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CHEBYSHEV APPROXIMATION BY A RATIONAL EXPRESSION FOR FUNCTIONS OF MANY VARIABLES

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Abstract. *The method of constructing the Chebyshev approximation by a rational expression for functions of many variables is proposed. The idea of the method is based on constructing the boundary mean-power approximation in* E^p *norm as* $p \to \infty$. *The least squares method with two variable weight functions is used to construct this approximation. One weight function ensures the construction of mean-power approximation, and the other one refines parameters of the rational expression by linearization scheme. The convergence of the method is provided by the original method of sequentially refining the values of the weight functions. Algorithms for calculating the parameters of the Chebyshev approximation of functions of many variables by a rational expression with absolute and relative errors is described.*

Keywords: *Chebyshev approximation by rational expression, functions of many variables, mean-power approximation, least squares method.*

INTRODUCTION

Let a continuous real function $f(X)$ of *n* variables $X = (x_1, x_2, \dots, x_n)$ be defined on the point set $\Omega = \{X_j\}_{j=1}^{s}$ from the bounded domain $D, \Omega \subset D$, where $D \subset R^n$ and R^n is a *n*-dimensional vector space. Function $f(X)$ needs to be approximated on the point set Ω by the irreducible rational expression:

$$R_{k,l}(a, b; X) = \frac{\sum_{i=0}^{k} a_i \varphi_i(X)}{\sum_{i=0}^{l-1} b_i \psi_i(X) + \psi_l(X)}, \tag{1}$$

where $\varphi_i(X)$, $i = \overline{0, k}$, and $\psi_i(X)$, $i = \overline{0, l}$, are systems of linearly independent real functions continuous on *D*, and a_i, $i = \overline{0, k}$, and b_i, $i = \overline{0, l-1}$, are unknown parameters: $\{a_i\}_{i=0}^{k} \in A$, $A \subseteq R^{k+1}$, $\{b_i\}_{i=0}^{l-1} \in B$, $B \subseteq R^l$.

To construct the Chebyshev approximation by the rational expression (1) for the function $f(X)$ on the point set Ω means to calculate the values of parameters a^* and b^* for which the condition is satisfied:

$$\max_{X \in \Omega} | f(X) - R_{k,l}(a^*, b^*; X) | = \min_{a \in A,\, b \in B} \max_{X \in \Omega} | f(X) - R_{k,l}(a, b; X) |. \tag{2}$$

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In many cases, for the same number of parameters, approximation by a rational expression provides a better approximation accuracy as compared with polynomial approximation [1]. Chebyshev approximation by a rational expression is used to represent elementary and special mathematical functions [1], to approximate solutions of differential and integral equations [1, 2], in neural networks [3], etc.

A number of methods have been developed to calculate parameters of the Chebyshev approximation of functions of one variable [1, 4]. In particular, a combined algorithm is proposed in [4] to construct the Chebyshev approximation of functions of one variable, which takes into account the advantages of the Remez and Werner methods. The paper [5] describes an algorithm for calculating the parameters of the Chebyshev approximation by a rational expression of functions of one variable based on the Remez scheme using differential correction, and paper [6] provides a variant of software implementation of determining the parameters of a rational expression with the use of this algorithm. In [1, 7], construction of the Chebyshev approximation of functions of many variables by a rational expression is reduced to sequential solution of a linear programming problem. However, nonlinear optimization methods are most often used to calculate parameters of the Chebyshev approximation of functions of many variables [8].

In this paper, we propose a method for constructing the Chebyshev approximation of functions of many variables by a rational expression as a boundary approximation in the norm of space L^p as $p \to \infty$. It is based on the method described in [8, 9] and implies sequential construction of mean-power approximations. Mean-power approximations by a rational expression are calculated by the least squares method using two variable weight functions whose values are specified with regard for all previous approximations [9]. Parameters of rational approximation by the least squares method can be found with the use of linearization [10, 11]. The method of constructing the Chebyshev approximation by a rational expression based on mean-power approximations is described in [12] for functions of one variable and in [13] for functions of two variables.

