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MULTI-OBJECTIVE OPTIMIZATION PROBLEM: STABILITY AGAINST PERTURBATIONS OF INPUT DATA IN VECTOR-VALUED CRITERION

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Abstract. *The conditions of stability against input data perturbations in vector-valued criterion for multi-objective optimization problem with continuous partial criterion functions and feasible set of arbitrary structure are established. The sufficient and necessary conditions of three types of stability for the problem of finding Pareto optimal solutions are proved.*

Keywords: *vector optimization problem, vector-valued criterion, stability, Pareto optimal solutions, Slater set, Smale set, perturbations of input data.*

INTRODUCTION

The paper pertains to the theoretical field of studies in the problem of stability of multi-objective (vector) optimization problems. This field is related to finding and analyzing the conditions whereby a set of Pareto, Slater or Smale optimal solutions of a problem possesses some predetermined property that characterizes in a certain way its stability against small perturbations of input data. The paper continues studies in the correctness of vector optimization problems, including their solvability and stability, presented in particular in [1–9]. The results described therein expand the well-known class of vector optimization problems, stable with respect to input data perturbations for vector-valued criterion. Another well-known field in the analysis of the stability problem is oriented toward obtaining and investigating the quantitative characteristics of feasible modifications in input data of the problem, in particular, radius of the maximum stability sphere (see, for example, [10–12]).

PROBLEM STATEMENT. BASIC DEFINITIONS

Let us consider a vector optimization problem of the form

$$Q(F, X):\ \max\{F(x) \mid x \in X\}, \tag{1}$$

where X is a set from R^n of arbitrary structure, probably discrete; R^n is an n-dimensional real space; $F(x) = (f_1(x), \ldots, f_\ell(x))$, $\ell \geq 2$, $f_i : R^n \to R^1$ is a continuous function, $i \in N_\ell = \{1, \ldots, \ell\}$, and $X \neq \varnothing$. Let the problem (1) be: find elements of the set of Pareto optimal solutions

$$P(F, X) = \{x \in X \mid \pi(x, F, X) = \varnothing\},$$

where $\pi(x, F, X) = \{y \in X \mid F(y) \geq F(x),\ F(y) \neq F(x)\}$.

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Let us also consider the sets of Slater optimal solutions:

$$Sl(F,X)=\{x\in X\mid \sigma(x,F,X)=\varnothing\},$$

where $\sigma(x,F,X)=\{y\in X\mid F(y)>F(x)\}$, and of Smale optimal ones:

$$Sm(F,X)=\{x\in X\mid \eta(x,F,X)=\varnothing\},$$

where $\eta(x,F,X)=\{y\in X\mid y\neq x, F(y)\geq F(x)\}$.

It can be easily seen that

$$Sm(F,X)\subset P(F,X)\subset Sl(F,X) \tag{2}$$

and $\forall x\in X\;\; \sigma(x,F,X)\subset \pi(x,F,X)\subset \eta(x,F,X)$.

A Pareto set $P(F,X)$ is called externally stable if for any nonoptimal solution $x\in X\setminus P(F,X)$ there is an optimal solution $x'\in P(F,X)$ for which $F(x')\geq F(x)$. According to [13], finiteness of the nonempty subset X is a sufficient condition for the existence of Pareto optimal solutions of the vector problem and external stability of the Pareto set. However, in case of infinite feasible area X, the Pareto set may be not externally stable and be empty. According to the Podinovskii theorem [13], a Pareto set is non-empty and externally stable if the feasible set X of the problem is a nonempty compact set, i.e., it is bounded and closed, and criterion vector function $F(x)$ of the problem is (component-wise) upper semicontinuous on X.

Noteworthy is also the well-known result about closedness of the set of Slater optimal solutions for the problem of optimization of a continuous vector function on a closed feasible set [13]. The statement below, which is related to problem (1), follows from it.

Statement 1. Let feasible set X of the problem $Q(F,X)$ be closed. Then set $Sl(F,X)$ is also closed.

