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THE PROBLEM OF APPROACH OF CONTROLLED OBJECTS IN DYNAMIC GAME PROBLEMS WITH A TERMINAL PAYOFF FUNCTION

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Abstract. *To solve the problem of convergence of controlled objects in dynamic game problems with the terminal payoff function, the author proposes a method that systematically uses the Fenchel–Moreau ideas as applied to the general scheme of the method of resolving functions. The essence of the method is that the resolving function can be expressed in terms of the function conjugate to payoff function and, using the involution of the conjugation operator for a convex closed function, a guaranteed estimate of the terminal value of the payoff function is obtained, which can be presented in terms of the payoff value at the initial instant of time and integral of the resolving function. The concepts of upper and lower resolving functions of two types are introduced and sufficient conditions for a guaranteed result in a differential game with a terminal payoff function are obtained for the case where the Pontryagin condition does not hold. Two schemes of the method of resolving functions are considered, the corresponding control strategies are generated, and guaranteed times are compared. The results are illustrated by a model example.*

Keywords: *terminal payoff function, quasilinear differential game, multi-valued mapping, measurable selector, stroboscopic strategy, resolving function.*

INTRODUCTION

In the paper, we will consider the problem of approach of controlled objects in dynamic game problems with terminal payoff function on the basis of the method of resolving functions [1]. We will introduce the concepts of upper and lower resolving functions of two types and will obtain sufficient conditions for the guaranteed result in the differential game with terminal payoff function in the case where the Pontryagin condition is not satisfied. We will propose two schemes of the method of resolving functions, generate appropriate control strategies, and compare the guaranteed times. The results will be illustrated by a modeling example.

The paper continues the studies from [1, 2], is related to the publications [3–22], extends the class of solvable game problems of approach of controlled objects, and reveals new capabilities of applying convex analysis to the theory of conflict-controlled processes.

THE GENERAL SCHEME OF THE METHOD. RESOLVING FUNCTIONS OF THE FIRST TYPE

Let us consider a conflict-controlled process whose evolution can be described by the equality

$$z(t) = g(t) + \int_0^t \Omega(t, \tau)\varphi(u(\tau), \upsilon(\tau))\, d\tau, \ \ t \geq 0\,. \tag{1}$$

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Here, $z(t) \in R^n$, function $g(t)$, $g: R_+ \to R^n$, is Lebesgue measurable [8] and is bounded for $t > 0$, the matrix function $\Omega(t, \tau)$, $t \geq \tau \geq 0$, is measurable with respect to t, and is summable with respect to τ for each $t \in R_+$. The control unit is specified by the function $\varphi(u, \upsilon)$, $\varphi: U \times V \to R^n$, which is considered continuous in the set of variables on the direct product of nonempty compact sets U and V; m, l, and n are natural numbers.

Players' controls $u(\tau)$, $u: R_+ \to U$, and $\upsilon(\tau)$, $\upsilon: R_+ \to V$, are measurable functions of time. Along with the process (1), an eigenfunction $\sigma(z)$, $\sigma: R^n \to R^1$, is given, closed and bounded from below with respect to z and whose values on the trajectories of process (1) determine the time of the game termination. If $z(t)$, $t \geq 0$, is the trajectory of the system (1), then the game is considered terminated at time $t_1 > 0$ if

$$\sigma(z(t_1)) \leq 0. \tag{2}$$

The objectives of the first player u and of the second player υ are opposite. The first player (we will call it pursuer) tries to satisfy inequality (2) on the appropriate trajectory of process (1) in the shortest time, and the second player tries to postpone as much as possible the time of satisfying this inequality or to avoid it.

Let us take the first player's side and be guided by opponent's choice of the control as an arbitrary measurable function that takes values from V. We assume that if the game (1), (2) proceeds on the interval $[0, T]$, then we choose the control of the first player at time t based on the information about $g(T)$ and $\upsilon_t(\cdot)$, i.e., in the form of a measurable function

$$u(t) = u(g(T), \upsilon_t(\cdot)),\ t \in [0, T],\ u(t) \in U, \tag{3}$$

where $\upsilon_t(\cdot) = \{\upsilon(s): s \in [0, t]\}$ is the previous history of control of the second player by the time t, or in the form of countercontrol

$$u(t) = u(g(T), \upsilon(\cdot)),\ t \in [0, T],\ u(t) \in U. \tag{4}$$

If, in particular, $g(t) = e^{At} z_0$, $\Omega(t, \tau) = e^{A(t-\tau)}$, $z(0) = z_0$ and e^{At} is its matrix exponent, then control $u(t) = u(z_0, \upsilon_t(\cdot))$ is considered to implement a quasistrategy [6], and countercontrol [3] $u(t) = u(z_0, \upsilon(\cdot))$ is a manifestation of the Hajek stroboscopic strategy [7].

According to the definition of a conjugate function and taking into account the Fenchel–Moreau theorem [9], we get

$$\sigma(z) = \sup_{\psi \in R^n} [(\psi, z) - \sigma^*(\psi)],$$

where

$$\sigma^*(\psi) = \sup_{z \in R^n} [(\psi, z) - \sigma(z)]. \tag{5}$$

Function $\sigma^*(\psi)$ is a closed and convex eigenfunction [9]. The effective set of function $\sigma^*(\psi)$ has the form $\operatorname{dom} \sigma^* = \{\psi \in R^n : \sigma^*(\psi) < +\infty\}$. Since the eigenfunction $\sigma(z)$ is bounded from below and due to the relation (5), we obtain $\sigma^*(0) = -\inf_{z \in R^n} \sigma(z)$; therefore, $0 \in \operatorname{dom} \sigma^*$.

Let L be the linear hull of set $\operatorname{dom} \sigma^*$ (intersection of all linear subspaces that contain set $\operatorname{dom} \sigma^*$). Then it is a linear subspace. Denote by π operator of orthogonal projection from R^n onto L. The relation holds:

$$\sigma(z) = \sigma(\pi z),\ z \in R^n.$$

Denote $\varphi(U, \upsilon) = \{\varphi(u, \upsilon): u \in U\}$ and consider the multi-valued mapping on set $\Delta \times V$:

$$W(t, \tau, \upsilon) = \operatorname{co} \pi \Omega(t, \tau) \varphi(U, \upsilon),$$

where $\operatorname{co} A$ is convexification of set A [9], $\Delta = \{(t, \tau): 0 \leq \tau \leq t < \infty\}$.

Assume that mapping $\pi \Omega(t, \tau) \varphi(U, \upsilon)$ has closed values and that boundaries of sets $W(t, \tau, \upsilon)$ and $\pi \Omega(t, \tau) \varphi(U, \upsilon)$ coincide on the set $\Delta \times V$. Taking into account the assumptions about the matrix function $\Omega(t, \tau)$, we can conclude that for any fixed $t > 0$, vector function $\pi\Omega(t, \tau)\varphi(u, v)$ is $\mathfrak{L} \otimes \mathfrak{B}$-measurable with respect to $(\tau, v) \in [0, t] \times V$ and continuous with respect to $u \in U$. Therefore, for any fixed $t > 0$, the multi-valued mappings $\pi\Omega(t, \tau)\varphi(U, v)$ and $W(t, \tau, v)$ have closed values and are $\mathfrak{L} \otimes \mathfrak{B}$-measurable with respect to $(\tau, v) \in [0, t] \times V$ [8].

Condition 1 (Pontryagin's condition). Multi-valued mapping $\bigcap_{v \in V} \pi\Omega(t, \tau)\varphi(U, v)$ takes nonempty values on set Δ, where $\Delta = \{(t, \tau) : 0 \le \tau \le t < \infty\}$.

In convex analysis [9], support functions $C^*(X, \psi) = \sup_{x \in X} (\psi, x)$ and $C_*(X, \psi) = \inf_{x \in X} (\psi, x)$ play the key role in description of sets, where set X is an element of space R^n. We will call functions $C^*(X, \psi) = \sup_{x \in X} (\psi, x)$ upper and functions $C_*(X, \psi) = \inf_{x \in X} (\psi, x)$ lower support functions.