To estimate the error of mean-power approximation of the function $f(X)$ defined on the point set Ω, the norm of the space E^p $(1 \le p < \infty)$

$$||\Delta||_{E^p} = \left(\sum_{X \in \Omega} |\Delta(X)|^p \right)^{1/p} \tag{3}$$

is used, where $\Delta(X) = f(X) - R_{k,l}(a, b; X)$. The boundary value of the norm $||\Delta||_{E^p}$ as $p \to \infty$ similarly to the norm of space L^p as $p \to \infty$ corresponds to the norm in the space of continuous functions $||\Delta||_C$ [1].

CALCULATING THE PARAMETERS OF THE CHEBYSHEV APPROXIMATION BY THE RATIONAL EXPRESSION

If a continuous Chebyshev approximation by the rational expression $R_{k,l}(a, b; X)$ (1) for the function $f(X)$ on the point set Ω exists, then construction of such an approximation is based on the idea of sequential calculation of mean-power approximations for $p = 2, 3, 4, \ldots$ in the space E^p with the norm (3). To construct the mean-power approximation of function $f(X)$ by the rational expression (1) in the space E^p, the iteration scheme based on the least squares method is used [9]

$$\sum_{X \in \Omega} \rho_r(X)(f(X) - R_{k,l}(a, b; X))^2 \xrightarrow[a \in A,\, b \in B]{} \min, \quad r = 0, 1, \ldots, p-2, \tag{4}$$

with sequential refining of the values of the weight function $\rho_r(X)$

$$\rho_0(X) = 1, \quad \rho_r(X) = \prod_{i=1}^{r} |\Delta_i(X)|, \quad r = 1, \ldots, p-2, \quad p = 3, 4, \ldots, \tag{5}$$

where $\Delta_s(X) = f(X) - R_{k,l,s-1}(a, b; X)$, $s = \overline{1, r}$, $R_{k,l,s}(a, b; X)$ is the least squares approximation of the function $f(X)$ with the weight function $\rho_s(X)$, which corresponds to the mean-power approximation of power $p = s + 2$.

The possibility of obtaining the Chebyshev approximation as a boundary approximation in the space L^p as $p \to \infty$ was studied in detail in [14], where E. Ya. Remez theoretically substantiated the convergence of computational schemes for constructing the Chebyshev approximation on the basis of mean-power approximation.

Least squares approximation by a rational expression is a nonlinear problem. To construct such an approximation, linearization is used, which implies iterative refinement of the approximation by the rational expression (1) using a variable weight function [10, 11]. According to this method of linearization, for each fixed value p, approximation of the function $f(X)$ by the rational expression $R_{k,l}(a,b;X)$ (1) by the least squares method is calculated:

$$\sum_{X\in\Omega}\rho_r(X)\,\upsilon_{r,t}(X)(\Phi_{r,t}(a,b;X))^2 \xrightarrow[a\in A,\, b\in B]{} \min,\; r=p-2,\; t=0,1,\dots, \tag{6}$$

where

$$\Phi_{r,t}(a,b;X)=f(X)\left(\sum_{i=0}^{l-1}b_{i,r,t}\psi_i(X)+\psi_l(X)\right)-\sum_{i=0}^{k}a_{i,r,t}\varphi_i(X). \tag{7}$$

We calculate the value of the weight function $\rho_r(X)$ by formula (5) and the value of the weight function $\upsilon_{r,t}(X)$ by the formula

$$\upsilon_{r,t}(X)=\begin{cases}1 & \text{if } r=0,\; t=0,\\ \left(\sum_{i=0}^{l-1}b_{i,r,t-1}\psi_i(X)+\psi_l(X)\right)^{-2} & \text{if } t>0.\end{cases} \tag{8}$$

Approximation by the rational expression (1) using the iteration scheme (6), (8) can be refined by controlling the accuracy ε_1 of the condition

$$|\eta_{r,t-1}-\eta_{r,t}|\le\varepsilon_1\eta_{r,t}, \tag{9}$$

where

$$\eta_{r,t}=\sum_{X\in\Omega}\rho_r(X)\upsilon_{r,t}(X)(\Phi_{r,t}(a,b;X))^2. \tag{10}$$

