DOI 10.1007/s10559-020-00320-y
Cybernetics and Systems Analysis, Vol. 56, No. 6, November, 2020

A SYSTEMATIC APPROACH TO MULTIOBJECTIVE OPTIMIZATION

A. N. Voronin[1†] **and A. S. Savchenko**[1‡]

UDC 519.9

Abstract. *A systematic approach to solving multiobjective optimization problems is proposed. It allows combining the models of individual schemes of compromises into an integrated structure that adapts to the situation of making multicriteria decisions. An advantage of the concept of nonlinear scheme of compromises is the possibility of making a multicriteria decision formally, without a direct human participation. The apparatus of the nonlinear scheme of compromises, developed as a formalized tool for the analysis of control systems with conflicting criteria, makes it possible to solve practically multicriteria problems of a broad class.*

Keywords: *system, optimization, multicriteria, utility function, scalar convolution, nonlinear scheme of compromises.*

INTRODUCTION

The informative content of many practical problems in different subject domains lies in choosing the conditions that allow the object under study to exhibit its best properties (optimization problems). The conditions on which the object properties depend are qualitatively expressed by certain variables $x_1, x_2, \ldots$, and x_n, specified in the domain X and are called optimization arguments. The external influences r do not depend on them; however, they can take their values from the compact set R. It is surmised that calculations are performed when the external influence vector $r^0 \in R$ on which the decision-making situation depends is known and specified.

Conversely, each property of the object in the domain M is quantitatively described using the variable y_k, $k \in [1, s]$, whose value characterizes the property of the object O in respect to this property.

In the general case, the properties $y_1, y_2, \ldots, y_s$, which are called quality criteria, form a vector $y = \{y_k\}_{k=1}^{s} \in M$. Its components quantitatively express the object properties with the specified set of optimization arguments $x = \{x_i\}_{i=1}^{n} \in X$.

PROBLEM STATEMENT

The ideas of system optimization are laid out in [1, 2]. The term "systematic approach" postulates that the real object represented by a system is described as a set of components interacting with each other while realizing a certain objective. From the variety of components of the real object, the final but ordered set of elements and relationships between them is "carved out." It can be surmised that the system is a model of the real object only in the terms of the objective being realized by it. The object requiring certain functions to be realized determines the composition and structure of the system in terms of them.

The objective isolates and defines the outlines of the system in the object. Only the information necessary and sufficient to achieve the objective will be included in this system (the object model). If one object is able to realize

[1]National Aviation University, Kyiv, Ukraine, †*alnv@ukr.net*; ‡*a.v.savchenko@ukr.net*. Translated from Kibernetika i Sistemnyi Analiz, No. 6, November–December, 2020, pp. 160–174. Original article submitted September 10, 2019.

 1060-0396/20/5606-1000

a couple of objectives, then it acts as a separate system in relation to each of them. The systematic approach implies that not only the object but the investigation process itself acts like a complex system, whose objective is to integrate different models of the object [3].

Thus, under the systematic approach, a researcher receives the information only about the real object necessary and sufficient to solve the stated problem.

OPTIMIZATION PROBLEM

If the object realizes only one objective, then the efficiency of realization of the stated objective is quantitatively expressed by the unique optimality criterion *y*. Solving the optimization problem implies the achievement of the extreme criterion value by choosing a set of optimization arguments.

Extremization of the optimality criterion is often associated with the objective realization. However, these notions are different. It can be surmised that the criterion and the objective relate to one another in the same way as the model and the original with all the resulting consequences. This is explained by the fact that the original is usually associated not with a single model but a few of them representing its certain aspects. Some objectives are hard or sometimes impossible to describe using quantitative criteria. At any rate, a criterion is just a substitute. The objective is always characterized by criteria, sometimes with better or worse results, only implicitly and approximately [4, 5].

Solving optimization problems implies a certain estimate of the quality of the system performance, from which it is possible to determine to which degree one system's performance is better than that of the other. The main problem of the quantitative estimate of objects and processes lies in the necessity to associate the notions of "better" and "worse" with the notions of "more" and "fewer." For the sake of certainty, it is assumed that, for example, the notion "better" means "fewer."

If a description allows us to employ measurements, then discussions can be substituted with calculations. Applying this approach to the problems under study means that in the case of the existence of the substantiated quantitative quality criteria of a complex system, its investigation can be performed by means of the formalized mathematical apparatus. Subjective estimates, ambiguous interpretations, and random decisions are otherwise inevitable.

The function $y = f(x)$ associates the quality criterion with the optimization arguments. In estimation problems, the function $f(x)$ is called an evaluation function, and an objective function in optimization problems. Under certain reservations, an optimization problem is formed as the means of searching for such a combination of values of arguments from their domain of definition, under which the objective function takes the following extreme value:

$$x^* = \arg \operatorname*{extr}_{\substack{x \in X \\ y \in M}} f(x) \Big|_{r^0 \in R} .$$

If the term "better" signifies "fewer," then in practice

$$x^* = \arg \min_{x \in X} f(x)$$

for the fixed $r^0 \in R$ and the guaranteed $y \in M$.

