DOI 10.1007/s10559-020-00293-y
Cybernetics and Systems Analysis, Vol. 56, No. 5, September, 2020

DIFFUSION PROCESS WITH EVOLUTION AND ITS PARAMETER ESTIMATION

V. S. Koroliuk,[1] D. Koroliouk,[2†] and S. O. Dovgyi[2‡]

UDC 519.24

Abstract. *A discrete Markov process in an asymptotic diffusion environment with a uniformly ergodic embedded Markov chain can be approximated by an Ornstein–Uhlenbeck process with evolution. The drift parameter estimation is obtained using the stationarity of the Gaussian limit process.*

Keywords: *discrete Markov process, diffusion approximation, asymptotic diffusion environment, Ornstein–Uhlenbeck process, phase merging, drift parameter estimation.*

We consider a random evolution $\zeta(t)$, $t \ge 0$, that depends on a random environment $Y(t)$, $t \ge 0$, which in turn, is switched by an embedded Markov chain X_k, $k \ge 0$. Below, we will explain the relation between the continuous-time $t \ge 0$ and the discrete-time $k \ge 0$.

The purpose of this study is to prove the convergence (in distribution) of the process $\zeta(t)$, $t \ge 0$, to the Ornstein–Uhlenbeck process under some scaling of the process and its time parameter.

The limit will be considered by a small series parameter $\varepsilon > 0$, $\varepsilon \to 0$.

ASYMPTOTIC DIFFUSION ENVIRONMENT

Consider a discrete Markov process in a semi-Markov asymptotic diffusion environment, defined by the solution of the following scaled difference stochastic equation:

$$\zeta^\varepsilon(t_{n+1}^\varepsilon) = -\varepsilon^2 V(Y_n^\varepsilon)\zeta^\varepsilon(t_n^\varepsilon) + \varepsilon\sigma(Y_n^\varepsilon)\Delta\mu^\varepsilon(t_{n+1}^\varepsilon), \tag{1}$$

where $t_n^\varepsilon := n\varepsilon^2$; hence $t_{n+1}^\varepsilon = t_n^\varepsilon + \varepsilon^2$, $n > 0$, $\varepsilon > 0$, for the process increments $\Delta\zeta^\varepsilon(t_{n+1}^\varepsilon) := \zeta^\varepsilon(t_{n+1}^\varepsilon) - \zeta^\varepsilon(t_n^\varepsilon)$, $n \ge 0$.

The asymptotic diffusion environment Y_n^ε, $n \ge 0$, is also a random evolution process generated by the solution of the following scaled difference evolutionary equation:

$$\Delta Y^\varepsilon(t_{n+1}^\varepsilon) = \varepsilon A_0(Y_n^\varepsilon; X_n^\varepsilon) + \varepsilon^2 A(Y_n^\varepsilon; X_n^\varepsilon), \ \ n \ge 0, \tag{2}$$

with the embedded Markov chain $X_n^\varepsilon := X(t_n^\varepsilon)$, $n \ge 0$.

The terms $A_0(y;x)$ and $A(y;x)$ are Lipschitz functions, together with the first derivative $A'_{0y}(y;x)$.

Here, the predictable evolutional component in (1) is defined by the following conditional expectation [1]:

$$V(Y_n^\varepsilon)\zeta^\varepsilon(t_n^\varepsilon) := E[\Delta\zeta^\varepsilon(t_{n+1}^\varepsilon) \mid Y_n^\varepsilon, \zeta^\varepsilon(t_n^\varepsilon)]\zeta^\varepsilon(t_n^\varepsilon),$$

where the drift regression function $V(z)$ is assumed to be positive: $V(z) > 0 \ \forall z$.

[1]Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, Ukraine. [2]Institute of Telecommunications and Global Information Space, National Academy of Sciences of Ukraine, Kyiv, Ukraine, [†]*dimitri.koroliouk@ukr.net*; [‡]*pryjmalnya@gmail.com*. Translated from Kibernetika i Sistemnyi Analiz, No. 5, September–October, 2020, pp. 55–62. Original article submitted March 22, 2020.