In the test, we used the value $\varepsilon_1=0.003$, which provided the convergence of two to three significant digits of the sum of squared deviations (10) on the point set Ω. Satisfying the condition (9) means that the mean-power approximation of power $p=r+2$ by the rational expression $R_{k,l,r}(a,b;X)$ is calculated with the accuracy ε_1. The values of the approximation parameters $R_{k,l,r}(a,b;X)$ are as follows:

$$a_{j,r}=a_{j,r,t}\;\;(j=\overline{0,k}),\;\; b_{j,r}=b_{j,r,t}\;\;(j=\overline{0,l-1}). \tag{11}$$

Thus, to construct the Chebyshev approximation by the rational expression (1) is to apply two iterative processes: nested iterations (6), (8) and external iterations (4), (5). Completion of the iterations (4), (5) can be monitored by attaining some prescribed accuracy ε

$$\mu_{r-1}-\mu_r\le\varepsilon\mu_r, \tag{12}$$

where

$$\mu_r=\max_{X\in\Omega}|f(X)-R_{k,l,r}(a,b;X)|. \tag{13}$$

In the solution of the test examples, the accuracy $\varepsilon=0.003$ in the case of approximation of functions of one and two variables was attained in eight to twelve iterations (4), (5). In the case of approximation of functions of more than three variables, twenty to twenty-five iterations were required to attain the accuracy $\varepsilon=0.003$. This accuracy ensured the convergence of two or three significant digits of the error of the Chebyshev approximation with a rational expression. The accuracy $\varepsilon_1=0.003$ of determining intermediate approximations by the rational expression was attained in three to four internal iterations (6)–(8). If for $r\ge1$ the value of the weight function $\upsilon_{r,0}(X)$ is assumed to be equal to $\upsilon_{r-1,t}(X)$, which corresponds to the values of this weight function at the previous iteration

$$\upsilon_{r,0}(X)=\upsilon_{r-1,t}(X),$$

then only two or three iterations (6), (8) were enough to refine the rational expression.

Convergence of iterations (4), (5) provides sequential refinement of the values of the weight function with regard for its previous values. According to (5), the values of the weight function vary in proportion to the modules of the values of the obtained error of the function approximation, namely, the largest increase in the value of the weight function corresponds to the point with the largest approximation error. This refinement of the values of the weight function provides sequential decrease in the value of the largest approximation error as a result of the next iteration.

ALGORITHM FOR CALCULATING THE PARAMETERS OF THE CHEBYSHEV APPROXIMATION BY THE RATIONAL EXPRESSION

Let us describe the process of calculating the values of the parameters of the Chebyshev approximation by the generalized rational expression (1) of the continuous function $f(X)$ defined on the point set Ω.

Let r be the number of iteration of the iteration process (4), (5), t be the number of iteration of the iteration process (6), (8), ε be the accuracy of constructing the Chebyshev approximation, and ε_1 be the accuracy of refining the parameters of the rational expression by the linearization scheme.

Implementation of the iteration processes (4), (5) and (6), (8) implies the following steps.

1. Assume $r=0$, $t=0$, $\mu_{r-1}=10^{10}$, $\eta_{0,t-1}=0$, $\rho_0(X)=1$, $\upsilon_{0,0}(X)=1$ for all $X\in\Omega$.

2. Find the solution of problem (6), i.e., calculate the values of the parameters of the mean-power approximation of power $p=r+2$ by the rational expression $R_{k,l,r,t}(a,b;X)$ at the tth iteration.

3. Test the obtained expression $R_{k,l,r,t}(a,b;X)$ for continuity on the set of points of the function. This can be done as tracking the sign of the value of the denominator of the rational expression $R_{k,l,r,t}(a,b;X)$ at points of the set Ω. If the sign of the value of the denominator of the rational expression $R_{k,l,r,t}(a,b;X)$ varies or is equal to zero, the algorithm stops. This means that for the given function there is no continuous approximation by the rational expression (1) for the specified values of the power of the numerator and denominator.

4. Using formula (10), calculate the error $\eta_{r,t}$ of the process of refining the values of parameters of the mean-power approximation of power $p=r+2$ by the rational expression $R_{k,l,r,t}(a,b;X)$ at the tth iteration.

5. Test the condition (9). If the value of the error $\eta_{r,t}$ satisfies condition (9), then the mean-power approximation of power $p=r+2$ by the rational expression $R_{k,l,r}(a,b;X)$ is considered to be constructed. Parameters of this approximation can be found by formulas (11). Continue the calculation from Step 6.

