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MULTISTAGE APPROACH TO SOLVING THE OPTIMIZATION PROBLEM OF PACKING NONCONVEX POLYHEDRA

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Abstract. *The paper considers the problem of packing nonconvex polyhedra into a container of minimum volume. An exact mathematical model of the problem of packing nonconvex polyhedra with continuous translations and rotations is constructed. Characteristics of the mathematical model are analyzed and are used as the basis to develop a multistage solution approach to obtain a nearly optimal solution, which is not the global minimum but is a proved local minimum. Numerical examples are given.*

Keywords: *packing, nonconvex polyhedra, Φ-function, nonlinear optimization.*

INTRODUCTION

Optimization problems of packing 3D objects are a part of the operations research theory and have a wide range of practical applications, for example, to solve modern problems in biology, mineralogy, medicine, materials science, nanotechnology, robotics, pattern recognition, 3D printing.

Solving such problems is important since they allow replacing full-scale expensive experiments with computer modeling of real processes and structures of materials. This saves significant time and financial resources.

For example, three-dimensional modeling of microstructures of different materials (including nanomaterials) is an innovative application of the problem of allocation of polyhedra. Recent achievements in this field are related to the development of a computer technology of 3D tomographic analysis of mineral particles [1]. The paper [2] describes application of the problem of packing polyhedra in powder metallurgy. The same problems are used for efficient solution of the problem of hazardous waste utilization and automation of crucible packing in production of semiconductor plates.

Problems of packing 3D objects are NP-hard, and various heuristics are usually used to solve such problems. The well-known approaches to solution of three-dimensional packing problems can be divided into the following groups:

— heuristic methods (heuristics based on relaxation of information about the form of objects [3]; genetic algorithms [4]; algorithms based on the idea of simulated annealing [5]; ant algorithms [6]; algorithms that use advanced pattern search [7]);

— traditional methods of linear and nonlinear programming [8];

— combined approaches that use heuristics and mathematical programming methods [9].

In the majority of studies devoted to allocation of three-dimensional bodies, their continuous rotations are impossible. For example, the paper [10] only uses translation transformation. The paper [11] considers orthogonal rotations of objects. The study [12] proposes the HAPE3D algorithm, which enables rotations of polyhedra around each coordinate axis discretely by angles multiple of 45°.

Since problems of packing three-dimensional bodies that allow continuous rotations along with continuous translations are the least investigated, there is a need to develop a methodology for mathematical and computer modeling of optimization of packing of such three-dimensional objects.

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PROBLEM STATEMENT

Let there be given a set of nonconvex polyhedra P_i, $i \in I = \{1, 2, \dots, n\}$, and a container in the form of a cuboid $\Omega = \{X \in R^3,\ 0 \le w_1 \le x \le w_2, 0 \le l_1 \le x \le l_2, 0 \le \eta_1 \le x \le \eta_2\}$, where w_1, w_2, l_1, l_2, η_1, and η_2 are variables, i.e., vector $u_\Omega = (w_1, w_2, l_1, l_2, \eta_1, \eta_2)$ determines dimension of Ω. Polyhedra P_i are a union of convex polyhedra $P_i = \bigcup_{k=1}^{\varepsilon_i} P_{ik}$, $i \in I_n$, and polyhedra P_{ik} are defined by the vertices $p_{ikt} = (p_{ikt}^1, p_{ikt}^2, p_{ikt}^3)$, $i \in I$, $k \in K_i = \{1, 2, \dots, \varepsilon_i\}$, $t \in T_{ik} = \{1, 2, \dots, \rho_{ik}\}$. Allocation of polyhedron P_i in the Euclidean arithmetic space R^3 is defined by the translation vector $\upsilon_i = (x_i, y_i, z_i)$ and rotation angles $\theta_i = (\alpha_i, \beta_i, \gamma_i)$, $i \in I$. Thus, vector $u_i = (\upsilon_i, \theta_i) = (x_i, y_i, z_i, \alpha_i, \beta_i, \gamma_i)$ denotes allocation of P_i in R^3. Hence, vector $u = (u_1, u_2, \dots, u_n) \in R^{6n}$ determines location of P_i, $i \in I$, in R^3; thus, the complete set of variables of the problem will be defined by the vector $X = (u, u_\Omega) = (u_1, u_2, \dots, u_n, u_\Omega) \in R^m$, where $m = 6n + 6$. In what follows, we will denote by $P_i(u_i)$ the polyhedron P_i translated by vector υ_i and rotated by angles α_i, β_i, and γ_i and denote by $\Omega(u_\Omega)$ container Ω with variable dimensions defined by the vector u_Ω.