If set X is convex and closed, then there is a one-to-one correspondence between it and its upper and lower support functions [9], and

$$X = \{x : (x, \psi) \le C^*(X, \psi) \ \ \forall \psi \in R^n\} = \{x : (x, \psi) \ge C_*(X, \psi) \ \ \forall \psi \in R^n\}.$$

Remark 1. In [23], the concept of lexicographic minimum with respect to orthogonal basis $e_1, \ldots, e_n$ of the compact set $A \in K(R^n)$ is introduced by the formula

$$\operatorname*{lex\,min}_{e_1, \ldots, e_n} A = \bigcap_{k=0}^{n} A_k,$$

where $A_0 = A$, $A_k = \{x \in A_{k-1} : (x, e_k) = C_*(A_{k-1}, \psi)\}$, $C_*(A_{k-1}, \psi)$ is the lower support function of set A_{k-1}, $k = 1, \ldots, n$ [24]. The set $\operatorname*{lex\,min}_{e_1, \ldots, e_n} A$ consists of one point belonging to the set of extreme points of the convex hull of set A [24]. Given the compact-valued $\mathfrak{L} \otimes \mathfrak{B}$-measurable multi-valued mapping $U(\tau, v)$ and orthogonal basis such that $e_1 = \psi$, $\psi \in R^m$, $\psi \ne 0$, [24] the equality holds

$$\left(\operatorname*{lex\,min}_{e_1, \ldots, e_n} U(\tau, v), \psi \right) = C_*(U(\tau, v), \psi), \ \ \tau \in [0, t], \ \ v \in V, \ \ t > 0.$$

LEMMA 1 [24]. Let the multi-valued mapping $U(\tau, v)$ be compact-valued, $\mathfrak{L} \otimes \mathfrak{B}$-measurable, and $\psi \in R^m$, $\psi \ne 0$. Then there exists a $\mathfrak{L} \otimes \mathfrak{B}$-measurable selector $u(\tau, v)$ of the multi-valued mapping $U(\tau, v)$ such that $(u(\tau, v), \psi) = C_*(U(\tau, v), \psi)$, and which is a superpositionally measurable function [1], $\tau \in [0, t]$, $t > 0$, $v \in V$.

Let $\gamma(t, \tau)$, $\gamma : \Delta \to L$, $\Delta = \{(t, \tau) : 0 \le \tau \le t < \infty\}$, be some almost everywhere bounded function, measurable with respect to t and summable with respect to τ, $\tau \in [0, T]$, for each $t > 0$. Following [1], we will call it a shift function.

Condition 2. For some shift function $\gamma(t, \tau)$, $\gamma : \Delta \to L$, on the set $\Delta \times V$ the inequality holds $\max_{\psi \in \mathrm{dom}\, \sigma^*} C_*(W(t, \tau, v) - \gamma(t, \tau), \psi) \le 0$.

Remark 2. Condition 2 is equivalent to the inclusion $0 \in \mathrm{co}[W(t, \tau, v) - \gamma(t, \tau)]$ for all $\tau \in [0, t]$, $v \in V$, which does not guarantee that Condition 1 is satisfied. And if $W(t, \tau, v) = \pi\Omega(t, \tau)\varphi(U, v)$, then Condition 2 guarantees that Condition 1 is satisfied.

Let us fix some shift function $\gamma(t, \tau)$ and suppose that

$$\xi(t) = \xi(t, g(t), \gamma(t, \cdot)) = \pi g(t) + \int_0^t \gamma(t, \tau) d\tau.$$

Consider the set

$$\mathrm{P}(g(\cdot), \gamma(\cdot, \cdot)) = \{t \ge 0 : \sigma(\xi(t, g(t), \gamma(t, \cdot)) \le 0\}.$$

If the inequality in curly brackets holds for none $t \ge 0$, we suppose that $\mathrm{P}(g(\cdot), \gamma(\cdot, \cdot)) = \varnothing$.

THEOREM 1. Let for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$, which is a convex closed eigenfunction bounded from below with respect to z, Condition 2 be satisfied and for the respective shift function $\gamma(t, \tau)$ the set $\mathrm{P}(g(\cdot), \gamma(\cdot, \cdot))$ be non-empty and $\mathrm{P} \in \mathrm{P}(g(\cdot), \gamma(\cdot, \cdot))$. Then the game can be terminated at time P with the use of control (4).

Proof. Let $\upsilon(\tau)$ be an arbitrary measurable selector of the compact set V, $\tau \in [0, \mathrm{P}]$. Let us specify the method of the choice of control by the the pursuer.

Let us consider the multi-valued mapping $W(\mathrm{P}, \tau, \upsilon) - \gamma(\mathrm{P}, \tau)$ for $\tau \in [0, \mathrm{P}]$, $\upsilon \in V$. By virtue of Lemma 1 and Condition 2, there exists a $\mathfrak{L} \otimes \mathfrak{B}$-measurable selector $u_0(\tau, \upsilon)$ of the multi-valued mapping $W(\mathrm{P}, \tau, \upsilon) - \gamma(\mathrm{P}, \tau)$ such that for $\psi \in \operatorname{dom} \sigma^*$, $\psi \ne 0$, the inequality holds

$$(u_0(\tau, \upsilon) - \gamma(\mathrm{P}, \tau), \psi) = C_*(W(\mathrm{P}, \tau, \upsilon) - \gamma(\mathrm{P}, \tau), \psi) \le 0 . \tag{6}$$

Note that selector $u_0(\tau, \upsilon)$ is a superpositionally measurable function [1], $\tau \in [0, \mathrm{P}]$, $\upsilon \in V$. Let the control of the first player be $u_0(\tau) = u_0(\tau, \upsilon(\tau))$, $\tau \in [0, \mathrm{P}]$.

Taking into account the equality $\sigma(z(\mathrm{P})) = \sigma(\pi z(\mathrm{P}))$, formula (1), and definition of a conjugate function, we obtain

$$\sigma(z(\mathrm{P})) = \max_{\psi \in \operatorname{dom} \sigma^*} \left[(\psi, \xi(\mathrm{P}) + \int_0^{\mathrm{P}} (\psi, \pi\Omega(\mathrm{P}, \tau)\varphi(u_0(\tau), \upsilon(\tau)) - \gamma(\mathrm{P}, \tau)) d\tau - \sigma^*(\psi) \right] . \tag{7}$$

Then (6) and (7) determine the relation

$$\sigma(z(\mathrm{P})) \le \sigma(\xi(\mathrm{P}, g(\mathrm{P}), \gamma(\mathrm{P}, \cdot))) \le 0 ,$$

whence the inequality (2) follows at time P.

Remark 3. From Condition 1 it follows that there exists a measurable selector $\gamma(t, \tau)$, $\gamma(t, \tau) \in W(t, \tau)$, for which Condition 2 is satisfied and Theorem 1 is true.

Let us consider the multi-valued mapping

$$\mathfrak{A}(t, \tau, \upsilon) = \left\{ \alpha \ge 0 \colon \sup_{\psi \in \operatorname{dom} \sigma^*} [C_*(W(t, \tau, \upsilon) - \gamma(t, \tau), \psi) + \alpha[(\psi, \xi(t)) - \sigma^*(\psi)]] \le 0 \right\} .$$

Condition 3. On the set Δ, the inequality holds

$$\sup_{\upsilon \in V} \sup_{\psi \in \operatorname{dom} \sigma^*} [C_*(W(t, \tau, \upsilon) - \gamma(t, \tau), \psi) + \mathfrak{A}(t, \tau, \upsilon)[(\psi, \xi(t)) - \sigma^*(\psi)]] \le 0 .$$

If Condition 3 is satisfied, the multi-valued mapping $\mathfrak{A}(t, \tau, \upsilon)$ is non-empty on set $\Delta \times V$ and generates the upper and lower scalar resolving functions of the first type

$$\alpha^*(t, \tau, \upsilon) = \sup \{\alpha : \alpha \in \mathfrak{A}(t, \tau, \upsilon)\}, \ \alpha^*(t, \tau, \upsilon) = \inf \{\alpha : \alpha \in \mathfrak{A}(t, \tau, \upsilon)\}, \ \tau \in [0, t], \ \upsilon \in V.$$