Otherwise, assume $t=t+1$, calculate new values of the weight function $\upsilon_{r,t}(x,y)$ by formula (8) and go to Step 2.

6. Using formula (13), calculate the value of error μ_r of the process of refining the Chebyshev approximation by the rational expression.

7. Test the condition (12). If the value of the error μ_r satisfies condition (12), then iterations (4), (5) are stopped. Assume the values of the parameters of the rational expression (1) to be equal to

$$a_j=a_{j,r}\ (j=\overline{0,k}),\ \text{and}\ b_j=b_{j,r}\ (j=\overline{0,l-1}),\tag{14}$$

and go to Step 8.

Otherwise, assume $r=r+1$, $t=0$, $\eta_{r,0}=0$ and calculate the new values of the weight function $\rho_r(X)$ by formula (5). Assume the values of the weight function $\upsilon_{r,0}(X)$ to be equal to

$$\upsilon_{r,0}(X)=\left(\sum_{i=0}^{l-1}b_{i,r-1}\psi_i(X)+\psi_l(X)\right)^{-2},\tag{15}$$

i.e., the values of this weight function are left the same as those used at the previous iteration. Continue the calculations from Step 2.

8. For the obtained approximation by the rational expression (1) whose values of parameters correspond to (14), carry out symmetrization adjustment [14]. Determine the value of the additive correction

$$\bar{a}_0=(\mu_{\max}+\mu_{\min})/2,\tag{16}$$

where

$$\mu_{\max}=\max_{X\in\Omega}(f(X)-R_{k,l}(a,b;X)),\ \mu_{\min}=\min_{X\in\Omega}(f(X)-R_{k,l}(a,b;X)).$$

As a result, the unknown Chebyshev approximation of the continuous function $f(X)$ by the rational expression (1) will have the form

$$R_{k,l}(a,b;X)=R_{k,l}(a,b;X)+\bar{a}_0.\tag{17}$$

CALCULATING THE PARAMETERS OF THE CHEBYSHEV APPROXIMATION BY THE RATIONAL EXPRESSION WITH RELATIVE ERROR

If the continuous function $f(X)$ does not take zero values at the point set Ω and the Chebyshev approximation $f(X)$ by the generalized rational expression (1) exists, then such approximation with relative error can be calculated by a scheme similar to that described in the algorithm above. The algorithm for calculating the Chebyshev approximation by a rational expression with relative error assumes the following changes.

In Step 1 of the algorithm described above, assume the initial values of the weight function $\rho_0(X)$ to be equal to

$$\rho_0(X) = \frac{1}{f(X)^2}. \tag{18}$$

At Step 6, calculate the value of the error μ_r of the Chebyshev approximation by a rational expression in the process of its refinement by the formula

$$\mu_r = \max_{X \in \Omega} \left| 1 - \frac{R_{k,l,r-1}(a, b; X)}{f(X)} \right|. \tag{19}$$

At Step 7, calculate new values of the weight function $\rho_r(X)$ by the formula (5), where

$$\Delta_s(X) = 1 - \frac{R_{k,l,s}(a, b; X)}{f(X)}. \tag{20}$$

At Step 8, calculate the value of the correction by the formula

$$c = \frac{2f(X_{\max})f(X_{\min})}{R_{k,l,r}(a, b; X_{\min})f(X_{\max}) + R_{k,l,r}(a, b; X_{\max})f(X_{\min})}, \tag{21}$$

where $X_{\max}$ is a point at which the relative approximation error μ_r (19) attains the greatest value at the point set Ω, and $X_{\min}$ attains the least value. With regard for the correction, Chebyshev approximation of the continuous function $f(X)$ defined on the point set Ω by the rational expression (1) with relative error has the form

$$R_{k,l}(a, b; X) = cR_{k,l}(a, b; X). \tag{22}$$

The value of the correction c (21) can be found as the solution of the one-parameter problem of Chebyshev approximation of the function $f(X)$ by the expression $cR_{k,l}(a, b; X)$ at the point set Ω with the relative error

$$\max_{X \in \Omega} \left| \frac{f(X) - cR_{k,l}(a, b; X)}{f(X)} \right| \xrightarrow[c]{} \min. \tag{23}$$

Example 1. Find the Chebyshev approximation of the function $y(x) = e^x$ defined at points x_i, $i = \overline{0, 30}$, where $x_i = -1 + 0.1i$, by the rational expression $R_{2,1}(a, b; x)$, where the numerator and denominator are polynomials of the second and first degree, respectively, with respect to the variable x.