Let V_{i_r}, $r \in J_i = \{1, 2, \dots, \vartheta_i\}$, be a vertex of the convex hull P_i. Thus, $V_{i_r}(u_i) = R_i^T V_{i_r} + \upsilon_i$, $i \in I$, $r \in J_i$, $p_{ikt}(u_i) = R_i^T p_{ikt} + \upsilon_i$, $i, j \in I$, $k \in K_i$, $t \in T_{ik}$, where R_i is rotation operator.

Problem. Find a vector $u \in R^m$ that ensures disjoint allocation $P_i(u_i)$, $i \in I$, in the container $\Omega(u_\Omega)$ so as to minimize the volume $H(u_\Omega)$.

THE MATHEMATICAL MODEL AND ITS PROPERTIES

With the use of the method of Φ-functions [13], we can present the mathematical model of the problem as follows:

$$H(u_\Omega^*) = \min_{X \in W} H(u_\Omega), \tag{1}$$

$$W = \{X \in R^m : \Phi_{ij}(u_i, u_j) \ge 0,\ i < j \in I,\ \Phi_i(u_i, u_\Omega) \ge 0,\ i \in I,\ F(u_\Omega) \ge 0\}, \tag{2}$$

where

$$H(u_\Omega) = (w_2 - w_1)(l_2 - l_1)(\eta_2 - \eta_1),\ F(u_\Omega) = \min\{w_2 - w_1, l_2 - l_1, \eta_2 - \eta_1\},$$

$$\Phi_i(u_i, u_\Omega) = \min\{\phi_{io}^r(u_i, u_\Omega),\ o \in O = \{1, 2, \dots, 6\},\ r \in J_i\},$$

$$\phi_{i1}^r(u_i, u_\Omega) = V_{1ir}(u_i) - w_1,\ \phi_{i2}^r(u_i, u_\Omega) = w_2 - V_{1ir}(u_i),\ \phi_{i3}^r(u_i, u_\Omega) = V_{2ir}(u_i) - l_1,$$

$$\phi_{i4}^r(u_i, u_\Omega) = l_2 - V_{2ir}(u_i),\ \phi_{i5}^r(u_i, u_\Omega) = V_{3ir}(u_i) - \eta_1,\ \phi_{i6}^r(u_i, u_\Omega) = \eta_2 - V_{3ir}(u_i).$$

Also, the inequality $\Phi_{ij}(u_i, u_j) \ge 0$ ensures non-intersection of P_i i P_j and the inequality $\Phi_i(u_i, u_j) \ge 0$ ensures finding P_i in $\Omega(u_\Omega)$, i.e., $\Phi_i(u_i, u_\Omega)$ is an Φ-function P_i and $B(u_\Omega) = R^3 \setminus \operatorname{int} \Omega(u_\Omega)$. Noteworthy is that $\Phi_{ij}(u_i, u_j)$ $= \min\{\Phi_{ij}^{sp}(u_i, u_j), s \in K_i, p \in K_j\}$, where $\Phi_{ij}^{sp}(u_i, u_j)$ is the Φ-function for a pair of convex polyhedra [14].

Let us consider some important features of the mathematical model (1), (2) that influence the development of the problem solution methodology.

1. The domain W of feasible solutions of the problem in the general case is a non-connected set, and each its connected component is multiply connected and has a “ravine” nature.