MULTICRITERIA PROBLEMS

It is impossible to characterize a complex object under study by a certain unique (for example, "the most important" or "typical") attribute since multiple properties are to be considered simultaneously during its description. In other words, in order to investigate complex objects, the modern systematic approach necessitates the involvement of the entire range of its properties. A complex object and any of its fragments is not to be considered in isolation, but instead in multiple contradictory relations and, which is very important, in various possible situations.

Complex systems exhibit various system properties under different conditions (situations and states), and the ones not compatible with any of the situations taken separately are among them. When investigating such systems, an approach consisting of the creation and of the simultaneous existance of multiple theoretical models of the same phenomenon is applied, where some of the models are conceptually contradictory. Neither one of them can be disregarded, however, since each of them characterizes a certain property of the phenomenon under study, and any of them can be considered unique since none of them expresses the full scope of all properties. Let us associate the aforesaid with the complementarity principle announced by Niels Bohr: to reconstruct a phenomenon in its entirety, one needs to apply the incompatible "complimentary" concept classes, where each of them can be utilized under its own special conditions, and only considered together can they exhaust all the information lending itself to definition.

In certain situations, multiple properties of a complex system in the time of its operation are quantitatively estimated by the corresponding partial criteria. In the course of different situations, different properties and, respectively, different partial criteria achieve the "most important" status. Thus, separate theoretical models in the role of incompatible "complimentary" concept classes are characterized by contradictory partial criteria, each of them is applicable under certain special conditions. Only a complete partial criteria set (a vector criterion) allows the possibility of adequate assessment of the complex system operation as a manifestation of the contradictory unity of all its properties. Therefore, it can be assumed that the notion of multicriteria embodies the complimentarily principle in the investigation technique of complex systems. However, this possibility represents only the necessary and not the sufficient condition of the vector estimate of a system in its entirety. Indeed, let the quantitative values of all the partial criteria be known, and even knowing these values will not allow us to estimate the efficiency of a system in its entirety.

To perform a comprehensive assessment, it is necessary to ascend to the next level, i.e., to carry out a criteria arrangement. Let us compare it to the following Gödel's description of the incompleteness theorem: in any sufficiently complex noncontradictory first-order theory, there exists a statement that cannot be neither proven nor disproven by the means of the theory itself. Nevertheless, the noncontradictory nature of one theory can be determined by the means of another stronger formal second-order theory. However, then the question arises about the noncontradictory nature of the second theory, etc. Gödel's theorem serves as a methodological basis of the criteria arrangement and constitutes a sufficient condition for the vector estimate of the system in its entirety.

Scalar convolution can be implemented as a tool for criteria arrangement representing a mathematical mean for data compression and quantitative assessment of its integral properties by a single number.

A simultaneous description of the phenomenon (object) from different positions always yields a qualitatively new and more complete overview of the phenomenon (object) being described compared to any "one-sided" description. For example, two flatshots forming a stereogram form a three-dimensional image of an object even without the use of the holography. The multicriteria approach "stereoscopically" assessing system performance opens new possibilities for improving complex control and decision-making systems. Thus, in order to perceive a complex system in its entirety under its different operation conditions, it is necessary to employ the multicriteria approach.

In practical problems, a real object usually realizes not a single but multiple objectives, thus, being characterized by a number of partial efficiency (quality) criteria. Note that efficiency criteria are usually contradictory. A skilled scientist is able to systematically associate models characterized by contradictory criteria. For example, Jean-Baptiste Colbert (the First Minister of the State in the court of King Louis XIV) said the following in 1665: "The art of taxation consists in so plucking the goose as to obtain the largest amount of feathers with the least possible amount of hissing." Under the systematic approach, a problem of uniting different models of an object arises. The problem is solved by performing the criteria arrangement.

For the purpose of systematic association in multicriteria problems, instead of $y = f(x)$ in the form of objective (evaluation) function the scalar convolution of the partial criteria $Y = f[y(x)]$ is used, where y is not a scalar but the s-dimentional criterion vector $y = \{y_k\}_{k=1}^{s}$. The scalar convolution acts as an instrument for criteria arrangement [6].

Aside from criteria, within the notion of optimization a none less important role is played by constraints both in relation to the optimization $x \in X$, as well as in relation to the efficiency criteria of the decision $y \in M$. Even its smallest changes can have a profound effect on the decision [7]. Excluding one set of constraints and introducing the other one under the same criteria system can also lead to serious consequences. Such situation can result in a risk of yielding unpredictable results when optimizing complex systems. This phenomenon has been described by Norbert Wiener in his first published papers in cybernetics and can be explained by the fact that without setting constraints, one can simultaneously yield unpredictable and undesired results with the extremization of the objective function.