With the use of the proposed method for $\varepsilon = 0.003$ at eight iterations (4) with the weight function (5) for the function $y(x)$, the following rational expression was obtained:

$$R_{2,1}(a, b; x) = \frac{1.043227312x^2 + 3.029780712x + 3.863602586}{3.903847747 - x}; \tag{24}$$

with regard for the correction $\bar{a}_0 = 0.000010784$ it provides the absolute approximation error 0.015695232. In the calculation of the Chebyshev approximation of the function $y(x)$, the approximation error at iterations (4), (5) took the following values:

0.023871778, 0.016934248, 0.01628043, 0.016008242, 0.015879723,

0.015801285, 0.015747203, 0.015706016.

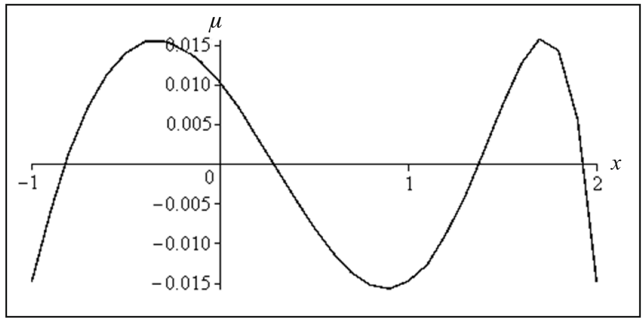

Fig. 1. Curve of the error of approximation of function $y(x)$ by the rational expression (24).

Chebyshev approximation of the function $y(x)$ by the rational expression $R_{2,1}(a, b; x)$, obtained by the iterative Remez scheme with alternation points refined by the Vallee–Poussin [1, 15] algorithm, provides the approximation error of 0.0155. The excess of the error of approximation by the rational expression (24) as compared with the error of the Chebyshev approximation obtained by the Remez scheme is 0.000195232, which is 1.26% of the error of the Chebyshev approximation obtained by the Remez scheme.

Figure 1 shows the curve of the error of approximation of function $y(x)$ by the rational expression (24).

The curve of the approximation of function $y(x)$ by the rational expression (24) shown in the figure corresponds to the characteristic property of the Chebyshev approximation [11, 15]: it has five extreme points at which it attains the deviation with the largest absolute value, and the values of the absolute values of these deviations coincide (within the given accuracy) and the sign of deviations at these points alternates:

$$(-1,\ -0.014891218),\ (-0.3,\ 0.015635164),\ (0.9,\ -0.01569526),$$

$$(1.7,\ 0.015695258),\ (2.0,\ -0.0149359).$$

These extreme points coincide with the alternation points obtained in the approximation of function $y(x)$ by the Remez scheme, with refinement of the alternation points by the Vallee–Poussin algorithm [11, 15]. At the end extreme points, the value of the approximation error is slightly smaller in the absolute value. To attain a better alignment of the absolute values of the approximation errors at extreme points, it is possible to increase the accuracy of calculation of the Chebyshev approximation by reducing the values of ε in the condition (12).

Chebyshev approximation of the function $y(x)$ with the absolute error 0.0155 was obtained by the method (4), (5) for $\varepsilon = 0.00003$ in 71 iterations:

$$\bar{R}_{2,1}(a, b; x) = \frac{1.04479306565x^2 + 3.02875502095x + 3.86394680668}{3.9043274993 - x}. \tag{25}$$

For the approximation of function $y(x)$ by the expression (25) at the extreme points, the following values of the errors were observed:

$$(-1,\ -0.015452392123),\ (-0.3,\ 0.015529413659),\ (0.9,\ -0.01552941366),$$

$$(1.7,\ 0.01546524431),\ (2.0,\ -0.01546288028).$$

As the accuracy of calculating the approximation by the rational expression increased, the extreme points did not change, and absolute values of the Chebyshev approximation error at these points almost leveled off (two or three significant digits coincided). The approximation (25) was calculated using 12 bits in the Maple medium.