In [1] it is shown that multi-valued mapping $\mathfrak{A}(t, \tau, \upsilon)$ is closed-valued, $\mathfrak{L} \otimes \mathfrak{B}$-measurable in the set of (τ, υ), $\tau \in [0, t]$, $\upsilon \in V$, and the upper and lower resolving functions (being respectively the upper and lower support functions of the multi-valued mapping $\mathfrak{A}(t, \tau, \upsilon)$ in the direction +1) are $\mathfrak{L} \otimes \mathfrak{B}$-measurable in the set of (τ, υ), $\tau \in [0, t]$, $\upsilon \in V$. Therefore, they are superpositionally measurable [1], i.e., $\alpha^*(t, \tau, \upsilon(\tau))$ and $\alpha_*(t, \tau, \upsilon(\tau))$ are measurable with respect to τ, $\tau \in [0, t]$, for any measurable function $\upsilon(\cdot) \in V(\cdot)$, where $V(\cdot)$ is the set of measurable functions $\upsilon(\tau)$, $\tau \in [0, +\infty]$, with values from V. Note also that the upper resolving function is upper semicontinuous and the lower one is lower semicontinuous with respect to the variable υ, and functions $\inf_{\upsilon \in V} \alpha^*(t, \tau, \upsilon)$ and $\sup_{\upsilon \in V} \alpha_*(t, \tau, \upsilon)$ are measurable with respect to τ, $\tau \in [0, t]$.

Let us consider the set

$$\mathrm{P}_*^1(g(\cdot),\gamma(\cdot,\cdot))=\left\{t\ge 0:\ \sigma(\xi(t,g(t),\gamma(t,\cdot)))\le 0,\ \int_0^t \sup_{\upsilon\in V}\alpha_*(t,\tau,\upsilon)d\tau<1\right\}.$$

If the inequalities in curly brackets hold for none $t\ge 0$, then we assume $\mathrm{P}_*^1(g(\cdot),\gamma(\cdot,\cdot))=\varnothing$.

THEOREM 2. Let for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$, which is a convex closed eigenfunction bounded from below in z, Condition 3 be satisfied and for the respective shift function $\gamma(t,\tau)$ the set $\mathrm{P}_*^1(g(\cdot),\gamma(\cdot,\cdot))$ be non-empty and $\mathrm{P}_*^1\in\mathrm{P}_*^1(g(\cdot),\gamma(\cdot,\cdot))$. Then the game can be terminated at time P_*^1 with the use of control (4).

Proof. Let $\upsilon(\tau)$ be an arbitrary measurable selector of the compact set V, $\tau\in[0,\mathrm{P}_*^1]$. Let us specify a method to choose the control by the pursuer.

By virtue of Condition 3 and Lemma 1, there exists a $\mathfrak{L}\otimes\mathfrak{B}$-measurable selector of the $u_*^1(\tau,\upsilon)$ multi-valued mapping $W(\mathrm{P}_*^1,\tau,\upsilon)-\gamma(\mathrm{P}_*^1,\tau)$ such that for $\psi\in\operatorname{dom}\sigma^*$, $\psi\ne 0$, $\upsilon\in V$, $\tau\in[0,\mathrm{P}_*^1]$ the following relations hold:

$$(u_*^1(\tau,\upsilon)-\gamma(\mathrm{P}_*^1,\tau),\psi)=C_*(W(\mathrm{P}_*^1,\tau,\upsilon)-\gamma(\mathrm{P}_*^1,\tau),\psi),$$

$$\sup_{\psi\in\operatorname{dom}\sigma^*}[(u_*^1(\tau,\upsilon)-\gamma(\mathrm{P}_*^1,\tau),\psi)+\alpha_*(\mathrm{P}_*^1,\tau,\upsilon)[(\psi,\xi(\mathrm{P}_*^1))-\sigma^*(\psi)]]\le 0. \tag{8}$$

Let the control of the first player have the form $u_*^1(\tau)=u_*^1(\tau,\upsilon(\tau))$, $\tau\in[0,\mathrm{P}_*^1]$. Adding and subtracting in square brackets of expression (7) $[(\psi,\xi(\mathrm{P}_*^1))-\sigma^*(\psi)]\int_0^{\mathrm{P}_*^1}\alpha_*(\mathrm{P}_*^1,\tau,\upsilon(\tau))d\tau$, we get

$$\sigma(z(\mathrm{P}_*^1))=\max_{\psi\in\operatorname{dom}\sigma^*}\left\{[(\psi,\xi(\mathrm{P}_*^1))-\sigma^*(\psi)]\left(1-\int_0^{\mathrm{P}_*^1}\alpha_*(\mathrm{P}_*^1,\tau,\upsilon(\tau))d\tau\right)\right.$$
$$+\int_0^{\mathrm{P}_*^1}[(\psi,\pi\Omega(\mathrm{P}_*^1,\tau)\varphi(u_*^1(\tau),\upsilon(\tau))-\gamma(\mathrm{P}_*^1,\tau))d\tau$$
$$\left.+\alpha_*(\mathrm{P}_*^1,\tau,\upsilon(\tau))[(\psi,\xi(\mathrm{P}_*^1))-\sigma^*(\psi)]]d\tau\right\}. \tag{9}$$

In view of the relations (8) and (9), the pursuer can guarantee that at time P_*^1 the inequality holds

$$\sigma(z(\mathrm{P}_*^1))\le\sigma(\xi(\mathrm{P}_*^1,g(\mathrm{P}_*^1),\gamma(\mathrm{P}_*^1,\cdot)))\left(1-\int_0^{\mathrm{P}_*^1}\alpha_*(\mathrm{P}_*^1,\tau,\upsilon(\tau))d\tau\right).$$

By the definition of P_*^1, we get $\sigma(\xi(\mathrm{P}_*^1,g(\mathrm{P}_*^1),\gamma(\mathrm{P}_*^1,\cdot)))\le 0$ and

$$1-\int_0^{\mathrm{P}_*^1}\alpha_*(\mathrm{P}_*^1,\tau,\upsilon(\tau))d\tau\ge 1-\int_0^{\mathrm{P}_*^1}\sup_{\upsilon\in V}\alpha_*(\mathrm{P}_*^1,\tau,\upsilon)d\tau>0.$$

Therefore,

$$\sigma(z(\mathrm{P}_*^1))\le\sigma(\xi(\mathrm{P}_*^1,g(\mathrm{P}_*^1),\gamma(\mathrm{P}_*^1,\cdot)))\left(1-\int_0^{\mathrm{P}_*^1}\alpha_*(\mathrm{P}_*^1,\tau,\upsilon(\tau))d\tau\right)\le 0,$$

which completes the proof of the theorem.

Remark 4. If for some shift function $\gamma(t,\tau)$, $\gamma:\Delta\to L$, on set $\Delta\times V$ Condition 2 is satisfied, then $0\in\mathfrak{A}(t,\tau,\upsilon)$, $\tau\in[0, \mathrm{P}_*^1]$, $\upsilon\in V$. Therefore, Condition 3 is satisfied and $\alpha_*(t,\tau,\upsilon)=\inf\{\alpha:\alpha\in\mathfrak{A}(t,\tau,\upsilon)\}=0$ on set $\Delta\times V$.

Condition 4. On set Δ Condition 3 is satisfied and the inequality holds:

$$\sup_{\upsilon\in V}\sup_{\psi\in\mathrm{dom}\,\sigma^*}[C_*(W(t,\tau,\upsilon)-\gamma(t,\tau),\psi)+\sup_{\upsilon\in V}\alpha_*(t,\tau,\upsilon)[(\psi,\xi(t))-\sigma^*(\psi)]]\le 0.$$

Remark 5. If for some shift function $\gamma(t,\tau)$, $\gamma:\Delta\to L$, on set $\Delta\times V$ Condition 2 is satisfied, then Condition 4 is true and $\sup_{\upsilon\in V}\alpha_*(t,\tau,\upsilon)=0$ on set Δ.