The musings of Norbert Wiener about the fact that we are ultimately unable to determine all of the conditions and constraints ensuring the absence of undesired optimization effects allowed him to make a grim prediction about the disastrous consequences of the cybernetization of society.

Nevertheless, from the position of system analysis the relation towards optimization can be formulated as follows: it is a powerful tool allowing us to increase efficiency; however, it should be used relative to the increase of problem complexity.

Let the set of possible decisions $X \subset E^n$ formed from vectors $x = \{x_i\}_{i=1}^{n}$ of the n-dimensional Euclidian space be given. In accordance with the physical nature of the problem, a vector holonomic (in statics) or nonholonomic (in dynamics) constraint $B(x) \leq 0$ is given. The decision is made under external influences described by the vector r that is given on the set of possible factors R.

The result of the decision-making is estimated over the set of contradictory partial criteria forming the s-dimentional vector $y(x) \in F \subset R^s$ specified on the feasible set X. The dependence $y \in F$ implies that the vector y belongs to the class F of feasible efficiency vectors. The partial criteria vectors $y \in M$ are constrained by the feasible domain $M \subset R^s$. The question about the existence of different types of optimal solutions of multicriteria optimization problems is examined in depth in [8]. Let us assume the existence of the feasible set $X \neq \varnothing$, as well as that of optimal solutions. The situation arising as a result of making the multicriteria decision x under the given external conditions r is characterized by the Cartesian product $S = X \times R$.

A problem is stated: to determine such a decision of $x^* \in X$ that will optimize the efficiency vector $y(x)$ under the given conditions and constraints, and with the given relations.

To find a constructive solution to the stated problem, it is necessary to structure some notions with respect to different specific problem statements. In order to do this, we need to make additional specific assumptions facilitating the solution process of the following vector optimization problems:

— determine the domain of Pareto-optimal solutions;
— choose the compromise scheme;
— normalize partial criteria;
— specify priorities.

The difficulties in solving vector optimization problems are not of a computational nature, but of a conceptual one (the question is not about a method of finding an optimal solution but rather about what should be interpreted as such). Therefore, the development of a formal technique of multicriteria problem solution represents one of the most difficult problems of the modern decision theory and management. This is important in both theoretical and applied sense.

CHOOSING THE COMPROMISE SCHEME

Considering vector optimization problems, special emphasis should be placed upon the problem of choosing a compromise scheme. One of the most important postulates of the decision theory with respect to multiple criteria lies in the notion that the best decision in some absolute sense does not exist. The decision made can be considered as the best one only for a specific decision maker (DM) according to the objective set by him under a certain situation. Normative models of multicriteria problem solving are based on the hypothesis of the existence of a certain utility function [9] in the mind of a DM that can be measured both in nominal and ordinal scales. This function is reflected by the compromise scheme and its model in a given situation, namely, in the form of the scalar convolution of the partial criteria $Y[y(x)]$ that allows us to find a constructive solution to multicriteria optimization.

The definition of multicriteria solution implies a compromise due to its nature and is ultimately based on the use of subjective information. Having acquired the information from a DM and having chosen a compromise scheme, we can pass from the general vector expression to the scalar convolution of partial criteria, which is the basis of development of constructive apparatus for multicriteria problem solving. Using the scalar convolution method, the solution model for vector optimization problems is mathematically expressed in the form of function extremization $Y[y(x)]$. This is a scalar function that is also a scalar convolution of the partial criteria vector with its form depending on the chosen compromise scheme.

The additive (linear) scalar convolution

$$Y[y(x)] = \sum_{k=1}^{s} a_k y_k(x),$$

where a_k is the weight coefficients determined by a DM based on their utility function in the given situation is applied most often. The Laplace principle in the decision theory lies in the extremization of the linear scalar convolution. A particular characteristic (disadvantage) of the use of the linear scalar convolution lies in the possibility to "compensate" one criterion with the other criteria.

The multiplicative scalar convolution

$$Y[y(x)] = \prod_{k=1}^{s} y_k(x)$$

is bereft of this disadvantage. Pascal's principle lies in the extremization of the multiplicative scalar convolution.

Pascal's principle has been historically presented in his work "Pensees" in 1670 for the first time. This work is considered to have laid the groundwork for the entire decision theory. Here, the following two main notions of the theory are introduced: (i) partial criteria with each of them estimating a specific side of the solution efficiency; (ii) the principle of optimality, i.e., rules that allow calculating a certain unique numerical measure of solution efficiency (criteria arrangement) over criteria values.

Pascal's principle is adequate when applied to problems with cumulative effect, i.e., when the actions of one set of efficiency factors seems to increase or decrease the influence of another set of factors. When maximizing partial factors, any of the factors with zero value balances the contribution of all the other ones for the sake of general efficiency of decision-making. In the aerospace field, this approach can be justified to a certain extent when each criterion (for example, criteria of reliability and safety) is crucial, and no improvement of other criteria can compensate a low indicator value. If a single partial criterion is equal to zero, then the global criterion is also equal to zero.