The martingale difference $\Delta\mu^\varepsilon(t_{n+1}^\varepsilon)$, $n \ge 1$, generated by the process $\Delta\zeta^\varepsilon(t_{n+1}^\varepsilon)$, $n \ge 1$, is defined by the following conditional second moment:

$$-\varepsilon^2\sigma^2(Y_n^\varepsilon) := E\,[(\Delta\,\zeta^\varepsilon(t_{n+1}^\varepsilon) + \varepsilon^2 V(Y_n^\varepsilon)\zeta^\varepsilon(t_n^\varepsilon))^2 \,|\, Y_n^\varepsilon].$$

The embedded Markov chain $X_n^\varepsilon := X(t_n^\varepsilon)$, $t_n^\varepsilon := n\varepsilon^2$, $n \ge 0$, is supposed to be a homogeneous ergodic Markov chain with transition probabilities $P(x, B)$, $x \in E$, $B \in \mathcal{E}$, having a stationary distribution $\rho(B)$, $B \in \mathcal{E}$, which satisfies the condition $\rho(B) = \int_E \rho(dx)P(x, B)$; $\rho(E) = 1$.

The stochastic difference equations (1), (2) generate a discrete stochastic basis [2, Ch. 1] with filtration $\mathrm{F}_m(\zeta^\varepsilon, Y^\varepsilon) = \sigma\{\zeta^\varepsilon(t_n^\varepsilon), Y^\varepsilon(t_n^\varepsilon), n \le m\}$, $m \ge 0$.

Now we consider three components $(\zeta^\varepsilon(t_n^\varepsilon), Y^\varepsilon(t_n^\varepsilon), X_n^\varepsilon)$, $n \ge 0$, as piecewise constant functions with continuous time:

$$\left.\begin{array}{l} \zeta^\varepsilon(t) = \zeta^\varepsilon(t_n^\varepsilon) \\ Y^\varepsilon(t) = Y^\varepsilon(t_n^\varepsilon) \\ X_t^\varepsilon = X_n^\varepsilon \end{array}\right\} \text{ for } n\varepsilon^2 < t \le (n+1)\varepsilon^2.$$

In what follows, solution of Eqs. (1), (2) is given by martingale characterization [3, Sec. 4.4] of the three-component Markov process $(\zeta^\varepsilon(t), Y^\varepsilon(t), X_t)$, $t \ge 0$:

$$M^\varepsilon(t) = \varphi(\zeta^\varepsilon(t), Y^\varepsilon(t), X_t) - \varphi(\zeta^\varepsilon(0), Y^\varepsilon(0), X_0)$$
$$-\int_0^{\varepsilon^2[t/\varepsilon^2]} L^\varepsilon\varphi(\zeta^\varepsilon(s), Y^\varepsilon(s), X_s^\varepsilon)ds,$$

and the generator of three-component Markov process $(\zeta^\varepsilon(t), Y^\varepsilon(t), X_t)$, $t \ge 0$, is represented as follows [4, Ch. 5]:

$$L^\varepsilon\varphi(c, y, x) := \varepsilon^2 E\,[\varphi(c + \Delta\zeta^\varepsilon(t_{n+1}^\varepsilon, Y^\varepsilon(t_{n+1}^\varepsilon), X_{n+1}^\varepsilon)) \,|\, \zeta^\varepsilon(t_n^\varepsilon) = c, Y^\varepsilon(t_n^\varepsilon) = y, X_n^\varepsilon = x].$$

APPROXIMATION OF A DISCRETE MARKOV PROCESS IN ASYMPTOTIC DIFFUSION ENVIRONMENT

Let the singular term $A_0(y; x)$ satisfy the balance condition

$$\int_E \rho(dx)A_0(y; x) \equiv 0. \tag{3}$$

The theorem below gives the approximation of a discrete Markov process in asymptotic diffusion environment.