Let us consider set

$$T(g(t),\gamma(\cdot,\cdot))=\left\{t\ge 0:\int_0^t\inf_{\upsilon\in V}\alpha^*(t,\tau,\upsilon)\,d\tau\ge 1,\ \int_0^t\sup_{\upsilon\in V}\alpha_*(t,\tau,\upsilon)d\tau<1\right\}. \tag{10}$$

If for some $t>0$ $\alpha^*(t,\tau,\upsilon)\equiv+\infty$ for $\tau\in[0,t]$, $\upsilon\in V$, then it is natural to suppose the value of the respective integral in curly brackets in (10) to be equal to $+\infty$, and $t\in T(g(t),\gamma(\cdot,\cdot))$ if the second inequality in curly brackets in relation (10) is true for this t. If both inequalities in (10) do not hold for all $t>0$, put $T(g(t),\gamma(\cdot,\cdot))=\varnothing$.

THEOREM 3. Let for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$, which is a convex closed eigenfunction bounded from below in z, Condition 4 be satisfied and for the respective shift function $\gamma(\cdot,\cdot)$ set $T(g(\cdot),\gamma(\cdot,\cdot))$ be non-empty, and $T\in T(g(\cdot),\gamma(\cdot,\cdot))$. Then the game can be terminated at time T with the use of control (3).

Proof. Let $\upsilon(\tau)$ be an arbitrary measurable selector of the compact set V, $\tau\in[0,T]$.

First, let us consider the case $\sigma(\xi(T,g(T),\gamma(T,\cdot)))>0$ and introduce the control function

$$h(t)=1-\int_0^t\alpha^*(T,\tau,\upsilon(\tau))d\tau-\int_t^T\sup_{\upsilon\in V}\alpha_*(T,\tau,\upsilon)d\tau,\ t\in[0,T].$$

By the definition of T,

$$h(0)=1-\int_0^T\sup_{\upsilon\in V}\alpha_*(T,\tau,\upsilon)\,d\tau>0,$$

$$h(T)=1-\int_0^T\alpha^*(T,\tau,\upsilon(\tau))\,d\tau\le 1-\int_0^T\inf_{\upsilon\in V}\alpha^*(t,\tau,\upsilon)d\tau\le 0.$$

Therefore, since the function $h(t)$ is continuous, there exists a time t_*, $t_*\in(0,T]$ such that $h(t_*)=0$. Note that switching time t_* depends on the previous history of control of the second player $\upsilon_{t_*}(\cdot)=\{\upsilon(s):s\in[0,t_*]\}$.

We will call the time intervals $[0,t_*)$, $[t_*,T]$ active and passive, respectively. Let us describe the method of control for the first player on each interval.

By Condition 3 and Lemma 1, there exists a $\mathfrak{L}\otimes\mathfrak{B}$-measurable selector $u_1^*(\tau,\upsilon)$ of the multi-valued mapping $W(T,\tau,\upsilon)-\gamma(T,\tau)$ such that for $\psi\in\mathrm{dom}\,\sigma^*$, $\psi\ne 0$, $\upsilon\in V$, $\tau\in[0,t_*)$ the relations hold

$$(u_1^*(\tau,\upsilon)-\gamma(T,\tau),\psi)=C_*(W(T,\tau,\upsilon)-\gamma(T,\tau),\psi),$$

$$\sup_{\psi\in\mathrm{dom}\,\sigma^*}[(u_1^*(\tau,\upsilon)-\gamma(T,\tau),\psi)+\alpha^*(T,\tau,\upsilon)[(\psi,\xi(T))-\sigma^*(\psi)]]\le 0. \tag{11}$$

Let the first player's control on the active time interval be $u_1^*(\tau)=u_1^*(\tau,\upsilon(\tau))$, $\tau\in[0,t_*)$.

By virtue of Condition 4 and Lemma 1, there exists a $\mathfrak{L}\otimes\mathfrak{B}$-measurable selector $u_*^1(\tau,\upsilon)$ of the multi-valued mapping $W(T,\tau,\upsilon)-\gamma(T,\tau)$ such that for $\psi\in\mathrm{dom}\,\sigma^*$, $\psi\ne 0$, $\upsilon\in V$, $\tau\in[t_*,T]$ the relations hold:

$$(u_*^1(\tau, \upsilon) - \gamma(T, \tau), \psi) = C_*(W(T, \tau, \upsilon) - \gamma(T, \tau), \psi),$$

$$\sup_{\psi \in \mathrm{dom}\sigma^*} [(u_*^1(\tau, \upsilon) - \gamma(T, \tau), \psi) + \sup_{\upsilon \in V} \alpha_*(T, \tau, \upsilon)[(\psi, \xi(t)) - \sigma^*(\psi)]] \le 0. \qquad (12)$$

Let control of the first player on the passive time interval be $u_*^1(\tau) = u_*^1(\tau, \upsilon(\tau))$, $\tau \in [t_*, T]$.

In view of the equality $\sigma(z(T)) = \sigma(\pi z(T))$, formula (1), and definition of a conjugate function, we obtain

$$\sigma(z(T)) = \max_{\psi \in \mathrm{dom}\,\sigma^*} \left[(\psi, \xi(T)) + \int_0^{t_*} (\psi, \pi\Omega(T, \tau)\varphi(u_1^*(\tau), \upsilon(\tau)) - \gamma(T, \tau)) d\tau \right.$$

$$\left. + \int_{t_*}^{T} (\psi, \pi\Omega(T, \tau)\varphi(u_*^1(\tau), \upsilon(\tau)) - \gamma(T, \tau)) d\tau - \sigma^*(\psi) \right]. \qquad (13)$$

Adding and subtracting in square brackets of expression (13)

$$[(\psi, \xi(T)) - \sigma^*(\psi)] \left[\int_0^{t_*} \alpha^*(T, \tau, \upsilon(\tau))\, d\tau + \int_{t_*}^{T} \sup_{\upsilon \in V} \alpha_*(T, \tau, \upsilon)\, d\tau \right]$$

yield

$$\sigma(z(T)) = \max_{\psi \in \mathrm{dom}\,\sigma^*} \left\{ [(\psi, \xi(T)) - \sigma^*(\psi)] h(t_*) \right.$$

$$+ \int_0^{t_*} [(\psi, \pi\Omega(T, \tau)\varphi(u_1^*(\tau), \upsilon(\tau)) - \gamma(T, \tau)) + \alpha^*(T, \tau, \upsilon(\tau))[(\psi, \xi(T)) - \sigma^*(\psi)]] d\tau$$

$$\left. + \int_{t_*}^{T} [(\psi, \pi\Omega(T, \tau)\varphi(u_*^1(\tau), \upsilon(\tau)) - \gamma(T, \tau)) + \sup_{\upsilon \in V} \alpha_*(T, \tau, \upsilon)[(\psi, \xi(T)) - \sigma^*(\psi)]] d\tau \right\}.$$

From here, with regard for (11) and (12), it follows that the pursuer can guarantee that at time T the inequality holds:

$$\sigma(z(T)) \le \sigma(\xi(T, g(T), \gamma(T, \cdot))) h(t_*) = 0.$$

For the case $\sigma(\xi(T, g(T), \gamma(T, \cdot))) \le 0$, it will suffice to apply Theorem 2 and thus to complete the proof of the theorem.

Condition 5. On the set Δ, Condition 3 is satisfied and the inequality holds:

$$\sup_{\upsilon \in V} \sup_{\psi \in \mathrm{dom}\sigma^*} [C_*(W(t, \tau, \upsilon) - \gamma(t, \tau), \psi) + \inf_{\upsilon \in V} \alpha^*(t, \tau, \upsilon)[(\psi, \xi(t)) - \sigma^*(\psi)]] \le 0.$$

THEOREM 4. Let for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$, which is a convex closed eigenfunction bounded from below in z, Conditions 4 and 5 be satisfied and for the respective shift function $\gamma(\cdot, \cdot)$ set $T(g(\cdot), \gamma(\cdot, \cdot))$ be non-empty and $T \in T(g(\cdot), \gamma(\cdot, \cdot))$. Then the game can be terminated at time T with the use of control (4).