One disadvantage in applying the multiplicative scalar convolution lies in the fact that an expensive and quite efficient system can have the same estimate as a cheap and inefficient one. Let us compare such "weapon system" as, for example, an atomic bomb and a slingshot possessing a certain killing effect along with a low cost. Applying the multiplicative scalar convolution, it is possible to choose a slingshot when equipping an army.

Similarly to the Laplace principle, let us generalize Pascal's principle by introducing the following weight coefficients:

$$Y[y(x)] = \prod_{i=1}^{s} [y_i(x)]^{a_i}.$$

Let us consider the scalar convolution according to the Charnes–Cooper theory based on the principle of "closer to the ideal (utopian) point." The ideal and a priori unknown vector y^{id} is specified in the criteria domain under the given conditions and constrains. In this case, the optimization problem is solved s times (according to the number of partial criteria); moreover, it is always solved with respect to a single (latest) criterion as if the other ones do not exist. The order of single-criterion solutions of the initial multicriteria problem determines the coordinates of the unreachable ideal vector $y^{id} = \{y_k^{id}\}_{k=1}^{s}$. Then, the criterion function $Y(y)$ is introduced as a measure of approximation to the ideal vector in the domain of vectors to be optimized as a certain non-negative function of the vector $(y^{id} - y)$ in the form of, for example, a squared Euclidean norm of the following vector:

$$Y(y) = \left\| \frac{y^{id} - y}{y^{id}} \right\| = \sum_{k=1}^{s} \left[\frac{y_k^{id} - y_k}{y_k^{id}} \right]^2.$$

A disadvantage of this method lies in the awkwardness of the determination process of the ideal vector coordinates. Likewise, a constraint violation remains possible.

The choice of the compromise scheme is performed by a decision maker and is of a conceptual nature.

FORMALIZATION IN SOLVING MULTICRITERIA PROBLEMS

Depending on the existence and the type of information, approaches to solving multicriteria problems may differ. If such information is not available, then the solution is reduced to finding any vector of the solution x^* providing only the fulfillment of a certain condition according to the constraints $A = (A_k)_{k=1}^{s}$:

$$y^* \in M = \{y \mid 0 \le y_k(x^*) \le A_k,\ k \in [1, s]\},\ \ x^* \in X.$$

(Here, the structurization process of the notion of the domain of constraints M is presented.)

The disadvantages are apparent since the obtained solution is often coarse and, as a rule, not Pareto-optimal. Therefore, the system's possibilities are not fully utilized in this case. This approach is recommended for application when optimizing quite complex systems and when performing even the easiest association of contradictory criteria (just to be within the boundaries of the constraints) is difficult. A variation of such an approach is a widespread method used when a DM chooses only one criterion (for example, only the first one) from the set y_k, $k \in [1, s]$, for the sake of optimization, and the other criteria pass into the register of constraints. Thus, the original multicriteria problem is artificially substituted with a single-criterion problem with the following constrains:

$$x^* = \arg\min_{x \in X} y_1(x),\ \ 0 \le y_k(x) \le A_k,\ k \in [1, s].$$

The above-described approach results in a solution in the form of a polar point of the Pareto domain, i.e., an openly coarse and subjective solution.

In the case of the scalar convolution with respect to the criteria to be minimized, the use of the following formula is provided for:

$$x^* = \arg\min_{x \in X} Y[y(x)].$$

SCALAR CONVOLUTION ANALYSIS

The analysis of the scalar convolution lies in the fact that the form of the function $Y(y)$ depends on the situation of the multicriteria decision making and is, therefore, usually unknown. Since it is difficult to obtain the function $Y(y)$ in the entire domain of definition, its analysis is often limited to its behavior in the neighborhood of the argument domain point that corresponds to the most typical situation. Since it is a case of small neighborhoods of the operating point, the applied smooth criteria function hypothesis is substituted with a tangent hyperplane to the surface of equal values $Y(y)$ at the operating point. In this case, the approximate dependence $Y[\alpha,\ y(x)]$ takes the form of the linear scalar convolution

$$Y^0[a,\ y(x)] = \sum_{k=1}^{s} a_k^0 y_k(x),$$

where α_k^0 is the regression coefficient that is a partial derivative of the criterion function with respect to the kth criterion calculated at the basic operating point. To calculate the coefficients α with the use of the information provided by the DM, we can solve a general problem using the least squares method. However, it is more expedient to use the heuristic modeling method described in [10].

When using the obtained expression, we need to consider the fact that it is only a linear approximation of the scalar convolution of criterion functions, and it can lead to significant distortions in situations differing from the standard one.

To obtain a criteria function in the entire domain, it is necessary to specify the type of the approximating dependence. As a rule, a positive result in the approximation practice depends on the level of adequacy by which the type of the specified function reflects the physical properties of the phenomenon under study. If the information about the workings of the phenomena is used, then the specified model is informative. In the case of absence of such information, a black box is used and standard formal regression models (polynomial, power, etc.) are specified for the sake of approximation. The quality of informative models usually exceeds the quality of the formal ones.