THEOREM 1. Let the Markov chain X_n, $n \ge 0$, be uniformly ergodic with the stationary distribution $\rho(B)$, $B \in \mathcal{E}$.

The finite-dimensional distributions of the discrete Markov process (1), together with asymptotic diffusion $Y^\varepsilon(t)$, $t \ge 0$, converge, as $\varepsilon \to 0$, to a diffusion Ornstein–Uhlenbeck process with evolution:

$$(\zeta^\varepsilon(t), Y^\varepsilon(t)) \xrightarrow{D} (\zeta^0(t), Y^0(t)),\ \varepsilon \to 0,\ 0 \le t \le T.$$

The limit two-component diffusion process with evolution $(\zeta^0(t), Y^0(t))$, $t \ge 0$, is set by the generator

$$\mathfrak{L}^0(y)\varphi(c,y) = -V(y)c\varphi'_c(c,y) + \frac{1}{2}\sigma^2(y)\varphi''_c(c,y)$$

$$+\hat{A}(y)\varphi'_y(c,y) + \frac{1}{2}\hat{B}^2(y)\varphi''_y(c,y),\ \varphi \in C_0^2(R,\mathfrak{B}), \tag{4}$$

where by definition

$$\hat{A}(y) := \int_E \rho(dx)[A(y;x) + A_1(y;x)], \tag{5}$$

$$A_1(y;x) := A_0(y;x)PR_0A'_{0y}(y;x), \tag{6}$$

$$\hat{B}^2(y) = \int_E \rho(dx)B(y;x),$$

$$B(y;x) = A_0(y;x)P\left[R_0 + \frac{1}{2}\mathbb{I}\right]A_0(y;x). \tag{7}$$

Here, $\mathbb{I}$ is the standard identity matrix, P is the transition operator of the Markov chain X_t, $t \geq 0$, and the potential kernel R_0 is defined as in [3, Sec. 5.2]:

$$R_0 = (\mathbb{Q} + \Pi)^{-1} - \Pi,\ \mathbb{Q} := P - \mathbb{I},\ \Pi\varphi(x) := \int_E \rho(dx)\varphi(x). \tag{8}$$

Remark 1. The limit two-component diffusion process $(\zeta^0(t), Y^0(t))$, $t \geq 0$, defined by the generator (4)–(8) has a stochastic representation by the stochastic differential equation

$$d\zeta^0(t) = -V(Y^0(t))\zeta^0(t)dt + \sigma(Y^0(t))dW(t),$$

$$dY^0(t) = \hat{A}(Y^0(t))dt + \hat{B}(Y^0(t))dW_0(t).$$

Consequently, the parameters of the limit diffusion $\zeta^0(t)$, $t \geq 0$, depend on the diffusion process $Y^0(t)$, $t \geq 0$.

Proof of Theorem 1. The basic idea is that any Markov process is determined by its generator on the class of real-valued test functions defined on the set of values of the Markov process [5].

First of all, the extended three-component Markov chain is used

$$(\zeta^\varepsilon(t_n), Y^\varepsilon(t_n), X^\varepsilon(t_n) = X_n),\ t_n = n\varepsilon^2,\ \varepsilon > 0, \tag{9}$$

with operator characterization in the following form.

LEMMA 1. The extended Markov chain is defined by the generator

$$\mathbb{L}^\varepsilon(x)\varphi(c,y,x) = \varepsilon^{-2}[\Gamma^\varepsilon(y)A^\varepsilon(x)\mathbb{P} - \mathbb{I}]\varphi(c,y,x), \tag{10}$$

where the transition operators are defined as follows:

$$\Gamma^\varepsilon(y)\varphi(c) := E[\varphi(c + \Delta\zeta^\varepsilon(t;y)) \mid \zeta^\varepsilon(t) = c, Y^\varepsilon(t) = y, X^\varepsilon(t) = x],$$

$$\mathbb{A}^\varepsilon(x)\varphi(y) := E[\varphi(y + \Delta Y^\varepsilon(t;x)) \mid Y^\varepsilon(t;x) = y, X^\varepsilon(t) = x], \tag{11}$$

$$\mathbb{P}\varphi(x) := \int_E P(x,dz)\varphi(z).$$

The assertion of Lemma 1 follows from the argumentation below. The extended three-component Markov chain (9), under the additional condition $Y^\varepsilon(t) = y$, $X^\varepsilon(t) = x$, has independent components. So, its transition probabilities are given by the product of transition probabilities of each component.