Chebyshev approximation of the function $y(x)$ by the rational expression $R_{2,1}(a, b; x)$ with the relative error with the use of iterations (4), (5) and with regard for (20) for $\varepsilon = 0.003$ was obtained in ten iterations. The rational expression

$$\bar{R}_{2,1}(a, b; x) = -\frac{0.7972125368x^2 + 2.746838210x + 3.678106394}{-3.659074535 + x} \tag{26}$$

with regard for the correction $c = 0.9999941962$ provides the relative approximation error of 0.874%. In calculating the approximation (26), relative error of the function $y(x)$ at iterations (4), (5) obtained the following values (in %):

2.543313149, 1.096813398, 1.002098232, 0.9440792358, 0.9157450516,

0.899304183, 0.8888750491, 0.8818886266, 0.8769965656, 0.8746088798.

The graph of the relative approximation error (26) also corresponds to the characteristic features of the Chebyshev approximation [11, 15]: the relative error had the following values at the extreme points:

(–1, – 0.008461594347), (– 0.6, 0.008740234325), (0.3, – 0.008740234112),

(1.5, 0.00870258476), (2.0, – 0.008290903896).

Example 2. Let us find Chebyshev approximation of the function $z(x, y) = e^{-(x^2+y^2)}$ defined at points (x_i, y_j), $i = \overline{0,10}$, $j = \overline{0,10}$, where $x_i = -1 + 0.2\,i$ and $y_j = -1 + 0.2\,j$, by the rational expression $R_{2,2}(a, b; x, y)$, where the numerator and denominator are second-degree polynomials with respect to the variables x and y.

With the use of the proposed method for $\varepsilon = 0.003$ in seven iterations (4), (5) for function $z(x, y)$, the rational expression

$$R_{2,2}(a, b; x, y) = \frac{P_2(a; x, y)}{Q_2(b; x, y)} \tag{27}$$

was obtained, where

$$P_2(a; x, y) = 1.007258776 - 0.115894128\,{}_{10}^{-8}x - 0.234478414\,{}_{10}^{-8}y$$
$$-0.3393352184x^2 - 0.3393352234y^2 + 0.1445200801\,{}_{10}^{-9}xy,$$
$$Q_2(b; x, y) = 1 + 0.2634009507\,{}_{10}^{-8}x - 0.5167223229\,{}_{10}^{-8}y$$
$$+0.7853630535x^2 + 0.7853630273y^2 + 0.4766678537\,{}_{10}^{-9}xy.$$

The rational expression (27) with regard for the correction $\overline{a}_0 = -0.00014942155$ provides the absolute approximation error of the function $z(x, y)$, which equals 0.007665. In calculating the approximation (27), the approximation error of function $z(x, y)$ at the iterations (4), (5) obtained the following values:

0.0153866457, 0.010443066, 0.009504082, 0.0085679,

0.007935789, 0.0078146481, 0.0078186119.

Figure 2a shows the surface of the approximation error of function$z(x, y)$ by the rational expression (27). This example is taken from the paper [7], where the method of reducing the nonlinear problem (2) to sequential solution of linear programming problems is used to obtain the Chebyshev approximation of the function $z(x, y)$. Chebyshev approximation of the function $z(x, y)$ in [7] is obtained with the error of 0.007666 in seven calls of the procedure of solving the linear programming problem.

Chebyshev approximation of the function $z(x, y)$ by the rational expression $R_{2,2}(a, b; x, y)$ with the relative error with the use of iterations (4), (5) with regard for (20) for $\varepsilon = 0.003$ was obtained in ten iterations. The rational expression

$$\overline{R}_{2,2}(a, b; x, y) = \frac{\overline{P}_2(a; x, y)}{\overline{Q}_2(b; x, y)}, \tag{28}$$

where

$$\overline{P}_2(a; x, y) = 1.020134607 - 2.378435424\,{}_{10}^{-9}y - 1.017216096\,{}_{10}^{-9}x$$
$$-0.3263939092x^2 - 7.456299599\,{}_{10}^{-10}xy - 0.32639402y^2,$$
$$\overline{Q}_2(b; x, y) = 1 - 9.919727530\,{}_{10}^{-9}y - 4.896131881\,{}_{10}^{-9}x$$
$$+0.8842411113x^2 - 4.219121297\,{}_{10}^{-9}xy + 0.8842407983y^2,$$

provides the relative approximation error of 2% with the correction $c = 1.00009989$.