Note that the sets $P(F,X)$ and $Sm(F,X)$ of Pareto and Smale optimal solutions (for example, for the partially integer problem $Q(F,X)$) can be non-closed even if the feasible set X is closed. Appropriate examples for the problem with linear partial criteria are presented in [1].

For problem (1), as input data that can be subject to perturbations, let us consider coefficients of the vector-valued criterion F. Denote the set of such input data by $u\in U$, U is the space of input data of the problem. When necessary, along with the introduced notation $F(x)=(f_1(x),\dots,f_\ell(x))$ for the vector objective function and partial criteria of the problem $Q(F,X)$, we will also use the notation $F_u(x)=(f_1^u(x),\dots,f_\ell^u(x))$, which specifies which element u from the space U of input data corresponds to the considered problem.

For any natural number q, we will consider the real vector space R^q as normed one. We will specify the norm in R^q by the formula

$$||z||=\sum_{i\in N_q}||z_i||, \tag{3}$$

where $z=(z_1,\dots,z_q)\in R^q$, $N_q=\{1,\dots,q\}$. By the norm of some matrix $B=[b_{ij}]_{m\times k}\in R^{m\times k}$, we will understand the norm of vector $(b_{11},b_{12},\dots,b_{mk})$.

As is known [14], in the finite-dimensional space R^q, any two norms $||\cdot||^{(1)}$ and $||\cdot||^{(2)}$ are equivalent, i.e., there are numbers $\alpha>0$ and $\beta>0$ such that $\forall\, z\in R^q$ the inequalities are true: $\alpha||z||^{(1)}\leq ||z||^{(2)}\leq \beta||z||^{(1)}$. According to this equivalence, the results below are also true for other norms introduced in finite-dimensional space.

For a set of input data $u\in U$ and any number $\delta>0$, let us define the set of perturbed input data $O_\delta(u)=\{u(\delta)\in U\mid ||u(\delta)-u||<\delta\}$.

Let us consider the problem with perturbed input data of the vector-valued criterion $Q(F_{u(\delta)},X):\max\{F_{u(\delta)}(x)|x\in X\}$, where $u(\delta)\in O_\delta(u)$, $F_{u(\delta)}(x)=(f_1^{u(\delta)}(x),\dots,f_\ell^{u(\delta)}(x))$.

Let us define different types of stability against perturbations of input data for the vector-valued criterion of the problem (1), by propagating, to this class of problems, the concepts of T_1-, T_2-, T_3-, T_4-, and T_5-stability with respect to vector-valued criterion, introduced in [4] for completely integer problem of finding Pareto optimal solutions with quadratic partial criteria.

Definition 1. Problem $Q(F_u, X)$ is called T_1-stable in a vector-valued criterion if there is a number $\delta > 0$ such that for any perturbed set $u(\delta) \in O_\delta(u)$ of input data of the problem, the inequality holds:

$$P(F_u, X) \bigcap P(F_{u(\delta)}, X) \neq \varnothing.$$

Definition 2. Problem $Q(F_u, X)$ is called T_2-stable in a vector-valued criterion if there is a number $\delta > 0$ for which the inequality holds:

$$\bigcap_{u(\delta) \in O_\delta(u)} P(F_{u(\delta)}, X) \neq \varnothing.$$

Definition 3. Problem $Q(F_u, X)$ is called T_3-stable (T_4-, T_5-stable) in a vector-valued criterion if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that $\forall\, u(\delta) \in O_\delta(u)$ the condition is satisfied:

$$P(F_u, X) \bigcap O_\varepsilon(x(\delta)) \neq \varnothing \quad \forall\, x(\delta) \in P(F_{u(\delta)}, X) \tag{4}$$

(respectively, the condition

$$P(F_{u(\delta)}, X) \bigcap O_\varepsilon(x) \neq \varnothing \quad \forall\, x \in P(F_u, X) \tag{5}$$

for T_4-stability and both conditions, (4) and (5), for T_5-stability), where $O_\varepsilon(x) = \{x' \in R^n \mid \|x - x'\| < \varepsilon\}\ \ \forall\, x \in R^n$.