An essential step in the proof of Theorem 1 is implemented in the following lemma.

LEMMA 2. The generator (10), (11) of the three-component Markov chain (9) on the class of real-valued test functions $\varphi(c, y, x)$, having bound derivatives up to the third order inclusively, admits the asymptotic representation

$$\mathbb{L}^{\varepsilon}(x)\varphi(c, y, x) = [\varepsilon^{-2}\mathbb{Q} + \varepsilon^{-1}\mathbb{A}_0(x)\mathbb{P} + \mathbb{A}(x)\mathbb{P} + \mathbb{L}^0(y)P + \mathbb{R}_{\varepsilon}(y;x)]\varphi(c, y, x); \tag{12}$$

$$\mathbb{A}_0(x)\varphi(y) = A_0(y;x)\varphi'(y), \quad \mathbb{A}(x)\varphi(y) = A(y;x)\varphi'(y);$$
$$\mathbb{L}^0(y)\varphi(c) = -V(y)c\varphi'(c) + \frac{1}{2}\sigma^2(y)\varphi''(c). \tag{13}$$

The residual term is expressed as

$$\mathbb{R}_{\varepsilon}(y;x)\varphi(c, y, x) \to 0, \ \varepsilon \to 0, \ \varphi \in C^3(R^2).$$

Here, one intends the uniform convergence for all the arguments.

Proof of Lemma 2. We use transformation of generator (12) by the formula

$$\mathbb{L}^{\varepsilon}(x)\varphi(c, y, x) = \varepsilon^{-2}[\mathbb{Q} + (A^{\varepsilon}(x) - \mathbb{L})\mathbb{P} + (\Gamma^{\varepsilon}(y) - \mathbb{I})\mathbb{P} + \mathbb{R}_{\varepsilon}(y;x)]\varphi(c, y, x). \tag{14}$$

The residual term has the following form:

$$\mathbb{R}_{\varepsilon}(y;x)\varphi(c, y, x) = (\Gamma^{\varepsilon}(y) - \mathbb{I})(A^{\varepsilon}(x) - \mathbb{I})\mathbb{P}\varphi(c, y, x).$$

Then we calculate

$$\varepsilon^{-2}[\Gamma^{\varepsilon}(y) - \mathbb{I}]\varphi(c) = \varepsilon^{-2}\{E[\varphi(c + \Delta\zeta^{\varepsilon}(t; y)) \mid \zeta^{\varepsilon}(t; y) = c] - \varphi(c)\} = [\mathbb{L}^0(y) + \mathbb{R}_{\varepsilon}(y;c)]\varphi(c).$$

The next term in (14) has the following representation:

$$\varepsilon^{-2}[A^{\varepsilon}(x) - \mathbb{I}]\varphi(y) = \varepsilon^{-2}\{E[\varphi(y + \Delta Y^{\varepsilon}(t; y)) \mid Y^{\varepsilon}(t; y) = y] - \varphi(y)\}$$

$$= \varepsilon^{-2}\left[E[\Delta Y^{\varepsilon}(t; y)]\varphi'(y) + \frac{1}{2}E[\Delta Y^{\varepsilon}(t; y)]^2\varphi''(y) + \varepsilon^2\mathbb{R}_{\varepsilon}(x)\varphi(y)\right]$$