INFORMATIVE ANALYSIS OF THE UTILITY FUNCTION

To increase the investigation efficiency of a function, a priori information about the physical properties of the phenomenon under study should always be used, passing from the formal models to the informative ones, if possible. In this case, the object of our investigations is quite an elusive matter, i.e., an imaginary utility function appearing in the mind of a DM when solving a specific multicriteria problem. Even if such a function were to exist, every DM would have his own utility function. Nonetheless, it is possible to acquire information to specify the type of informative model of the criterion function by determining and analyzing some common patterns observed in the process of making multicriteria decisions by different DMs in different situations.

A comparison of partial criteria of different physical nature is possible only in a normalized (dimensionless) domain. Let us normalize the efficiency vector y by the constraint vector A obtaining the following vector of relative partial criteria (the normalized effectiveness vector)

$$y_0(x) = \{y_k(x) / A_k\}_{k=1}^{s} = \{y_{0k}(x)\}_{k=1}^{s}.$$

This is a monotone procedure, and according to the well-known Germeier's theorem, each monotone transformation does not change the results of the comparison. Therefore, let us replace the solution model of the vector optimization problem with the initial criteria functions by the model

$$x^* = \arg\min_{x \in X} Y[y_0(x)],\ \ y_{0k}(x) \in [0; 1],\ \ k \in [1, s],$$

where the practically applicable compromise schemes possess a physical meaning. The form of the function $Y(y_0)$ depends on the chosen compromise scheme.

The compromise scheme determines the perspective, depending on which the obtained multicriteria solution exceeds the other Pareto-optimal ones. At the present time, the choice of the compromise scheme is not defined by a theory, and is rather performed heuristically, based on individual preferences and the professional experience of the developers, as well as on the information about the situation in which a multicriteria choice is made.

The difficulty of passing from the vector criterion of efficiency to the scalar convolution lies in the fact that the convolution has to represent a conglomerate of partial criteria, the value (importance) of each changing in the general estimate depending on the situation. The "most important" status can be obtained by different partial criteria in different situations. In other words, the scalar convolution of the partial criteria should be an expression of the compromise scheme dependant on the situation. This postulate will be the cornerstone of our analysis of the formalization possibilities when choosing a compromise scheme.

There is an assumption that there exist certain invariants or rules that are usually common for all the DMs irrespective of their individual characteristics and that have to be upheld in some situations. The inevitable subjectivity of DMs has its limits [11]. When making business decisions, a person has to be rational, be able to explain the motive behind their choice and the logic behind their subjective model. Therefore, any preferences of a DM have to be within the framework of a certain rational system. This makes the process of formalization possible.

The concept of a situation expressed by the pair $S = \langle r, x \rangle$ formed from the Cartesian product $R \times X$ is essential for the vector optimization theory since it, by being objective, is the only factor in the attempt to formalize the choice of a compromise scheme. Let us introduce the following notion of the tension of a situation [12] as a measure of approximation of the relative partial criteria to their limiting value (unity):

$$\rho_k(r, x) = 1 - y_{0k}(r, x),\ \ \rho_k \in [0; 1],\ \ k \in [1, s].$$

This system is a structured characteristic of the concept of the situation $S = \langle r, x \rangle,\ r \in R,\ x \in X$.

If a multicriteria solution is applied in the complex situation $S_1 = \{S \mid \rho_k \approx 0,\ \ k \in [1, s]\}$, then under the specified external conditions r one or a few partial criteria $y_{0k}(r, x),\ k \in [1, s]$, can be found in a dangerous proximity to their limiting values ($\rho_k = 0$) as a result of the solution x. In the case of one of them reaching or exceeding the limit, this event will not be compensated by the possible low level of the rest of the criteria (it is impermissible to violate any of the constraints in the usual case).

In this situation, it is crucial not to allow the growth of the most unfavorable (i.e., the one that is the closest to its limit) partial criterion without even taking into account the behavior of the other criteria. Therefore, a DM allows the maximum (the most important under the given conditions) partial criterion to degrade by the value of one in very tense situations (with low values of ρ_k) only in the case of compensating it with the other criteria that are improved by a high number of unity values. Not considering the other criteria, the attention of the DM is solely focused on this singular unfavorable partial criterion in very tense situations (first polar case $\rho_k \approx 0$).

Therefore, an adequate expression of a compromise scheme in the case of a tense situation is the minimax (Chebyshev) model

$$x^* = \arg\min_{x \in X} \max_{k \in [1,s]} y_{0k}(x). \tag{1}$$

In less tense situations, it is necessary to satisfy the other criteria at the same time and to take into consideration the contradictory unity of all the interests and the objective of the system. Thus, the DM modifies the gain estimate according to one type of criteria and the loss estimate according to the other type depending on the situation. In intermediate cases, such compromise schemes are chosen that allow for different levels of partial adjustment of partial criteria. With the easing of the tension of the situation, the preferences over certain criteria are adjusted.