Proof. Let $\upsilon(\tau)$ be an arbitrary measurable selector of the compact set V, $\tau \in [0, T]$.

First, let us consider the case $\sigma(\xi(T, g(T), \gamma(T, \cdot))) > 0$ and introduce the control function

$$h(t) = 1 - \int_0^t \inf_{\upsilon \in V} \alpha^*(T, \tau, \upsilon) d\tau - \int_t^T \sup_{\upsilon \in V} \alpha_*(T, \tau, \upsilon) d\tau, \quad t \in [0, T].$$

By the definition of T,

$$h(0) = 1 - \int_0^T \sup_{\upsilon \in V} \alpha_*(T, \tau, \upsilon)\, d\tau > 0, \quad h(T) = 1 - \int_0^T \inf_{\upsilon \in V} \alpha^*(T, \tau, \upsilon)\, d\tau \le 0.$$

Since the function $h(t)$ is continuous, there exists an instant of time t_*, $t_* \in (0, T]$, such that $h(t_*) = 0$. Note that switching time t_* does not depend on the previous history of control of the second player $\upsilon_{t_*}(\cdot) = \{\upsilon(s): s \in [0, t_*]\}$.

We will call the time intervals $[0, t_*)$, $[t_*, T]$ active and passive, respectively. Let us describe the method of control for the first player on each time interval.

By virtue of Condition 5 and Lemma 1, there exists a $\mathfrak{L} \otimes \mathfrak{B}$-measurable selector $\tilde{u}_1^*(\tau, \upsilon)$ of the multi-valued mapping $W(T, \tau, \upsilon) - \gamma(T, \tau)$ such that for $\psi \in \operatorname{dom} \sigma^*$, $\psi \neq 0$, $\upsilon \in V$, $\tau \in [0, t_*)$ the relations hold:

$$(\tilde{u}_1^*(\tau, \upsilon) - \gamma(T, \tau), \psi) = C_*(W(T, \tau, \upsilon) - \gamma(T, \tau), \psi),$$

$$\sup_{\psi \in \operatorname{dom} \sigma^*} [(\tilde{u}_1^*(\tau, \upsilon) - \gamma(T, \tau), \psi) + \inf_{\upsilon \in V} \alpha^*(T, \tau, \upsilon)[(\psi, \xi(T)) - \sigma^*(\psi)]] \leq 0. \tag{14}$$

Let control of the first player on the active time interval be $\tilde{u}_1^*(\tau) = \tilde{u}_1^*(\tau, \upsilon(\tau))$, $\tau \in [0, t_*)$.

In view of Condition 4 and Lemma 1, there exists a $\mathfrak{L} \otimes \mathfrak{B}$-measurable selector $\tilde{u}_*^1(\tau, \upsilon)$ of the multi-valued mapping $W(T, \tau, \upsilon) - \gamma(T, \tau)$ such that for $\psi \in \operatorname{dom} \sigma^*$, $\psi \neq 0$, $\upsilon \in V$, $\tau \in [t_*, T]$ the relations hold:

$$(\tilde{u}_*^1(\tau, \upsilon) - \gamma(T, \tau), \psi) = C_*(W(T, \tau, \upsilon) - \gamma(T, \tau), \psi),$$

$$\sup_{\psi \in \operatorname{dom} \sigma^*} [(\tilde{u}_*^1(\tau, \upsilon) - \gamma(T, \tau), \psi) + \sup_{\upsilon \in V} \alpha_*(T, \tau, \upsilon)[(\psi, \xi(T)) - \sigma^*(\psi)]] \leq 0. \tag{15}$$

Let the control of the first player on the passive time interval be $\tilde{u}_*^1(\tau) = \tilde{u}_*^1(\tau, \upsilon(\tau))$, $\tau \in [t_*, T]$.

In view of the equality $\sigma(z(T)) = \sigma(\pi z(T))$, formula (1), and definition of a conjugate function, we obtain

$$\sigma(z(T)) = \max_{\psi \in \operatorname{dom} \sigma^*} \left[(\psi, \xi(T)) + \int_0^{t_*} (\psi, \pi\Omega(T, \tau)\varphi(\tilde{u}_1^*(\tau), \upsilon(\tau)) - \gamma(T, \tau)) d\tau \right.$$
$$\left. + \int_{t_*}^{T} (\psi, \pi\Omega(T, \tau)\varphi(\tilde{u}_*^1(\tau), \upsilon(\tau)) - \gamma(T, \tau)) d\tau - \sigma^*(\psi) \right]. \tag{16}$$

Adding and subtracting in square brackets of (16)

$$[(\psi, \xi(T)) - \sigma^*(\psi)] \left[\int_0^{t_*} \inf_{\upsilon \in V} \alpha^*(T, \tau, \upsilon)\, d\tau + \int_{t_*}^{T} \sup_{\upsilon \in V} \alpha_*(T, \tau, \upsilon)\, d\tau \right]$$

yield

$$\sigma(z(T)) = \max_{\psi \in \operatorname{dom} \sigma^*} \left\{ [(\psi, \xi(T)) - \sigma^*(\psi)] h(t_*) \right.$$
$$+ \int_0^{t_*} [(\psi, \pi\Omega(T, \tau)\varphi(\tilde{u}_1^*(\tau), \upsilon(\tau)) - \gamma(T, \tau)) + \inf_{\upsilon \in V} \alpha^*(T, \tau, \upsilon)[(\psi, \xi(T)) - \sigma^*(\psi)]] d\tau$$
$$\left. + \int_{t_*}^{T} [(\psi, \pi\Omega(T, \tau)\varphi(\tilde{u}_*^1(\tau), \upsilon(\tau)) - \gamma(T, \tau)) + \sup_{\upsilon \in V} \alpha_*(T, \tau, \upsilon)[(\psi, \xi(T)) - \sigma^*(\psi)]] d\tau \right\}.$$

Whence, with regard for (14) and (15), it follows that the pursuer can guarantee that at time T the inequality holds:

$$\sigma(z(T)) \leq \sigma(\xi(T, g(T), \gamma(T, \cdot))) h(t_*) = 0.$$

For the case $\sigma(\xi(T, g(T), \gamma(T, \cdot))) \leq 0$, it will suffice to apply Theorem 2 and thus to complete the proof of the theorem.

MODIFICATION OF THE METHOD. RESOLVING FUNCTIONS OF THE SECOND TYPE

Let us consider the multi-valued mapping

$$\mathfrak{A}(t,\tau)=\bigcap_{v\in V}\mathfrak{A}(t,\tau,v),\ (t,\tau)\in\Delta .$$

Condition 6. On the set Δ, the inequality holds:

$$\sup_{v\in V}\sup_{\psi\in\operatorname{dom}\sigma^*}[C_*(W(t,\tau,v)-\gamma(t,\tau),\psi)+\mathfrak{A}(t,\tau)[(\psi,\xi(t))-\sigma^*(\psi)]]\le 0 .$$

If Condition 6 is satisfied, then the multi-valued mapping $\mathfrak{A}(t,\tau)$ is non-empty on the set Δ and generates the upper and lower scalar resolving functions of the second type

$$\alpha^*(t,\tau)=\sup\{\alpha:\alpha\in\mathfrak{A}(t,\tau)\},\ \alpha_*(t,\tau)=\inf\{\alpha:\alpha\in\mathfrak{A}(t,\tau)\},\ \tau\in[0,t],\ v\in V.$$

As is shown in [1], the multi-valued mapping $\mathfrak{A}(t,\tau)$ is closed-valued, $\mathfrak{L}$-measurable in τ, $\tau\in[0,t]$, $v\in V$, and the upper and lower resolving functions (being respectively the upper and lower support functions of the multi-valued mapping $\mathfrak{A}(t,\tau)$ in the direction +1) are $\mathfrak{L}$-measurable in τ, $\tau\in[0,t]$.