2. The inequality $\Phi_i(X) \ge 0$ is a system of continuous differentiable functions.

3. Since each function $\Phi_{ij}(X)$ is a system of maximin functions, the domain of feasible solutions can be presented as a union of subdomains, i.e., $W = \bigcup_{q=1}^{\varsigma} W_q$, where each subdomain W_q is determined by the system of

inequalities with continuously differentiable functions. Thus, problem (1), (2) can be reduced to a sequence of problems $F(X^*) = \text{extr}\ \{F(X^{*q}), q = 1, 2, \dots, \varsigma\}$, where $F(X^{*q}) = \underset{X \in W_q}{\text{extr}}\ F(X)$.

4. Each subproblem $F(X^*) = \text{extr}\ \{F(X^{*q}), q = 1, 2, \dots, \varsigma\}$ is a milti-extremum nonlinear programming problem.

5. Problem (1), (2) belongs to the class of NP-hard probems.

GENERAL STRATEGY OF THE MULTISTAGE APPROACH

To reduce large computing and time costs, let us decompose the problem solution into the preparatory stage and multiple-run stage.

At the preparatory stage, a number of nonlinear programming auxiliary problems are implemented, which allow to obtain data for the construction of the initial points of the main problem (1), (2). At the stage of multiple run, different initial feasible points are constructed and local minima corresponding to them are found. To solve these problems, a strategy based on homothetic transformations and construction of promising points is used [15, 16].

The best local minimum obtained as a result of the multiple-run phase is chosen as an approximation to the global minimum of the problem.

CONSTRUCTING A FEASIBLE INITIAL POINT

To construct a feasible initial point for the problem (1), (2), we propose a clustering method that has the following algorithm.

The given polyhedra are packed pairwise in cuboids (clusters) of minimum volume. The obtained set of clusters is used to form a subset of clusters that covers the set of given polyhedra, based on the criterion of the maximum coefficient of cluster filling by polyhedra. Then the problem of packing of the generated subset of clusters into a cuboid of minimum volume is solved and polyhedra allocation parameters are determined for each polyhedron, according to allocation of clusters (Fig. 1). To calculate the angles of rotation of each polyhedron, a nonlinear programming problem is solved.

Let us consider the proposed approach in more detail. Let the set of polyhedra P_i, $i \in I_n$, consist of k groups, each containing l_k identical polyhedra. At the first stage, each nonconvex polyhedron P_i and each its convex component are covered with balls S_i of minimum radius r_i^* and balls S_{ik} of minimum radius r_{ik}^*, $i \in I_n$, $k \in K_i$. To this end, the nonlinear programming problems are solved:

$$r_i^* = \min_{(v_i, r_i) \in D_i \subset R^4} r_i, \ i \in I_n,$$

$$D_i = \{(v_i, r_i) \in R^4 : \Psi_{ij} = r_i^2 - (x'_{ij} - x_i)^2 - (y'_{ij} - y_i)^2 - (z'_{ij} - z_i)^2 \geq 0, \ j \in T_i\}.$$

Let (v_i^*, r_i^*) be a point of local minimum of this problem. Pole of each polyhedron P_i and of each its convex component are shifted by vector v_i^*.

At the second stage, $C_n^2 + n$ problems of pairwise packing of P_i, $i \in I_n$, into cuboids C_{ij} of minimum volume D_{ij}^C are solved,

$$D_{ij}^C(u_{C_{ij}}^*) = \min_{(u_i, u_j, u_{C_{ij}}) \in W_{ij} \subset R^{18}} D_{ij}^C(u_{C_{ij}}), \tag{3}$$

$$W_{ij} = \{(u_i, u_j, u_{C_{ij}}) \in R^{18} : \Phi_{ij}(u_i, u_j) \geq 0, \Phi_i(u_i, u_{C_{ij}}) \geq 0, \ \Phi_j(u_j, u_{C_{ij}}) \geq 0, F(u_{C_{ij}}) \geq 0\}, \tag{4}$$