Note that condition (4) is equivalent to the inclusion $P(F_{u(\delta)}, X) \subset O_\varepsilon(P(F_u, X))$ and condition (5) to the inclusion $P(F_u, X) \subset O_\varepsilon(P(F_{u(\delta)}, X))$, where $O_\varepsilon(B) = \{x \in R^n \mid r(x, B) < \varepsilon\}$ is ε-neighborhood of some set $B \subset R^n$. Here, $r(x, B) = \inf_{y \in B} \|x - y\|$ is the distance between any point $x \in R^n$ and set B. Thus, T_3-stability (T_4-, T_5-stability) in the vector-valued criterion of problem $Q(F, X)$ means that the point–set mapping $P: U \to 2^X$, $u \to P(u) = P(F_u, X)$ is Hausdorff upper semicontinuous (respectively, lower semicontinuous, continuous) at point $u \in U$.

SUFFICIENT AND NECESSARY STABILITY CONDITIONS OF THE MULTI-OBJECTIVE PROBLEM

THEOREM 1. If set X is bounded and closed, then the equality

$$Sl(F, X) = \mathrm{cl}(P(F, X)), \tag{6}$$

where $\mathrm{cl}\, B$ is the closure of some set $B \subset R^n$, is a sufficient condition of T_3-stability in the vector-valued criterion of the problem $Q(F, X)$.

Proof. Assume the opposite. Let Eq. (6) be true but the problem $Q(F_u, X)$ be not T_3-stable in the vector-valued criterion. The latter means that $\exists\, \varepsilon > 0$ such that $\forall\, \delta > 0$ there is a perturbed set of input data $u(\delta) \in O_\delta(u)$ for which condition (4) is not satisfied. Then $\forall\, \delta > 0$ there is at least one solution $x_\delta \in P(F_{u(\delta)}, X)$ of the perturbed problem $Q(F_{u(\delta)}, X)$, which together with all the ε-neighborhood does not belong to the Pareto set of the problem $Q(F_u, X)$:

$$O_\varepsilon(x_\delta) \subset R^n \setminus P(F_u, X). \tag{7}$$

From the fact that $x_\delta \in P(F_{u(\delta)}, X)$ it follows that

$$\pi(x_\delta, F_{u(\delta)}, X) = \{z \in X \mid F_{u(\delta)}(z) \geq F_{u(\delta)}(x_\delta), F_{u(\delta)}(z) \neq F_{u(\delta)}(x_\delta)\} = \varnothing,$$

i.e., one of the relations is true $\forall\, z \in X$:

$$N^<(z) = \{i \in N_\ell \mid f_i^{u(\delta)}(z) < f_i^{u(\delta)}(x_\delta)\} \neq \varnothing,$$

$$N^=(z) = \{i \in N_\ell \mid f_i^{u(\delta)}(z) = f_i^{u(\delta)}(x_\delta)\} = N_\ell.$$

In that case, for any point $z \in X$ it is possible to select a stationary sequence $\{i_{\delta_r} \mid r \in N\}$ from the finite set $\{i_\delta \in N^<(z) \cup N^=(z) \mid \delta > 0\} \subset N_\ell$. And considering the Bolzano–Weierstrass theorem [14], it is also possible to select a converging sequence $\{x_{\delta_r} \mid r \in N\}$ from the bounded set $\{x_\delta \mid \delta > 0\} \subset X$, and $\lim_{r\to\infty} \delta_r = 0$. Denote $\tilde{x} = \lim_{r\to\infty} x_{\delta_r}$ and $i_0 = \lim_{r\to\infty} i_{\delta_r}$. With regard for the boundedness of the set X, we conclude that $\tilde{x} \in X$. As $r \to \infty$, we obtain the inequality $f_{i_0}^u(z) \le f_{i_0}^u(\tilde{x})$, which takes place for all $z \in X$. Thus, $\sigma(\tilde{x}, F_u, X) = \{z \in X \mid F_u(z) > F_u(\tilde{x})\} = \varnothing$ and $\tilde{x} \in Sl(F, X)$. Taking into account condition (6), we conclude that