Remark 6. If for some shift function $\gamma(t,\tau)$ on the set Δ Condition 4 is satisfied, then $\sup_{v\in V}\alpha_*(t,\tau,v)\in\mathfrak{A}(t,\tau)$, $\tau\in[0,t]$. Then Condition 6 is satisfied and the equality holds: $\sup_{v\in V}\alpha_*(t,\tau,v)=\alpha_*(t,\tau)$, $\tau\in[0,t]$. If for some shift function $\gamma(t,\tau)$ on the set Δ Condition 5 is satisfied, then $\inf_{v\in V}\alpha^*(t,\tau,v)\in\mathfrak{A}(t,\tau)$, $\tau\in[0,t]$. Then Condition 6 is satisfied and the equality holds: $\inf_{v\in V}\alpha^*(t,\tau,v)=\alpha^*(t,\tau)$, $\tau\in[0,t]$.

Let us consider the set

$$\mathrm{P}_*^2(g(\cdot),\gamma(\cdot,\cdot))=\left\{t\ge 0:\sigma(\xi(t,g(t),\gamma(t,\cdot)))\le 0,\ \int_0^t\alpha_*(t,\tau)\,d\tau<1\right\}.$$

If the inequalities in curly brackets hold for none $t\ge 0$, then we suppose that $\mathrm{P}_*^2(g(\cdot),\gamma(\cdot,\cdot))=\varnothing$.

THEOREM 5. Let for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$, which is a convex closed eigenfunction bounded from below in z, Condition 6 be satisfied and for the respective shift function $\gamma(t,\tau)$ set $\mathrm{P}_*^2(g(\cdot),\gamma(\cdot,\cdot))$ be non-empty and $\mathrm{P}_*^2\in\mathrm{P}_*^2(g(\cdot),\gamma(\cdot,\cdot))$. Then the game can be terminated at time P_*^2 with the use of control (4).

Proof. Let $v(\tau)$ be an arbitrary measurable selector of the compact set V, $\tau\in[0,\mathrm{P}_*^1]$. Let us specify the method of choosing the control by the pursuer.

In view of Condition 6 and Lemma 1, there exists a $\mathfrak{L}\otimes\mathfrak{B}$-measurable selector $u_*^2(\tau,v)$ of the multi-valued mapping $W(\mathrm{P}_*^2,\tau,v)-\gamma(\mathrm{P}_*^2,\tau)$ such that for $\psi\in\operatorname{dom}\sigma^*$, $\psi\ne 0$, $v\in V$, $\tau\in[0,\mathrm{P}_*^2]$ the relations hold:

$$(u_*^2(\tau,v)-\gamma(\mathrm{P}_*^2,\tau),\psi)=C_*(W(\mathrm{P}_*^2,\tau,v)-\gamma(\mathrm{P}_*^2,\tau),\psi),$$

$$\sup_{\psi\in\operatorname{dom}\sigma^*}[(u_*^2(\tau,v)-\gamma(\mathrm{P}_*^2,\tau),\psi)+\alpha_*(\mathrm{P}_*^2,\tau)[(\psi,\xi(\mathrm{P}_*^2))-\sigma^*(\psi)]]\le 0. \tag{17}$$

Let the control of the first player be $u_*^2(\tau) = u_*^2(\tau, \upsilon(\tau))$, $\tau \in [0, \mathrm{P}_*^2]$. Adding and subtracting in square brackets of (17) $[(\psi, \xi(\mathrm{P}_*^2)) - \sigma^*(\psi)] \int_0^{\mathrm{P}_*^2} \alpha_*(\mathrm{P}_*^2, \tau) d\tau$ yield

$$\sigma(z(\mathrm{P}_*^2)) = \max_{\psi \in \mathrm{dom}\,\sigma^*} \left\{ [(\psi, \xi(\mathrm{P}_*^2)) - \sigma^*(\psi)] \left(1 - \int_0^{\mathrm{P}_*^2} \alpha_*(\mathrm{P}_*^2, \tau) d\tau \right) \right.$$

$$\left. + \int_0^{\mathrm{P}_*^2} [(\psi, \pi\Omega(\mathrm{P}_*^2, \tau)\varphi(u_*^2(\tau), \upsilon(\tau)) - \gamma(\mathrm{P}_*^2, \tau)) + \alpha_*(\mathrm{P}_*^2, \tau)[(\psi, \xi(\mathrm{P}_*^2)) - \sigma^*(\psi)]] d\tau \right\}. \quad (18)$$

By virtue of (17) and (18), the pursuer can guarantee that at time P_*^2 the inequality holds:

$$\sigma(z(\mathrm{P}_*^2)) \le \sigma(\xi(\mathrm{P}_*^2, g(\mathrm{P}_*^2), \gamma(\mathrm{P}_*^2, \cdot))) \left(1 - \int_0^{\mathrm{P}_*^2} \alpha_*(\mathrm{P}_*^2, \tau) d\tau \right).$$

By the definition of P_*^2, we get $\sigma(\xi(\mathrm{P}_*^2, g(\mathrm{P}_*^2), \gamma(\mathrm{P}_*^2, \cdot))) \le 0$ and

$$1 - \int_0^{\mathrm{P}_*^2} \alpha_*(\mathrm{P}_*^2, \tau) d\tau > 0.$$

Thus,

$$\sigma(z(\mathrm{P}_*^2)) \le \sigma(\xi(\mathrm{P}_*^2, g(\mathrm{P}_*^2), \gamma(\mathrm{P}_*^2, \cdot))) \left(1 - \int_0^{\mathrm{P}_*^2} \alpha_*(\mathrm{P}_*^1, \tau) d\tau \right) \le 0,$$

which completes the proof of the theorem.

Remark 7. If for some shift function $\gamma(t, \tau)$, $\gamma : \Delta \to L$, on the set $\Delta \times V$ Condition 2 is satisfied, then $0 \in \mathfrak{A}(t, \tau)$, $\tau \in [0, t]$. Therefore, conditions 4 and 6 are satisfied and on the set Δ the equality holds: $\sup_{\upsilon \in V} \alpha_*(t, \tau, \upsilon) = \alpha_*(t, \tau) = 0$.

Let us consider the set

$$\Theta(g(t), \gamma(\cdot, \cdot)) = \left\{ t \ge 0 : \int_0^t \alpha^*(t, \tau)\, d\tau \ge 1, \int_0^t \alpha_*(t, \tau)\, d\tau < 1 \right\}. \quad (19)$$

If for some $t > 0$ we get $\alpha^*(t, \tau) \equiv +\infty$ for $\tau \in [0, t]$, $\upsilon \in V$, then it is natural to suppose the value of the respective integral in curly brackets in (19) to be equal to $+\infty$, and $t \in T(g(t), \gamma(\cdot, \cdot))$, if for this t the second inequality in curly brackets in (19) is true. If the inequalities in (19) do not hold for all $t > 0$, suppose $\Theta(g(t), \gamma(\cdot, \cdot)) = \varnothing$.

THEOREM 6. Let for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$, which is a convex closed eigenfucntion bounded from below in z, Condition 3 be satisfied and for the respective shift function $\gamma(\cdot, \cdot)$ the set $\Theta(g(\cdot), \gamma(\cdot, \cdot))$ be non-empty, and $\Theta \in \Theta(g(\cdot), \gamma(\cdot, \cdot))$. Then the game can be terminated at time Θ with the use of control (4).

Proof. Let $\upsilon(\tau)$ be an arbitrary measurable selector of the compact set V, $\tau \in [0, \Theta]$.

First, let us consider the case $\sigma(\xi(\Theta, g(\Theta), \gamma(\Theta, \cdot))) > 0$ and introduce the control function

$$h(t) = 1 - \int_0^t \alpha^*(\Theta, \tau) d\tau - \int_t^T \alpha_*(\Theta, \tau) d\tau, \; t \in [0, \Theta].$$

By the definition of Θ, we get

$$h(0) = 1 - \int_0^{\Theta} \alpha_*(\Theta, \tau)\, d\tau > 0, \; h(\Theta) = 1 - \int_0^{\Theta} \alpha^*(\Theta, \tau)\, d\tau \le 0.$$

Since the function $h(t)$ is continuous, there exists an instant of time t_*, $t_* \in (0, T]$ such that $h(t_*) = 0$. Note that the switching moment t_* does not depend on the previous history of second player's control $v_{t_*}(\cdot) = \{v(s) : s \in [0, t_*]\}$.