$$= [\varepsilon^{-1}\mathbb{A}_0(y;x) + \mathbb{A}_0(y;x)]\varphi'(y) + \frac{1}{2}\mathbb{A}_0^2(y;x)\varphi''(y) + \mathbb{R}_{\varepsilon}(x)\varphi(y),$$

gives the asymptotic expansion in Lemma 2:

$$\mathbb{L}^{\varepsilon}(x)\varphi(c, y, x) = \Big[\varepsilon^{-2}\mathbb{Q} + \varepsilon^{-1}\mathbb{A}_0(x)\mathbb{P} + \mathbb{A}(x)\mathbb{P}$$
$$+ \frac{1}{2}(\mathbb{A}_0(x)\mathbb{P})^2 + \mathbb{L}^0(y)\Big]\varphi(c, y, x) + \mathbb{R}_{\varepsilon}(y;x)\varphi(c, y, x).$$

Then we use the solution of singular perturbation problem for the truncated operator [4].

LEMMA 3. The solution of the singular perturbation problem for the truncated operator is realized on perturbed test functions:

$$\mathbb{L}_0^{\varepsilon}(x)\varphi^{\varepsilon}(c, y, x) = \Big[\varepsilon^{-2}\mathbb{Q} + \varepsilon^{-1}\mathbb{A}_0(x)\mathbb{P} + A(x)\mathbb{P}$$
$$+ \frac{1}{2}(\mathbb{A}_0(x)\mathbb{P})^2 + \mathbb{L}^0(y)\Big][\varphi(c, y) + \varepsilon\varphi_1(c, y, x) + \varepsilon^2\varphi_2(c, y, x)]$$
$$= \mathfrak{L}^0(y)\varphi(c, y) + \mathbb{R}_{\varepsilon}(x)\varphi(c, y). \tag{15}$$

The averaging parameters are determined by the formulas (4)–(8).

The limit operator is calculated by the formula

$$\mathfrak{L}^0(y)\varphi(c, y) = \mathbb{L}^0(y)\varphi(c, y) + \hat{A}(y)\varphi'_y(c, y) + \frac{1}{2}\hat{B}^2(y)\varphi''_y(c, y). \tag{16}$$

Proof of Lemma 3. To solve the singular perturbation problem for the truncated operator, consider the asymptotic representation by the powers of ε:

$$\mathbb{L}_0^{\varepsilon}(x)\varphi^{\varepsilon}(c, y, x) = \varepsilon^{-2}\mathbb{Q}\varphi(c, y) + \varepsilon^{-1}[\mathbb{Q}\varphi_1(c, y, x) + \mathbb{A}_0(x)\varphi(c, y)]$$

$$+\left[\mathbb{Q}\varphi_2(c, y, x) + \mathbb{A}_0(x)\mathbb{P}\varphi_1(c, y, x) + \left[\mathbb{A}(x) + \frac{1}{2}(\mathbb{A}_0(x)\mathbb{P})^2 + \mathbb{L}^0(y)\right]\varphi(c, y)\right] + \mathbb{R}_{\varepsilon}(x)\varphi(c, y).$$

Obviously, $\mathbb{Q}\varphi(c, y) = 0$.

The balance condition (3) is then used. The solution of the equation

$$\mathbb{Q}\varphi_1(c, y, x) + \mathbb{A}_0(x)\varphi(c, y) = 0$$

is given by the formula [4, Sec. 5.4]:

$$\varphi_1(c, y, x) = \mathbb{R}_0\mathbb{A}_0(x)\varphi(c, y).$$

Lemma 3 implies the following equation:

$$\mathbb{Q}\varphi_2(c, y, x) + [\mathbb{B}(x) + \mathbb{A}(x) + \mathbb{L}^0(y)]\varphi(c, y) = \mathfrak{L}^0(y)\varphi(c, y). \tag{17}$$

Here, by definition

$$\mathbb{B}(x) := \mathbb{A}_0(x)\,\mathbb{P}\,R_0\mathbb{A}_0(x) + \frac{1}{2}(\mathbb{A}_0(x)\,\mathbb{P})^2.$$

