2 \sigma_1 (\bar{\bar{u}}_1 + \alpha) < 0, \quad H(\tau_2) = -\psi_2 \sigma_2 (\bar{\bar{u}}_1 + \alpha) > 0,$$

i.e., function $H(t)$ changes its sign at least once on the interval (t', t''). However, from [10] it follows that under the conditions (3), (6), and (8) the value of $H(t)$ does not depend on time. From the obtained contradiction it follows that the assumption $t'' - t' > T$ is incorrect.

The lemma is proved.

3. OPTIMALITY OF THE CONSTRUCTED CONTROLS

In Sec. 2, we have constructed the controls and respective trajectories of system (3) by means of the maximum principle, which is the necessary, and in the linear case also the sufficient optimality condition. Since this system is nonlinear, it is necessary to prove that the constructed controls and trajectories are optimal. To prove the optimality, we will use the theory of regular synthesis [10].

Assume that as a result of numerical calculations described in Sec. 2, a finite nonzero number of closed curvilinear tetragons, beginning with the first one (omissions are not admitted) are constructed for each arc $\breve{A}$, $\breve{B}$, $\breve{C}$, $\breve{D}$. We consider that each such tetragon is generated by four different points, which are vertices, and by four arcs (sides), which only have common points at vertices. Each such tetragon is completely filled with arcs satisfying (5); all these arcs correspond to one stationary point from the set A, B, C, D, which does not belong to this tetragon. Such arcs are pairwise non-intersecting, only one arc passes through each point of the tetragon. The constructed tetragons have no common interior points. We will call the set of the constructed curvilinear tetragons regular if all of them satisfy the specified properties.

Denote by $\bar{G}$ the closure of the union of all the constructed curvilinear tetragons and let G be the interior of set $\bar{G}$. The trajectory belonging to the set G and transferring a phase point from state X to the origin of coordinates and the control corresponding to this trajectory are considered optimal in G if there is no other trajectory belonging to G and respective control such that transition time from X to the origin of coordinates is less. Let us show that the constructed trajectories are optimal in G.

Consider one of the constructed curvilinear tetragons. Let S_0 be its side, which is a switching line, after hitting which the phase points move inside the tetragon, and let S_1 be the opposite side of the tetragon, also representing a switching line, at the moment of hitting which the phase points abandon the tetragon. For example, in the tetragon OPP_2P_1 (see Fig. 2) side S_0 is the arc P_1P_2 and side S_1 is the arc OP.

LEMMA 2. If in the constructed curvilinear tetragon arc S_1 is a smooth line and each phase trajectory which approaches an internal point of this arc from the internal part of the tetragon forms a nonzero angle with it, then arc S_0 is also a smooth line.

The proof of this lemma (which we omit here) is based on the theorem about differentiability of the solution with respect to initial values [11] and on the implicit function theorem [12]. (A smooth line is defined in [12].) Since the arcs $\breve{A}$, $\breve{B}$, $\breve{C}$, and $\breve{D}$ satisfy (5), they are smooth lines. Therefore, it follows by induction from Lemma 2 that all the switching arcs in the constructed curvilinear tetragons the smooth. It is obvious that all the other sides of curvilinear tetragons are also smooth.

Let $A' = (a_1', a_2')$, $B' = (b_1', b_2')$, $C' = (c_1', c_2')$, and $D' = (d_1', d_2')$ be points belonging to the arcs $\breve{A}$, $\breve{B}$, $\breve{C}$, and $\breve{D}$, respectively.

LEMMA 3. The following inequalities are true:

$$a_1' \le -\bar{u}_2, \ b_2' \le \bar{\bar{u}}_1, \ c_1' \ge -\bar{\bar{u}}_2, \ d_2' \ge \bar{u}_1. \tag{9}$$

The proof of this lemma (which we omit here) is based on the constancy of function $H(t)$.

By means of Lemma 3 it is easy to prove that the function $H(t)$ is nonnegative. Indeed, since function $H(t)$ is constant, it is possible to check its nonnegativity when a phase point hits one of the arcs $\breve{A}$, $\breve{B}$, $\breve{C}$, $\breve{D}$. Let us perform the check for the arc $\breve{A}$ (in three other cases, the check is similar). Let at the instant of time t' the phase point hit the specified arc. It means that, beginning with the time t', control vector $(\bar{\bar{u}}_1, \bar{u}_2)$ is used, at the instant of time t' we get $\Psi(t') = (0, \sigma)$, where $\sigma < 0$. Using the first inequality in (9), we obtain

$$H(t') = \sigma(x_1(t') + \bar{u}_2)(x_2(t') + \alpha) \ge 0.$$

In [10], the concept of regular synthesis is introduced and theorem on the optimality of trajectories for which conditions of such synthesis are satisfied is proved. For the problem (3), conditions of regular synthesis in set G can be formulated as follows.

1. Sides of each constructed curvilinear tetragon are smooth lines. The control vector is constant inside such tetragon. On all the sides of the tetragon, the control vector is also constant, except for the vertices at which control can vary according to the condition of its right-continuity.

2. A unique phase trajectory of system (3) passes through each point of each curvilinear tetragon. If the trajectory passes inside the tetragon, it abandons it in a finite time, bumping into a switching line at a nonzero angle and approaching it with a nonzero velocity. If the trajectory on some time interval coincides with a side of the tetragon, which is not a switching line, then in a finite time it abandons the tetragon, approaching its vertex with a nonzero velocity.

3. Each constructed trajectory is ended at the origin of coordinates, intersecting curvilinear tetragons and switching lines a finite number of times.

4. All the constructed trajectories satisfy the maximum principle.

5. The value of the time of transition from point X to the beginning of coordinates is a continuous function of point X.

From Lemma 2 and from the control creation process it follows that the first condition of regular synthesis is satisfied. The second condition (except for the condition of nonzero angles) as well as the third and the fifth ones are undoubtedly true. It is obvious that the velocity of motion of a phase point cannot be equal to zero. For the constructed controls and trajectories, the equalities (3), (6), and (8) hold, and nonnegativity of function H is proved by means of Lemma 3; therefore, the fourth condition is satisfied. Thus, the following theorem is true.

THEOREM 1. Let the set of the constructed curvilinear tetragons be regular and the following condition be satisfied for each constructed trajectory passing through interior parts of the tetragons: it approaches switching lines from inside the tetragons at nonzero angles. Then all the constructed trajectories and respective vectors of control are optimal in G.

The proved theorem can used provided that trajectories bump into switching lines at nonzero angles. Theoretical check of this condition is difficult; therefore, it is proposed to use a numerical method. Naturally, for each switching arc, checking an angle for being nonzero can only be performed in a finite number of points of the arc. For example, in Fig. 2, a check was performed for each switching no less than at 1000 different points; as a result, a conclusion was made that nonzero angles are absent. Hence, the theorem is applicable in this case.

Thus, in the present paper by means of the Pontryagin maximum principle, we have constructed optimal controls and motion trajectories for the controlled Lotka–Volterra system. The controls are stepwise constant functions of time, and phase points move along spiral-like curves. Since the system is nonlinear, the optimality of the constructed controls and trajectories is proved; Boltyanskii's theory of regular synthesis is used here.

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