We will call the time intervals $[0, t_*)$ and $[t_*, \Theta]$ active and passive, respectively. Let us describe the method of control for the first player on each time interval.

By virtue of Condition 6 and Lemma 1, there exists a $\mathfrak{L} \otimes \mathfrak{B}$-measurable selector $\tilde{u}_2^*(\tau, v)$ of the multi-valued mapping $W(\Theta, \tau, v) - \gamma(\Theta, \tau)$ such that for $\psi \in \operatorname{dom} \sigma^*$, $\psi \neq 0$, $v \in V$, $\tau \in [0, t_*)$ the relations hold:

$$(\tilde{u}_2^*(\tau, v) - \gamma(\Theta, \tau), \psi) = C_*(W(\Theta, \tau, v) - \gamma(\Theta, \tau), \psi),$$

$$\sup_{\psi \in \operatorname{dom} \sigma^*} [(\tilde{u}_2^*(\tau, v) - \gamma(\Theta, \tau), \psi) + \alpha^*(\Theta, \tau)[(\psi, \xi(\Theta)) - \sigma^*(\psi)]] \leq 0. \tag{20}$$

Let the control of the first player on the active time interval be $\tilde{u}_2^*(\tau) = \tilde{u}_2^*(\tau, v(\tau))$, $\tau \in [0, t_*)$.

In view of Condition 6 and Lemma 1, there exists a $\mathfrak{L} \otimes \mathfrak{B}$-measurable selector $\tilde{u}_*^2(\tau, v)$ of the multi-valued mapping $W(\Theta, \tau, v) - \gamma(\Theta, \tau)$ such that for $\psi \in \operatorname{dom} \sigma^*$, $\psi \neq 0$, $v \in V$, $\tau \in [t_*, \Theta]$ the relations hold:

$$(\tilde{u}_*^2(\tau, v) - \gamma(\Theta, \tau), \psi) = C_*(W(\Theta, \tau, v) - \gamma(\Theta, \tau), \psi),$$

$$\sup_{\psi \in \operatorname{dom} \sigma^*} [(\tilde{u}_*^2(\tau, v) - \gamma(\Theta, \tau), \psi) + \alpha_*(\Theta, \tau)[(\psi, \xi(\Theta)) - \sigma^*(\psi)]] \leq 0. \tag{21}$$

Let the first player's control on the passive time interval be $\tilde{u}_*^2(\tau) = \tilde{u}_*^2(\tau, v(\tau))$, $\tau \in [t_*, \Theta]$.

With regard for the equality $\sigma(z(\Theta)) = \sigma(\pi z(\Theta))$, formula (1), and definition of a conjugate function, we obtain

$$\sigma(z(\Theta)) = \max_{\psi \in \operatorname{dom} \sigma^*} \left[(\psi, \xi(\Theta)) + \int_0^{t_*} (\psi, \pi\Omega(\Theta, \tau)\varphi(\tilde{u}_2^*(\tau), v(\tau)) - \gamma(\Theta, \tau)) d\tau \right.$$
$$\left. + \int_{t_*}^{\Theta} (\psi, \pi\Omega(\Theta, \tau)\varphi(\tilde{u}_*^2(\tau), v(\tau)) - \gamma(\Theta, \tau)) d\tau - \sigma^*(\psi) \right]. \tag{22}$$

Adding and subtracting in square brackets in (22)

$$[(\psi, \xi(\Theta)) - \sigma^*(\psi)] \left[\int_0^{t_*} \alpha^*(\Theta, \tau)\, d\tau + \int_{t_*}^{\Theta} \alpha_*(\Theta, \tau)\, d\tau \right]$$

yield

$$\sigma(z(\Theta)) = \max_{\psi \in \operatorname{dom} \sigma^*} \left\{ [(\psi, \xi(\Theta)) - \sigma^*(\psi)] h(t_*) \right.$$
$$+ \int_0^{t_*} [(\psi, \pi\Omega(\Theta, \tau)\varphi(\tilde{u}_2^*(\tau), v(\tau)) - \gamma(\Theta, \tau)) + \alpha^*(\Theta, \tau)[(\psi, \xi(\Theta)) - \sigma^*(\psi)]] d\tau$$
$$\left. + \int_{t_*}^{\Theta} [(\psi, \pi\Omega(\Theta, \tau)\varphi(\tilde{u}_*^2(\tau), v(\tau)) - \gamma(\Theta, \tau)) + \alpha_*(\Theta, \tau)[(\psi, \xi(\Theta)) - \sigma^*(\psi)]] d\tau \right\},$$

whence with regard for (20) and (21) it follows that the pursuer can guarantee that at time Θ the inequality holds:

$$\sigma(z(\Theta)) \leq \sigma(\xi(\Theta, g(\Theta), \gamma(\Theta, \cdot))) h(t_*) = 0.$$

For the case $\sigma(\xi(\Theta, g(\Theta), \gamma(\Theta, \cdot))) \leq 0$, it will suffice to apply Theorem 2 and thus to complete the proof of the theorem.

COMPARING THE GUARANTEED TIMES

LEMMA 2. Let for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$, which is a convex closed eigenfunction bounded from below in z, and for some shift function $\gamma(t,\cdot)$ Condition 6 be satisfied, and $\sigma(\xi(t, g(t), \gamma(t,\cdot))) > 0$. Then the inequalities hold:

$$\sup_{v \in V} \alpha_*(t,\tau,v) \le \alpha_*(t,\tau), \ (t,\tau) \in \Delta, \tag{23}$$

$$\inf_{v \in V} \alpha^*(t,\tau,v) \ge \alpha^*(t,\tau), \ (t,\tau) \in \Delta. \tag{24}$$

If Condition 4 is also satisfied, then inequality (23) is transformed to the equality. If Condition 5 is satisfied, then inequality (24) becomes the equality. If the multi-valued mapping $\mathfrak{A}(t,\tau,v)$ takes convex values on set $\Delta \times V$, then Conditions 4 and 5 are satisfied and the equality takes place in relations (23) and (24).

THEOREM 7. Let for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$, which is a convex closed eigenfunction bounded from below in z, for some shift function $\gamma(\cdot,\cdot)$ Condition 6 be satisfied. Then the inclusions takes place:

$$T(g(\cdot),\gamma(\cdot,\cdot)) \supset \Theta(g(\cdot),\gamma(\cdot,\cdot)) \supset \mathrm{P}_*^2(g(\cdot),\gamma(\cdot,\cdot)) \supset \mathrm{P}_*^1(g(\cdot),\gamma(\cdot,\cdot)) \supset \mathrm{P}(g(\cdot),\gamma(\cdot,\cdot)).$$

If Conditions 4 and 5 are also satisfied or if the multi-valued mapping $\mathfrak{A}(t,\tau,v)$ takes convex values on set $\Delta \times V$, then the equalities are true:

$$T(g(\cdot),\gamma(\cdot,\cdot)) = \Theta(g(\cdot),\gamma(\cdot,\cdot)), \ \mathrm{P}_*^2(g(\cdot),\gamma(\cdot,\cdot)) = \mathrm{P}_*^1(g(\cdot),\gamma(\cdot,\cdot)).$$

If Condition 2 is satisfied, then

$$\mathrm{P}_*^2(g(\cdot),\gamma(\cdot,\cdot)) = \mathrm{P}_*^1(g(\cdot),\gamma(\cdot,\cdot)) = \mathrm{P}(g(\cdot),\gamma(\cdot,\cdot)),$$

and if Condition 1 is satisfied, some Pontryagin selector can be chosen as $\gamma(\cdot,\cdot)$ [2].

The proof of Lemma 2 and Theorem 7 immediately follows from the structures of the respective definitions, remarks, and theorems.