$$\tilde{x} \in \mathrm{cl}(P(F_u, X)). \tag{8}$$

However, from inclusion (7) it follows that the inequalities $\|x - x_{\delta_r}\| \ge \varepsilon$, $r \in N$, are true $\forall x \in P(F, X)$, which in turn lead to the inequality $\|x - \tilde{x}\| \ge \varepsilon$ as $r \to \infty$. The latter means that $\tilde{x} \notin \mathrm{cl}(P(F_u, X))$ and it contradicts (8).

The theorem is proved.

THEOREM 2. Let the set X be bounded and closed. A sufficient condition of T_4-stability in the vector-valued criterion of the problem $Q(F, X)$ is the equality

$$\mathrm{cl}(P(F, X)) = \mathrm{cl}(Sm(F, X)). \tag{9}$$

Proof. Assume (by contradiction) that condition (9) is satisfied but the problem $Q(F_u, X)$ is not T_4-stable in the vector-valued criterion. The latter means that $\exists\, \varepsilon > 0$ such that $\forall\, \delta > 0$ there is a perturbed set of input data $u(\delta) \in O_\delta(u)$ for which condition (5) is not satisfied and hence there is at least one Pareto optimal solution $x_\delta^* \in P(F_u, X)$ that does not belong to the perturbed Pareto set $P(F_{u(\delta)}, X)$ together with its neighborhood $O_\varepsilon(x_\delta^*)$. Thus,

$$O_\varepsilon(x_\delta^*) \subset R^n \setminus P(F_{u(\delta)}, X) \tag{10}$$

and $\forall\, y \in P(F_{u(\delta)}, X) : \|x_\delta^* - y\| \ge \varepsilon$.

Since the set $\{x_\delta^* \mid \delta > 0\} \subset P(F_u, X) \subset X$ is bounded (due to the boundedness of X), according to the Bolzano–Weierstrass theorem, is possible to select a converging sequence $\{x_{\delta_r}^* \mid r \in N\}$ from it, for which $\lim_{r\to\infty} \delta_r = 0$. Let us introduce the notation $x^* = \lim_{r\to\infty} x_{\delta_r}^*$. Since the set X is closed, we conclude that

$$x^* \in \mathrm{cl}(P(F_u, X)) \subset X. \tag{11}$$

Let us consider the neighborhood $O_{\varepsilon/2}(x^*)$ for which (based on the definition of the limit of a sequence), we can specify a number $r_0 \in N$ such that $\forall\, r \ge r_0$ $x_{\delta_r}^* \in O_{\varepsilon/2}(x^*)$. Based on the inclusion (10), which takes place $\forall\, \delta > 0$, we conclude that $\forall\, r \ge r_0 : O_{\varepsilon/2}(x^*) \subset O_\varepsilon(x_{\delta_r}^*) \subset R^n \setminus P(F_{u(\delta_r)}, X)$, whence it follows that for any point $v \in O_{\varepsilon/2}(x^*) \bigcap X$ and number $r \ge r_0$, there is a Pareto optimal solution $\bar{x}_{\delta_r} = \bar{x}_{\delta_r}(v) \in P(F_{u(\delta_r)}, X)$ of the perturbed problem, for which the inequalities hold: $f_i^{u(\delta_r)}(\bar{x}_{\delta_r}) \ge f_i^{u(\delta_r)}(v)$, $i \in N_\ell$.

The last conclusion is made with regard for the external stability inherent in the set $P(F_{u(\delta_r)}, X)$ in view of the (component-wise) continuity of the criterion vector function of the problem $Q(F_{u(\delta_r)}, X)$ and of the fact that the nonempty feasible set X of this problem is a compact set [13].