where $i, j \in I_n$, $u_{C_{ij}} = (l_2, l_1, w_2, w_1, h_2, h_1)$, $D_{ij}^C(u_{C_{ij}}) = (l_2 - l_1)(w_2 - w_1)(h_2 - h_1)$, $F(u_{C_{ij}}) = \min\ \{l_2 - l_1, w_2 - w_1, h_2 - h_1\}$. The inequality $\Phi_{ij}(u_i, u_j) \geq 0$ guarantees that $\text{int}\, P_i \cap \text{int}\, P_j = \varnothing$ and the inequalities $\Phi_i(u_i, u_{C_{ij}}) \geq 0$ and $\Phi_j(u_j, u_{C_{ij}}) \geq 0$ guarantee that P_i and P_j, respectively, are allocated in $C_{ij}(u_{C_{ij}})$.

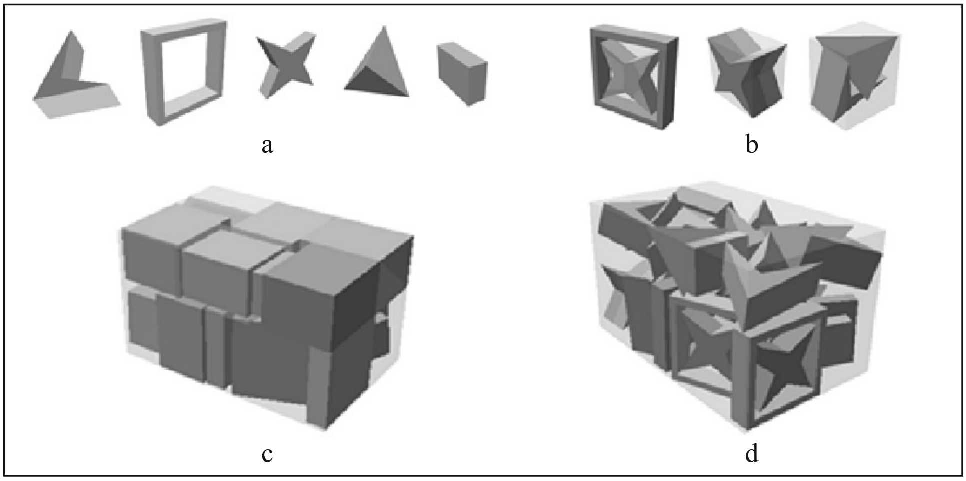

Fig. 1. Constructing the initial point by the clustering method: (a) given forms of polyhedra; (b) clusters selected by the criterion of maximum filling factor; (c) the result of packing the generated subset of clusters; (d) feasible initial point, which corresponds to the arrangement of clusters.

The result of solution of problems (3), (4) is the set K of clusters that consists of $C_n^2 + n$ parallelepipeds.

At the third stage, from the obtained set of K clusters, it is required to separate (by the criterion of maximum occupation coefficient) a subset of $\widetilde{K}$ clusters, where it is possible to allocate all the polyhedra P_i, $i \in I_n$.

Thus, each cluster Q_i contains a pair of polyhedra P_{k_i} and P_{t_i} with allocation parameters $u_{k_i}^Q$ and $u_{t_i}^Q$ with respect to its local coordinate system Q_i.

Then we solve the problem of packing of the generated subset of clusters Q_i, $i \in M$, into a cuboid Ω of minimum volume. On the basis of Eqs. (1), (2), let us write the mathematical model of the problem as follows:

$$H(\widetilde{u}_\Omega^*) = \min_{(\widetilde{u}, \widetilde{u}_\Omega) \in \widetilde{W} \subset R^{6\mu+6}} H(\widetilde{u}_\Omega), \tag{5}$$

$$\widetilde{W} = \{(\widetilde{u}, \widetilde{u}_\Omega) \in R^{6\mu+6} : \Phi_{ij}(\widetilde{u}_i, \widetilde{u}_j) \geq 0, \ i < j \in M,$$