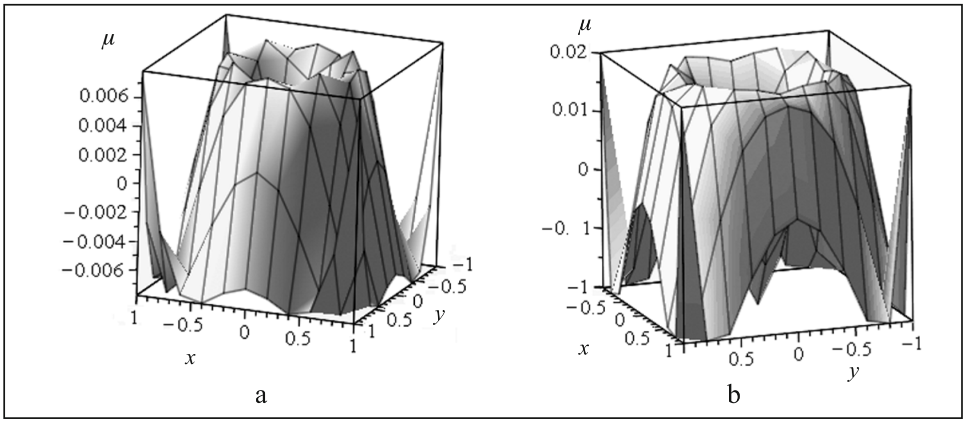

Fig. 2. Surfaces of the error of approximation of function $z(x, y)$:
by the rational expression (27) with absolute error (a),
by the rational expression (28) with relative error (b).

Figure 2b shows the surface of the error of approximation of function $z(x, y)$ by the rational expression (28) with the relative error.

Example 3. Find the Chebyshev approximation of the function $z_3(x, y, t) = e^{-(x+y+t)}$ defined at points (x_i, y_j, t_r), $i = \overline{0, 20}$, $j = \overline{0, 20}$, $r = \overline{0, 20}$, where $x_i = -1 + 0.1i$, $y_j = -1 + 0.1j$, and $t_r = -1 + 0.1r$, by the rational expression where the numerator and denominator are first-degree polynomials of the variables x, y, and t.

With the use of the proposed method (4), (5) for $\varepsilon = 0.003$, approximation is obtained in 22 iterations for the function $z_3(x, y, t)$ by the rational expression

$$R_{1,1}(a, b; x, y, t) = \frac{P_1(a; x, y, t)}{Q_1(b; x, y, t)}, \tag{29}$$

where

$$P_1(a; x, y, t) = 1.588802979 - 0.9411049363t - 0.9411010086y - 0.94109332x,$$

$$Q_1(a; x, y, t) = 1 + 0.2626767504t + 0.2626767831y + 0.2626785615x.$$

The rational expression (29) provides absolute approximation error for function $z_3(x, y, t)$, which equals 0.7402088392, with the correction $\bar{a}_0 = 0.0200267282$.

Chebyshev approximation of function $z_3(x, y, t)$ by a rational expression, where the numerator and denominator are first-degree polynomials of variables x, y, and t, with a relative error was not obtained. A discontinuous rational expression was obtained in calculating the approximation with relative error at iterations (6), (8).

Chebyshev approximation of function $z_3(x, y, t)$ by a rational expression in which the numerator and denominator are second-degree polynomials of variables x, y, and t, with the use of the method (4), (5) for $\varepsilon = 0.003$ was obtained in eleven iterations. Absolute error of this approximation with regard for the correction $\bar{a}_0 = 0.0002501987786$ is 0.0233863597314. Chebyshev approximation of function $z_3(x, y, t)$ by a rational expression in which the numerator and denominator are second-degree polynomials of variables x, y, and t with the relative error for $\varepsilon = 0.003$ was obtained in 19 iterations. With regard for the correction $c = 1.00006973092$ it provided the relative approximation error of 2.156%.