THEOREM 8. Let for the conflict-controlled process (1), (2) with the terminal functional $\sigma(z)$, which is a convex closed eigenfunction bounded from below in z, Condition 3 be satisfied and for the respective shift function $\gamma(\cdot,\cdot)$ set $T(g(\cdot),\gamma(\cdot,\cdot))$ be non-empty, $T \in T(g(\cdot),\gamma(\cdot,\cdot))$ and the multi-valued mapping $\mathfrak{A}(T,\tau,v)$ take convex values for all (τ, v), $\tau \in [0,T]$, $v \in V$. Then the game can be terminated at time T with the use of control (4).

The proof immediately follows from Lemma 2 and Theorems 6 and 7.

ILLUSTRATIVE EXAMPLE

Let us consider a simple motion $\dot{z} = u - v$, $z \in R^n$, $z(0) = z_0$, $v \in S$, $u \in aS^0$, $a > 1$, where S is a unit full-sphere with the center at zero, S^0 is its boundary.

Let us choose the shift function $\gamma(t,\tau) \equiv 0$. Since $\Omega(t,\tau) = E$, E is a unit matrix, $L = R^n$ and $\pi = E$, we get $\xi(t) = z_0$. Suppose $\sigma^*(\psi) = \varepsilon \|\psi\|$, $\psi \in R^n$, $\sigma(z) = \max_{\|\psi\|=1} [(z,\psi) - \varepsilon \|\psi\|]$.

The Pontryagin condition is not satisfied since $aS^0 \stackrel{*}{-} S = \varnothing$, $\stackrel{*}{-}$ is the Minkowsky geometrical difference [22]. Then the multi-valued mapping $\mathfrak{A}(t,\tau,v)$ does not depend on t, τ and has the form

$$\mathfrak{A}(t,\tau,v) = \mathfrak{A}(v, z_0) = \left\{ \alpha \ge 0 : \max_{\|\psi\|=1} [C_*(aS^0 - v, \psi) + \alpha[(z_0,\psi) - \varepsilon\|\psi\|]] \le 0 \right\}.$$

This set possesses nonempty images and therefore the inequality holds:

$$\max_{v\in S}\max_{\|\psi\|=1}[-a\|\psi\|-(v,\psi)+\mathfrak{A}(v,z_0)[(z_0,\psi)-\varepsilon\|\psi\|]]\le 0\,.$$

Thus, Condition 3 is satisfied.

The upper resolving function can be found from the relation

$$\alpha^*(t,\tau,v)=\alpha^*(v,z_0)=\sup\{\alpha\ge 0:\max_{\|\psi\|=1}[-a\|\psi\|-(v,\psi)+\alpha[(z_0,\psi)-\varepsilon\|\psi\|]]\le 0\}$$

$$=\sup\{\alpha>0:\|v-\alpha z_0\|=[a+\alpha\varepsilon]\}\,.$$

From here it follows that it is the larger positive root of the quadratic equation

$$(\|z_0\|^2-\varepsilon^2)\alpha^2-2[(v,z_0)+a\varepsilon]\,\alpha-(a^2-\|v\|^2)=0\,.$$

Thus, the formula is true:

$$\alpha^*(v,z_0)=\frac{(v,z_0)+a\varepsilon+\sqrt{[(v,z_0)+a\varepsilon]^2+(\|z_0\|^2-\varepsilon^2)(a^2-\|v\|^2)}}{\|z_0\|^2-\varepsilon^2}\,.$$

Here, $\min\limits_{v\in S}\alpha^*(v,z_0)=\dfrac{a-1}{\|z_0\|-\varepsilon}$ is attained for $v=-\dfrac{z_0}{\|z_0\|}$.

The lower resolving function can be found from the relation

$$\alpha_*(t,\tau,v)=\alpha_*(v,z_0)=\inf\{\alpha\ge 0:\max_{\|\psi\|=1}[-a\|\psi\|-(v,\psi)+\alpha[(z_0,\psi)-\varepsilon\|\psi\|]]\le 0\}$$

$$=\sup\{\alpha\ge 0:\max_{\|\psi\|=1}[-a\|\psi\|-(v,\psi)-\alpha[(z_0,-\psi)-\varepsilon\|\psi\|]]\le 0\}$$

$$=\sup\{\alpha\ge 0:\|v-\alpha z_0\|=[a-\alpha\varepsilon]\}\,.$$

Therefore, it is the larger positive root of the quadratic equation

$$(\|z_0\|^2-\varepsilon^2)\alpha^2-2[(v,z_0)-a\varepsilon]\alpha-(a^2-\|v\|^2)=0\,.$$

Hence, we obtain

$$\alpha_*(v,z_0)=\frac{(v,z_0)-a\varepsilon+\sqrt{[(v,z_0)-a\varepsilon]^2+(\|z_0\|^2-\varepsilon^2)(a^2-\|v\|^2)}}{\|z_0\|^2-\varepsilon^2}\,.$$

Here, $\max\limits_{v\in S}\alpha_*(v,z_0)=\dfrac{a+1}{\|z_0\|+\varepsilon}$ is attained for $v=\dfrac{z_0}{\|z_0\|}$.

Let us test Condition 4. By virtue of the construction of the upper and lower resolving functions, the condition should be satisfied: $\min\limits_{v\in S}\alpha^*(v,z_0)\ge\max\limits_{v\in S}\alpha_*(v,z_0)$, which leads to the inequality

$$[a-1]\ge\frac{a+1}{\|z_0\|+\varepsilon}[\|z_0\|-\varepsilon]\,. \tag{25}$$

The relations hold:

$$\max_{v\in S}\max_{\|\psi\|=1}[-a\|\psi\|+(v,\psi)+\max_{v\in S}\alpha_*(v,z_0)[(z_0,\psi)-\varepsilon\|\psi\|]]$$

$$=\max_{\|\psi\|=1}[-[a-1]\|\psi\|+\max_{v\in S}\alpha_*(v,z_0)[(z_0,\psi)-\varepsilon\|\psi\|]]=-[a-1]+\frac{a+1}{\|z_0\|+\varepsilon}[\|z_0\|-\varepsilon]\le 0\,.$$

Then by virtue of inequality (25) Condition 4 is satisfied.

Let us test Condition 5. The relations hold:

$$\max_{v\in S}\max_{\|\psi\|=1}[-a\|\psi\|+(v,\psi)+\inf_{v\in S}\alpha^*(v,z_0)[(z_0,\psi)-\varepsilon\|\psi\|]]$$

$$=\max_{\|\psi\|=1}[-[a-1]\|\psi\|+\inf_{v\in S}\alpha^*(v,z_0)[(z_0,\psi)-\varepsilon\|\psi\|]]$$

$$=-[a-1]+\frac{a-1}{\|z_0\|-\varepsilon}[\|z_0\|-\varepsilon]=0.$$

Therefore, Condition 5 is true.

In this example, we get

$$\int_0^T \min_{v\in V}\alpha^*(v,z_0)\,d\tau=\frac{a-1}{\|z_0\|-\varepsilon}T=1,\ T=T(z_0)=\frac{\|z_0\|-\varepsilon}{a-1}.$$

If the game parameters satisfy the condition

$$1>\frac{\varepsilon}{\|z_0\|}>\frac{1}{a},\ a>1,$$

then the inequality is true:

$$\int_0^T \max_{v\in V}\alpha_*(v,z_0)\,d\tau=\frac{a+1}{\|z_0\|+\varepsilon}T=\frac{(a+1)}{(\|z_0\|+\varepsilon)}\frac{(\|z_0\|-\varepsilon)}{(a-1)}<1.$$

Hence, for the example under study, all the conditions of Theorems 3 and 4 are satisfied. By virtue of Lemma 2 and Remark 6, the conditions of Theorem 6 are true for the example.

CONCLUSIONS

In the paper, we have considered quasilinear conflict-controlled processes of general form, with terminal payoff function. We have formulated sufficient conditions of game termination in a finite guaranteed time in the case where the Pontryagin condition is not satisfied. We have proposed two schemes of the method of resolving functions that ensure termination of the conflict-controlled process with terminal payoff function in the class of quasistrategies and countercontrols and have compared the guaranteed times. We have provided an illustrative example of approach of controlled objects with a simple motion in order to obtain the upper and lower resolving functions in explicit form, which allow us to make a conculsions if the game can be terminated when the Pontryagin condition does not hold.

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