Let us fix some point v from set $O_{\varepsilon/2}(x^*) \bigcap X$, consider a sequence $\{\bar{x}_{\delta_r} \mid r \ge r_0, r \in N\} \subset X$, and select a converging subsequence $\{\bar{x}(\delta_{r_k}) \mid k \in N\}$ from it, for which $\lim_{k\to\infty} \delta_{r_k} = 0$. Denote $\bar{x} = \lim_{k\to\infty} \bar{x}(\delta_{r_k})$. Since the set X is closed, we get $\bar{x} \in X$. It is obvious that $\|x^* - \bar{x}\| > \varepsilon/2$, $\bar{x} \notin O_{\varepsilon/2}(x^*)$, and hence, $\bar{x} \ne v$.

As $k \to \infty$, let us pass from the inequalities $f_i^{u(\delta_{r_k})}(\bar{x}(\delta_{r_k})) \ge f_i^{u(\delta_{r_k})}(v)$, $i \in N_\ell$, $k \in N$, to $f_i^u(\bar{x}) \ge f_i^u(v)$, $i \in N_\ell$.

The last inequalities together with the inequality $v \ne \bar{x}$ allow us to conclude that $\bar{x} \in \eta(v, F, X) \ne \varnothing$ and $v \notin Sm(F, X)$. Thus, $O_{\varepsilon/2}(x^*) \bigcap Sm(F, X) = \varnothing$ and point x^* is not a contact point of the set $Sm(F, X)$; therefore, it does not belong to the closure $\mathrm{cl}(Sm(F,X))$ of this set neither. Considering the assumption that $\mathrm{cl}(P(F,X))=\mathrm{cl}(Sm(F,X))$ is true, we conclude that $x^*\notin\mathrm{cl}(P(F,X))$, which contradicts (11).

The theorem is proved.

Theorems 1 and 2 obviously imply the following one.

THEOREM 3. If the set X is bounded and closed, then relations $Sl(F,X)=\mathrm{cl}(P(F,X))=\mathrm{cl}(Sm(F,X))$ are a sufficient condition of T_5-stability in the vector-valued criterion of the problem $Q(F,X)$.

Let us formulate the necessary conditions of T_3- and T_4-stability of problem (1) under the following additional conditions imposed on its objective vector function $F(x)=(f_1(x),\dots,f_\ell(x))$:

$$f_i(x)=g_i(x)+\langle c_i,x\rangle,\ i\in N_\ell, \tag{12}$$

where $f_i:R^n\to R^1$, $g_i:R^n\to R^1$, $c_i=(c_{i1},\dots,c_{in})\in R^n$. In particular, these can be quadratic and linear functions that compose the vector-valued criterion. Let us present the input data $u\in U$ for the considered vector-valued criterion as a pair $u=(u^g,C)$, where u^g is the set of all input data necessary to represent the functions $g_i(x)$, $i\in N_\ell$, $C=[c_{ij}]\in R^{\ell\times n}$.

THEOREM 4 [2]. Equality (6) is the necessary condition of T_3-stability in the vector-valued criterion of the problem $Q(F,X)$ with the linear partial criteria $f_i(x)=\langle c_i,x\rangle$, $i\in N_\ell$.

THEOREM 5. Let the set X be closed. Equality (9) is the necessary condition of T_4-stability in the vector-valued criterion of the problem $Q(F,X)$ with the partial criteria $f_i(x)$, $i\in N_\ell$, presented by formulas (12).

Proof. Assume (by contradiction) that for the problem $Q(F,X)$ that is T_4-stable by the vector-valued criterion the condition $\mathrm{cl}(P(F,X))=\mathrm{cl}(Sm(F,X))$ is not satisfied and hence, there is a point $v\in\mathrm{cl}(P(F,X))\setminus\mathrm{cl}(Sm(F,X))$. On the one hand, belonging of the point v to the closure $\mathrm{cl}(P(F,X))$ means that $\forall\varepsilon>0\ \exists\ y\in O_\varepsilon(v)\cap P(F,X)$. On the other hand, considering v as a point of the open set $R^n\setminus\mathrm{cl}(Sm(F,X))$, we conclude that $\exists\,\varepsilon'>0$: $O_{\varepsilon'}(v)\subset R^n\setminus\mathrm{cl}(Sm(F,X))$. This allows us to conclude that there exists the point $y=(y_1,\dots,y_n)\in O_{\varepsilon'}(v)\cap P(F,X)\subset R^n\setminus\mathrm{cl}(Sm(F,X))$. Since $y\in P(F,X)\setminus Sm(F,X)$, there is also a point $z=(z_1,\dots,z_n)\in\eta(y,F,X)\setminus\pi(y,F,X)$ for which the following relations are true:

$$F(z)=F(y),\ z\neq y. \tag{13}$$

To obtain a contradiction with the above-mentioned assumptions about T_4-stability of the problem $Q(F_u,X)$, let us show that $\exists\,\varepsilon''>0$ such that $\forall\delta>0$ there is a set of perturbed input data $u(\delta)\in O_\delta(u)$ for which $P(F_{u(\delta)},X)\cap O_{\varepsilon''}(y)=\varnothing$. In this connection, for an arbitrary $\delta>0$, let us introduce a perturbed set of input data $u(\delta)=(u^g,C(\delta))$, where the component u^g, which represents input data necessary to describe the functions g_i, $i\in N_\ell$, remains invariable as compared with the initial set $u=(u^g,C)\in U$ of input data, and let us construct the matrix $C(\delta)=[c_{ij}(\delta)]\in R^{\ell\times n}$ based on the following formulas for calculation of its separate elements:

$$c_{ij}(\delta)=c_{ij}+\alpha\,\mathrm{sgn}(z_j-y_j),\ i\in N_\ell,\ j\in N_n,\ 0<\alpha<\frac{\delta}{n\ell}.$$

Taking into account (3), it is easy to verify that $\|C(\delta)-C\|=\sum_{i\in N_\ell}\sum_{j\in N_n}|c_{ij}(\delta)-c_{ij}|<\delta$ and hence, $u(\delta)=(u^g,C(\delta))\in O_\delta(u)$. Moreover, with regard for formulas (3) and (13), the relations hold for all $i\in N_\ell$:

$$f_i^{u(\delta)}(z)-f_i^{u(\delta)}(y)=g_i(z)+\langle c_i(\delta),z\rangle-g_i(y)-\langle c_i(\delta),y\rangle=f_i^u(z)-f_i^u(y)$$

$$+\alpha\sum_{j\in N_n}(z_j-y_j)\mathrm{sgn}(z_j-y_j)=\alpha\sum_{j\in N_n}|z_j-y_j|=\alpha\|z-y\|>0.$$

Thus, $F_{u(\delta)}(z)-F_{u(\delta)}(y)>0$. Hence, $z\in\sigma(y,F_{u(\delta)},X)\neq\varnothing$, and according to the definition of a Slater set, point y does not belong to the perturbed set $Sl(F_{u(\delta)},X)$, which is closed according to Statement 1. Then point y belongs to the open set $R^n\setminus Sl(F_{u(\delta)},X)$ and there is a number $\varepsilon''>0$ for which the inclusion $O_{\varepsilon''}(y)\subset R^n\setminus Sl(F_{u(\delta)},X)$ holds.

Considering relations (2), we also obtain the inclusion $O_{\varepsilon''}(y) \subset R^n \setminus P(F_{u(\delta)}, X)$. However, the existence of such neighborhood $O_{\varepsilon''}(y)$ of point $y \in P(F_u, X)$ contradicts the assumption about T_4-stability of the problem $Q(F, X)$, because by the definition of T_4-stability, $\forall \varepsilon > 0 \ \exists \delta > 0$ such that $\forall u(\delta) \in O_\delta(u)$ condition (5) is satisfied.