In the second polar case ($\rho_k \approx 1$) with the situation as calm ($S_2 = \{S \mid \rho_k \approx 1,\ k \in [1, s]\}$) that the partial criteria are perceived as low, there is no threat of violating the constraints. In this case, the DM surmises that the deterioration of any partial criterion by the value of one can be easily compensated by an equivalent improvement of any other criterion by the value of one. This statement is congruent with the scheme of economic compromise ensuring minimal monetary losses over partial criteria under the specified conditions. This scheme is represented by the following integral optimality model:

$$x^* = \arg\min_{x \in X} \sum_{k=1}^{s} y_{0k}(x). \tag{2}$$

According to the analysis, compromise schemes can be grouped at two extremes representing different principles of optimization: (a) the egalitarian principle, the principle of uniformity; (b) the utilitarian principle, the principle of cost efficiency.

The principle of uniformity expresses a desire for a gradual, i.e., equal reduction of the level of all partial criteria during the operation of a control system. An important implementation of the uniformity principle is the Chebyshev’s model (1), i.e., a polar scheme of this group. This scheme requires the minimization of the most disadvantageous (the highest) relative criterion by reducing it to the level of the other criteria, therefore, adjusting all the partial criteria. The disadvantage of the egalitarian uniformity system lies in its low cost-efficiency. The insurance of the most equal in relation to each other level of criteria is often reached by means of significant elevation of their total level. Alternatively, even a slight deviation from the uniformity principle allows for a significant reduction of one or a few of important criteria.

The principle of cost efficiency based on the possibility to compensate a certain deterioration of quality over certain criteria by a certain improvement of quality over other criteria is bereft of this disadvantage. The polar scheme of this group is implemented by integral optimality model (2). The utilitarian scheme ensures the minimal total level of the relative criteria. A general disadvantage of cost efficiency schemes lies in the possibility of sharp differentiating of the levels of certain criteria.

The carried out analysis attests to the consisting pattern based on which DMs change their choice from the model of integral optimality (2) in calm situations to the minimax model (1) in tense situations. In intermediate cases, a DM chooses compromise schemes specifying different satisfaction levels of certain criteria according to their individual preferences and in accordance with the given situation. If we take the conclusions reached by the analysis as the logical basis for the formalization of the choice of compromise schemes, then we can recommend different practical concepts, one of them being the nonlinear compromise scheme.

NONLINEAR COMPROMISE SCHEME

Relative to the systematic approach, it is expedient to substitute the problem of choosing compromise schemes with the equivalent problem of synthesizing a certain single scalar convolution of partial criteria that would express different optimality principles in different situations. Single models of compromise schemes unite into one integral model with its structure adapting to the situation of multicriteria decision-making. The requirements towards the function $Y(y_0)$ to be synthesized are the following:

(i) it should be smooth and differential;

(ii) it should express the minimax principle in tense situations;

(iii) it should express the integral optimality principle in calm situations;

(iv) it should lead to Pareto-optimal solutions in intermediate cases estimating different methods of partial criteria satisfaction.

In other words, such a universal scalar convolution should be an expression of the compromise scheme that is able to adapt to a situation. Thus, the adaptation and the ability to adapt itself is the main informative content of the investigation of multicriteria systems. It is necessary for the scalar convolution to explicitly express the characteristics of a tense situation. It is possible to consider a few functions satisfying the above-stated requirements. The simplest of them is the following scalar convolution:

$$Y(\alpha, y_0) = \sum_{k=1}^{s} \alpha_k [1 - y_{0k}(x)]^{-1}; \ \alpha_k \geq 0, \ \sum_{k=1}^{s} \alpha_k = 1,$$

where $\alpha_k = \mathrm{const}$ are formal parameters specified on the simplex and possessing a dual physical meaning. On the one hand, these are coefficients expressing the DM preferences over certain criteria. On the other hand, these are regression coefficients of the informative regression model formed on the basis of the concept of the nonlinear compromise scheme.

Therefore, the following nonlinear compromise scheme associated with the vector optimization model explicitly depending on the tension characteristics of the situation is pivotal:

$$x^* = \arg\min_{x \in X} \sum_{k=1}^{s} \alpha_k [1 - y_{0k}(x)]^{-1}. \tag{3}$$

From here it follows that if any relative partial criterion, for example $y_{0i}(x)$, will start to come close to its limit (unity), i.e., the situation will become tense, then the corresponding element $Y_i = \alpha_i / [1 - y_{0i}(x)]$ will increase to such an extent in the minimized sum that the problem of minimization of the entire sum will be reduced to the minimization of this most unfavorable element, i.e., the criterion $y_{0i}(x)$. This is equal to the behavior of minimax model (1). If the relative partial criteria are far from the unity, i.e., the situation is calm, then model (3) behaves in the same way as integral optimality model (2). In the intermediate situations, we will obtain different levels of partial criteria alignment.