CONCLUSIONS

The proposed method of constructing the Chebyshev approximation by a rational expression of continuous tabular functions of many variables is to sequentially determine the mean-power approximations using the least squares method with two variable weight functions. One weight function provides construction of the mean-power approximation, and the second one ensures refinement of parameters of the rational expression according to the linearization scheme. The method converges due to the original method of sequential refinement of the values of the weight functions by formulas (5) and (8) for the absolute error and taking into account (18), (20) for the relative error. The algorithms presented in the paper for calculating the parameters of approximation by a rational expression are simple to implement, reliable, and efficient. They provide the possibility to calculate the parameters of approximation by the rational expression with a desired accuracy for the absolute and relative errors. The results of solving the test examples confirm a fairly rapid convergence of the proposed method in the case of approximation, by the rational expression, of functions of one, two, and three variables. When solving the test examples by this method, convergence of two or three significant digits of the error of approximation by the rational expression was attained with the use of eight to 22 iterations (4), (5).

REFERENCES

1. L. Collatz and W. Krabs, Approximationstheorie: Tschebyscheffsche Approximation mit Anwendungen (Teubner Studienbücher Mathematik) Vieweg+Teubner Verlag (1973).
2. T. Azizov, O. Melnyk, O. Orlova, A. Kalenchuk-Porkhanova, and L. Vakal, “Calculation of reinforced concrete ceilings with normal cracks accounting the Chebyshev approximation,” in: Proc. 6th Intern. Sci. Conf. “Reliability and Durability of Railway Transport Engineering Structures and Buildings” Transbud-2017 (April 19–21, 2017, Kharkiv, Ukraine), Kharkiv (2017), pp. 1–7.
3. V. Peiris, N. Sharon, N. Sukhorukova, and J. Ugon, “Rational approximation and its application to improving deep learning classifiers,” arXiv:2002.11330v1 [math.OC] 26 Feb 2020.
4. A. A. Kalenchuk-Porkhanova, “Best Chebyshev approximation of functions of one and many variables,” Cybern. Syst. Analysis, Vol. 45, No. 6, 988-996 (2009).
5. Y. Nakatsukasa, O. Sete, and L. N. Trefethen, “The AAA algorithm for rational approximation,” SIAM J. Sci. Comput., Vol. 40, No. 3, A1494–A1522 (2018).
6. S.-I. Filip, Y. Nakatsukasa, L. N. Trefethen, and B. Beckermann, “Rational minimax approximation via adaptive barycentric representations,” SIAM J. Sci. Comput., Vol. 40, No. 4, A2427–A2455 (2018).
7. L. V. Petrak, “Approximation of functions of many variables by rational fractions,” Tr. IMM UNTs AN USSR, Issue 6: Optimization Programs (Approximation of Functions), 130–144 (1975).
8. P. S. Malachivskyy, Y. N. Matviychuk, Y. V. Pizyur, and R. P. Malachivskyi, “Uniform approximation of functions of two variables,” Cybern. Syst. Analysis, Vol. 53, No. 3, 426–431 (2017).
9. P. S. Malachivskyy, Y. V. Pizyur, R. P. Malachivskyi, and O. M. Ukhanska, “Chebyshev approximation of functions of several variables,” Cybern. Syst. Analysis, Vol. 56, No. 1, 76–86 (2020).
10. N. N. Kalitkin, Numerical Methods [in Russian], Nauka, Moscow (1978).
11. P. S. Malachivskyy and Y. V. Pizyur, Solving Problems in the Maple Environment [in Ukrainian], RASTR–7, Lviv (2016).
12. P. S. Malachivskyy, Y. V. Pizyur, and R. P. Malachivskyi, “Uniform approximation by a rational expression,” Komp. Tekhnologii Drukarstva, No. 1 (39), 54–59 (2018).
13. P. S. Malachivskyy, B. R. Montsibovych, Y. V. Pizyur, and R. P. Malachivskyi, “Chebyshev approximation of functions of two variables by a rational expression,” Matematychne ta Komp. Modelyuvannya, Ser. Tekhnichni Nauky, Issue 19, 75–81 (2019).
14. E. Ya. Remez, Fundamentals of Numerical Methods of Chebyshev Approximation [in Russian], Naukova Dumka, Kyiv (1969).
15. P. S. Malachivskyy and V. V. Skopetskii, Continuous and Smooth Minimax Spline Approximation [in Ukrainian], Naukova Dumka, Kyiv (2013).