CONCLUSIONS

In the paper, we have analyzed qualitative characteristics of various concepts of stability of vector optimization problems with continuous partial criteria functions and set of feasible solutions of arbitrary structure. We have analyzed the stability for the case of possible perturbations of input data of vector-valued optimization criterion and established the conditions whereby the set of Pareto optimal solutions of the problem has a predetermined property of invariancy with respect to external actions on input data. We have proved the sufficient (Theorems 1–3) and necessary (Theorem 5) stability conditions of three different types, which guarante that rather small variations in input data of the vector-valued criterion either do not generate new Pareto optimal solutions or preserve all Pareto optimal solutions of the problem and admit ocurrence of new ones, or do not change the set of Pareto optimal solutions of the original problem. The obtained results extend the well-known class of vector optimization problems that are stable against input data perturbations for the vector-valued criterion.

REFERENCES

1. L. N. Kozeratskaya, T. T. Lebedeva, and T. I. Sergienko, "Mixed integer vector optimization. Stability issues," Cybern. Syst. Analysis, Vol. 27, No. 1, 76–80 (1991).
2. L. N. Kozeratskaya, "Vector optimization problems: Stability in the decision space and in the space of alternatives," Cybern. Syst. Analysis, Vol. 30, No. 6, 891–899 (1994).
3. I. V. Sergienko, L. N. Kozeratskaya, and T. T. Lebedeva, Stability Analysis and Parametric Analysis of Discrete Optimization Problems [in Russian], Naukova Dumka, Kyiv (1995).
4. T. T. Lebedeva, N. V. Semenova, and T. I. Sergienko, "Stability of vector problems of integer optimization: Relationship with the stability of sets of optimal and nonoptimal solutions," Cybern. Syst. Analysis, Vol. 41, No. 4, 551–558 (2005).
5. T. T. Lebedeva and T. I. Sergienko, "Stability of a vector integer quadratic programming problem with respect to vector criterion and constraints," Cybern. Syst. Analysis, Vol. 42, No. 5, 667–674 (2006).
6. T. T. Lebedeva and T. I. Sergienko, "Different types of stability of vector integer optimization problem: General approach," Cybern. Syst. Analysis, Vol. 44, No. 3, 429–433 (2008).
7. T. T. Lebedeva, N. V. Semenova, and T. I. Sergienko, "Qualitative characteristics of the stability vector discrete optimization problems with different optimality principles," Cybern. Syst. Analysis, Vol. 50, No. 2, 228–233 (2014).
8. I. V. Sergienko, T. T. Lebedeva, and N. V. Semenova, "Existence of solutions in vector optimization problems," Cybern. Syst. Analysis, Vol. 36, No. 6, 823–828 (2000).
9. T. I. Sergienko, "Conditions of Pareto optimization problems solvability. Stable and unstable solvability," in: S. Butenko, P. Pardalos, and V. Shylo (eds.), Optimization Methods and Applications, Vol. 130, Springer Optimization and Its Applications, Springer, Cham (2017), pp. 457–464.
10. V. A. Emelichev and K. G. Kuzmin, "Stability radius of a vector integer linear programming problem: Case of a regular norm in the space of criteria," Cybern. Syst. Analysis, Vol. 46, No. 1, 72–79 (2010).
11. V. A. Emelichev, V. M. Kotov, K. G. Kuzmin, T. T. Lebedeva, N.V. Semenova, and T.I. Sergienko, "Stability and effective algorithms for solving multiobjective discrete optimization problems with incomplete information," J. Autom. Inform. Sci., Vol. 46, No. 2, 27–41 (2014).
12. V. Emelichev and Yu. Nikulin, "On the quasistability radius for a multicriteria integer linear programming problem of finding extremum solutions," Cybern. Syst. Analysis, Vol. 55, No. 6, 949–957 (2019).
13. V. V. Podinovskii and V. D. Nogin, Pareto Optimal Solutions to Multi-objective Problems [in Russian], Nauka, Moscow (1982).
14. I. I. Lyashko, V. F. Emel'yanov, and O. K. Boyarchuk, Mathematical Analysis, Pt. 1 [in Ukrainian], Vyshcha Shkola, Kyiv (1992).