The limit operator is calculated using the balance condition

$$\mathfrak{L}^0(y)\varphi(c, y) = \{\Pi[\mathbb{B}(x) + \mathbb{A}(x)]\Pi + \mathbb{L}^0(y)\}\varphi(c, y).. \tag{18}$$

Recall projector's operation:

$$\Pi\mathbb{B}(x)\Pi = \int_E \rho(dx)B(y,x), \quad B(y,x) = A_0(y,x)\,\mathbb{P}\,\mathbb{R}_0 A_0(y,x) + \frac{1}{2}(A_0(y,x))^2.$$

With regard for the definition of evolutionary operators (13), the limit generator is defined by formula (16).

The limit operator (18) provides a solution of Eq. (17), which is a function of $\varphi_2(c, y, x)$. The existence of perturbing functions $\varphi_i(\cdot)$, $i = 1,2$, ensures the asymptotic representation (15). This completes the proof of Theorem 1.

The volatility is generated by introducing a random environment $Y^0(t)$, $t \ge 0$, into the diffusion parameter of the Ornstein–Uhlenbeck diffusion process $\zeta^0(t)$, $t \ge 0$.

PARAMETER ESTIMATION OF THE LIMIT PROCESS

The limit Ornstein–Uhlenbeck diffusion process parameters estimation is substantiated in this section without the assumption of volatility, which greatly changes the kind of estimates. The stationarity of the Gaussian statistical experiment is essentially used [6].

It is known [7] that diffusion-type processes are given by the stochastic differential

$$d\xi_t = \alpha_t(\xi_t)dt + dW_t. \tag{19}$$

The predictable component satisfies the conditions

$$P\left(\int_0^T \alpha_t^2(\xi_t)dt < \infty\right) = 1, \; T < \infty, \tag{20}$$

$$P\left(\int_0^\infty \alpha_t^2(\xi_t)dt = \infty\right) = 1,$$

which ensures absolute continuity of the measure $\mu_\xi(B) := P\{\omega : \xi \in B\}$ and the measure $\mu_W(B) := P\{\omega : W \in B\}$ for all $B \in \mathcal{B}_T = \sigma(\xi_t : 0 \le t \le T)$.

The Radon–Nicodym derivative specifies the density of the measure

$$\zeta_T(\xi) := \frac{d\mu_\xi}{d\mu_W}(\xi, T), \tag{21}$$

which has the following representation for processes of diffusion type (19).

THEOREM 2 [7]. The measure density (21) for the processes of diffusion type (19) with additional conditions (20) is given by the exponential martingale

$$\frac{d\mu_\xi}{d\mu_W}(\xi_t, T) = \exp\left[\int_0^T \alpha_t(\xi_t)d\xi_t - \frac{1}{2}\int_0^T \alpha_t^2(\xi_t)dt\right]. \tag{22}$$

In particular, the exponential martingale (21) is determined by the solution of the stochastic Doléans–Dade equation [2]

$$d\zeta_T(\xi) = \zeta_T(\xi)\alpha_T(\xi)d\xi_T\,,\ \zeta_0(\xi) = 1, \tag{23}$$

or in the equivalent form:

$$\zeta_T(\xi) - 1 = \zeta_T(\xi)\alpha_T(\xi)d\xi_T.$$

The relationship of the density (22) with the stochastic Doléans–Dade equation (23) can be explained, using the Itô formula for exponential function $\varphi(\xi) = \exp\left[\eta_T(\xi) - \frac{1}{2}\langle\eta(\xi)\rangle_T\right]$, with $\eta_T(\xi) := \int_0^T \alpha_t(\xi)dt$, $\langle\eta(\xi)\rangle_T := \int_0^T \alpha_t^2(\xi)dt$, namely (see [2]):

$$d\varphi(\xi_T) = \varphi'(\xi_T)\left[d\eta_T(\xi) - \frac{1}{2}d\langle\eta(\xi)\rangle_T\right] + \frac{1}{2}\varphi''(\xi_T)d\langle\eta(\xi)\rangle_T.$$