Thus, the nonlinear compromise scheme possesses the property of continuous adaptation to the situation of multicriteria decision making. From here, traditional compromise schemes can be viewed as a result of linearization of nonlinear schemes at different situation points. This fact explains the name given to the proposed nonlinear compromise scheme since it is no more "nonlinear" in other respects than the other schemes considered in the decision theory. Note that the adaptation of the nonlinear scheme to a situation is continuous, whereas the traditional process of choosing a compromise scheme is performed discontinuously, so that mistakes related to the quantization of compromise schemes are added to the subjective errors.

As stated above, the choice of a compromise scheme is the prerogative of a person, a reflection of its subjective utility function when solving certain multicriteria problems. Nonetheless, we have succeeded in uncovering some consistent patterns, and formed on this objective basis a scalar convolution of criteria whose form follows from the informative conception of the nature of the phenomenon under study. The phenomenon of the individual preferences of a DM are formally represented by the presence of the vector α in the structure of informative model (3).

The questions about the Pareto-optimal nonlinear compromise scheme and its axiomatics are investigated at length in [10, 13].

UNIFICATION IN MULTICRITERIA PROBLEMS

It is possible to state different estimates of the role of subjective factors in solving multicriteria problems. Subjectivity is admissible and even expedient if such a problem is solved for the benefit of a certain person. Therefore, individual preferences are often applied in practice when solving multicriteria problems. However, subjectivity in their solution is admissible and desirable only in the case when the result is intended for certain DMs or people with similar preferences. If the result is intended for general use, then it should be quite objective. The result of the multicriteria problem solution intended for general use is standardized, and individual preferences are negated according to the statistics. If the a priori information about the varying criteria values is nonexistent, then the Bernoulli–Laplace's principle of insufficient reason attests to the fact that all the weight coefficients in (3) should be accepted as equal between each other. By normalizing on the simplex, we obtain $\alpha_k \equiv 1/s \ \forall k \in [1, s]$. Then,

$$Y(\alpha, y_0) = \frac{1}{s}\sum_{k=1}^{s}[1 - y_{0k}(x)]^{-1}.$$

Taking into consideration that the process of multiplication by $1/s$ is a monotone transformation that does not change the results of the comparison according to the Germeier's theorem, we will pass to the unified (with no weight coefficients) expression for the scalar convolution of criteria

$$Y(y_0) = \sum_{k=1}^{s}[1 - y_{0k}(x)]^{-1}. \tag{4}$$

It is expedient to apply this formula when a multicriteria problem is solved not for the benefit of one certain DM, but for general use.

The unified scalar convolution over the nonlinear scheme possesses form (4) or the form

$$Y(y) = \sum_{k=1}^{s} A_k [A_k - y_k(x)]^{-1}$$

equivalent to it, i.e., a form with no preliminary normalization of partial criteria. The nonlinear compromise scheme concept aligns with the principle of "further from the constraints."

For the maximized criteria the unified scalar convolution takes the form

$$Y(y) = \sum_{k=1}^{s} B_k [y_k(x) - B_k]^{-1},$$

where B_k is the minimally available criteria values to be maximized.

DUAL METHOD

If a multicriteria problem is solved for the benefit of a certain DM, then a unified (initial) solution should be obtained and presented to a certain person. Only in the case when this solution does not satisfy this person and requires to be corrected, is it necessary to specify the weight coefficients reflecting the personal preferences of this person. The specification process begins not at an arbitrary point in the criteria domain, but at the general initial solution.

The practice of multicriteria problem solving demonstrates that the assumption about the existence of the complete and stable (even in an inexplicit form) utility function at the initial stage in the mind of DMs isn't always the case. When solving a multicriteria problem, a DM compares the sum of certain criterion values under different alternatives, makes some explorative steps (and possible mistakes), and interprets the relation between the needs and the possibilities to satisfy them by the specified object in a given situation. When there are contradictory criteria present, the nature of this relationship is inherently a compromise. However, a DM does not possess a realized a priori compromise scheme, or it is present only in its originating state. In the usual case, a realization of a compromise scheme necessary to solve the problem appears, and is gradually improved only due to the attempts of a DM to improve upon the multicriteria solution when making explorative steps. It is obvious that the existence of an interactive computer technology is implied since this process is impossible in reality.

Thus, a person adapts to the multicriteria problem to be solved by structuring his preferences and improving his comprehension of the utility function on the one hand, and progressively finds a number of optimal solutions with respect to the present function of the solution efficiency on the other hand. The interdependent processes of a DM adapting to the problem and finding the best results represent the dual nature and make a fundamental part of the human-machine solution method of multicriteria problems.

As has been stated above, not only the analytical description of the utility function but its complete a priori concept is practically absent in the mind of a DM at the initial stage of problem solving. Hence, the interactive process has to be organized in the dual form, and the search for the optimization scheme has to allow interactive programming in the form of order scales, as well as apply the minimal information about the utility function. Therefore, the sequential analog to the simplex planning [10, 13] is a process based on comparison of preferences with respect to specially calculated alternatives.

An important factor determining the efficiency of this procedure lies in the fact that the initial investigation point is not chosen as an arbitrary point in the Pareto set, but as an axiomatically proven initial solution to be corrected only with respect to the informal preferences of a certain DM. The correction process insures the mutual adaptation, where a person adapts to the given certain multicriteria problem, and the nonlinear compromise scheme model becomes a reflection of the individual preferences of this person.

The fundamental difference of the scalar convolution from the other well-known scalar convolutions lies in the natural connection with the situation of the multicriteria decision-making. Essentially, the proposed scalar convolution is a nonlinear regression function (linear with respect to parameters) chosen according to physical reasoning, and it is, therefore, effective. In the expression for the nonlinear scalar convolution, the coefficients α take the form of parameters of the nonlinear informative regression function, and, therefore, do not change from one situation to an other when determined as in the case of the linear and other well-known scalar convolutions that are not adaptable to the situation.

The problem of determination of coefficients α under the dual process can be considered as a synthesis problem of the decision rule, which, being formally applied, adequately represents the logic of certain DMs in any possible situation. Such a problem arises, for example, when a multicriteria system functions in the "advice-giver human operator" mode under the lack of time. It is necessary for the system to be able to immediately make the same decisions in any situation as its human operator who has the possibility to analyze it. Similar problems arise when developing the decision making system of an artificial intelligence robot that needs to achieve the objectives it has been assigned in the same way a person teaching it might have done.

CONCLUSIONS

As a result of the systematic approach, a model of multicriteria optimization has been obtained, allowing for the object to achieve all the set objectives in the entire range of possible situations. The systematic approach to the multicriteria optimization problem has allowed us to combine the models of single compromise schemes into a single cohesive structure adapting to the situation of multicriteria decision making. The advantage of the nonlinear compromise scheme lies in its ability to make multicriteria decisions without the direct involvement of humans. Furthermore, both the problems important for the general use and the problems with their main informative content being the satisfaction of individual preferences of a DM are solved on the same ideological basis. The nonlinear compromise scheme method developed as a formalized tool for the investigation of control systems with contradictory criteria allows for practical solution to a wide spectrum of multicriteria problems.

REFERENCES

1. V. M. Glushkov, “Systemwise optimization,” Cybern. Syst. Analesis, Vol. 16, No. 5, 731–733 (1980). https://doi.org/10.1007/BF01078504.
2. V. M. Glushkov, V. S. Mikhalevich, V. L. Volkovich, and G. A. Dolenko, “System optimization in multitest linear programming problems,” Cybern. Syst. Analysis, Vol. 18, No. 3, 271–277 (1982). https://doi.org/10.1007/BF01069751.
3. N. V. Semenova, “Methods of searching for guaranteeing and optimistic solutions to integer optimization problems under uncertainty,” Cybern. Syst. Analysis, Vol. 43, No. 1, 85–93 (2007).· https://doi.org/10.1007/s10559-007-0028-8.
4. A. V. Antonov, Systems Analysis [in Russian], Vyssh. Shk., Moscow (2004).
5. F. I. Peregudov and F. P. Tarasenko, Introduction to Systems Analysis [in Russian], Radio i Svyaz’, Moscow (1989).
6. A. N. Voronin, “Nested scalar convolutions of a vector criterion,” J. Autom. Inform. Sci., Vol. 35, Iss. 9, 5–14 (2003).
7. T. T. Lebedeva, N. V. Semenova, and T. I. Sergienko, “Qualitative characteristics of the stability vector discrete optimization problems with different optimality principles,” Cybern. Syst. Analysis, Vol. 50, No. 2, 228–233 (2014). https://doi.org/10.1007/s10559-014-9609-5.
8. I. V. Sergienko, T.T. Lebedeva, and N. V. Semenova, “Existence of solutions in vector optimization problems,” Cybern. Syst. Analysis, Vol. 36, No. 6, 823–828 (2000). https://doi.org/10.1023/A:1009401209157.
9. P. C. Fishburn, Utility Theory for Decision Making, Publications in Operations Research, John Wiley and Sons, New York (1970).
10. A. N. Voronin, Yu. K. Ziatdinov, and M. V. Kuklinsky, Multi-Criteria Decisions: Models and Methods [in Russian], NAU, Kyiv (2010).
11. O. I. Larichev, Science and Art of Decision Making [in Russian], Nauka, Moscow (1979).
12. A. N. Voronin, “A nonlinear compromise scheme in multicriteria evaluation and optimization problems,” Cybern. Syst. Analysis, Vol. 45, No. 4, 597–604 (2009). https://doi.org/10.1007/s10559-009-9127-z
13. A. Voronin, Multi-Criteria Decision Making for the Management of Complex Systems, IGI Global (2017).