Taking into account the equality $\varphi(\xi_T) = \varphi'(\xi_T) = \varphi''(\xi_T)$ for exponential function $\varphi(\xi)$, we get a stochastic Doléans–Dade differential equation for exponential martingale (22). According to the results of the previous section, the limit diffusion process for normalized discrete Markov processes is the Ornstein–Uhlenbeck process with a linear predictable component

$$d\alpha_t = -V_0\alpha_t dt + \sigma dW_t\,,\ 0 \le t \le T,$$

Without loss of generality, let us put $\sigma = 1$.

The maximum likelihood method for estimating the parameter V_0 of a diffusion process with a stochastic differential (19) ($\sigma = 1$) is implemented for the logarithm of the measure density (22):

$$L(V, T) := \ln\zeta_T(\xi) = -V\int_0^T \alpha_t dt - \frac{V^2}{2}\int_0^T \alpha_t^2 dt.$$

Therefore, the equation for estimating the maximum likelihood method is

$$\max_{0 \le V \le 2} \partial L(V, T)/\partial V = -\int_0^T \alpha_t dt - V_T\int_0^T \alpha_t^2 dt = 0,$$

and the estimate of the maximum likelihood method has the following form:

$$V_T = -\int_0^T \alpha_t d\alpha_t \Big/ \int_0^T \alpha_t^2 dt.$$

The least square method estimation of parameter V_0 of diffusion process with stochastic differential (19) ($\sigma = 1$) is implemented using the equality

$$\int_0^T \alpha_t d\alpha_t = -V_0 \int_0^T \alpha_t^2 dt + \int_0^T \alpha_t dW_t.$$

So, we have the relationship

$$V_0 - V_T = \int_0^T \alpha_t dW_t \Big/ \int_0^T \alpha_t^2 dt.$$

The estimation of the least squares method has the representation

$$V_T^0 = \int_0^T \alpha_t d\beta_t \Big/ \int_0^T \alpha_t^2 dt.$$

COROLLARY 1. The estimates of maximum likelihood and least squares coincide: $V_T = V_T^0$.

COROLLARY 2. Estimation by the least squares method, and hence estimation by the method of maximum likelihood of the parameter V_0 are strongly consistent:

$$P1 \lim_{T \to \infty} V_T^0 = V_0. \tag{24}$$

Remark 2. Under volatility (see [8]), the maximum likelihood estimate and the least squares estimate are different, but the property of strong consistency (24) is retained.

REFERENCES

1. D. Koroliouk, “Binary statistical experiments with persistent nonlinear regression,” Theor. Probability and Math. Statist., Vol. 91, 71–80 (2015).
2. Yu. V. Borovskikh and V. S. Korolyuk, Martingale Approximation, VSP, Utrecht (1997).
3. S. N. Ethier and T. G. Kurtz, Markov Processes: Characterization and Convergence, Willey, Hoboken (1986).
4. V. S. Koroliuk and N. Limnios, Stochastic Systems in Merging Phase Space, World Scientific, New Jersey–London (2005).
5. V. S. Korolyuk and D. Koroliouk, “Diffusion approximation of stochastic Markov models with persistent regression,” Ukrain. Matem. Zhurn., Vol. 47, No. 7, 928–935 (1995).
6. D. Koroliouk, “Stationary statistical experiments and the optimal estimator for a predictable component,” J. of Mathematical Sciences, Vol. 214, No. 2, 20–228 (2016).
7. S. N. Cohen and R. J. Elliott, Stochastic Calculus and Applications. Probabilitity and its Application, Birkhauser, Basel (2015).
8. M. Bel Hadj Khlifa, Yu. Mishura, K. Ralchenko, G. Shevchenko, and M. Zili, “Stochastic differential equations with generalized stochastic volatility and statistical estimators,” Teoriya Imovirnostei ta Matematychna Statystyka, Vol. 96, 8–20 (2017).