$$\Phi_i(\widetilde{u}_i, \widetilde{u}_\Omega) \geq 0, \ i \in M, \ F(\widetilde{u}_\Omega) \geq 0\}, \tag{6}$$

where

$$\widetilde{u}_\Omega = (l_2, l_1, w_2, w_1, h_2, h_1), \ H(\widetilde{u}_\Omega) = (l_2 - l_1)(w_2 - w_1)(h_2 - h_1),$$

$$F(\widetilde{u}_\Omega) = \min\{l_2 - l_1, w_2 - w_1, h_2 - h_1\}.$$

Let point $(\widetilde{u}^*, \widetilde{u}_\Omega^*) \in R^{6\mu+6}$ be an approximation to the point of global minimum of problem (5), (6). It corresponds to the packing of clusters $Q_i(u_i^*)$, $i \in M$, into the cuboid $\Omega(u_\Omega^*)$ and each cluster contains the pair of polyhedra P_{k_i} and P_{t_i} with allocation parameters $u_{k_i}^Q$ and $u_{t_i}^Q$ with respect to the local system of coordinates of cluster Q_i. To pass to the feasible initial point $(u^0, u_\Omega^0) \in W$ of problem (1), (2), it is necessary, according to the allocation of clusters Q_i, $i \in M$, to determine the parameters of allocation of polyhedra P_i, $i \in I_n$. Let us put $u_\Omega^0 = \widetilde{u}_\Omega^*$ and calculate the parameters of allocation of polyhedra P_i, $i \in I_n$, by the formula $v_i^0 = \widetilde{v}_i^* + v_i^Q$. To find the rotation angles θ_i^0 of the polyhedra P_i, $i \in I_n$, it is necessary to solve n nonlinear programming problems of the form

$$r_{13}^{i*} = \min_{R_i \in D_i \subset R^9} r_{13}^i, \tag{7}$$

$$D_i = \left\{ R_i \in R^9 : V_1^i R^i = \widetilde{V}_1^i,\ V_2^i R^i = \widetilde{V}_2^i,\ V_3^i R^i = \widetilde{V}_3^i, \right.$$

$$\left. \sum_i r_{ij} r_{ik} = \delta_{jk},\ \sum_i r_{ji} r_{ki} = \delta_{jk},\ i = 1, 2, 3 \right\}, \tag{8}$$

where $i \in I_n$, V_1^i, V_2^i, and V_3^i are vectors of output coordinates of the first three vertices of the polyhedron P_i; $\widetilde{V}_j^i = \widetilde{R}_i^* (R_i^Q V_j^i + v_i^Q)$, $j = 1, 2, 3$; R_i is the required rotation matrix. Let R_i^* be solution of problem (7), (8). Then the angles of rotation of the polyhedron P_i can be determined as $\beta_i = \arcsin r_{13}^{i*}$, $\alpha_i = \arcsin(-r_{23}^{i*} / \cos\beta_i)$, and $\gamma_i = \arccos(-r_{12}^{i*} / \cos\beta_i)$.

LOCAL OPTIMIZATION

Note that the domain of feasible solutions of problem (1), (2) can be described by a large number of nonlinear inequalities [17]. This requires methods to be developed that would efficiently solve the problem of high dimension. The idea of the local optimization method is based on decomposition of the main problem into subproblems of smaller dimension and with a significantly smaller number of constraints. To this end, the following steps are distinguished: sequential generation of subdomains for domains of feasible solutions that contain the initial point; determining the subsystem of ε-active constraints; finding local extrema on the selected subdomains using modern solvers of nonlinear programming problems of second order; organizing transition to other subdomains. Let us consider the developed methods in more detail.

Internal Point Method with the Decomposition Strategy. Let point $X^\bullet \in W$ be an initial point. Search for a local extremum begins with selecting a subdomain W_0 such that $X^\bullet \in W_0 \subset W$. To construct a subdomain W_0, it is necessary to substitute point $X^\bullet$ into the inequalities that generate system (2). Since the Φ-function has the form $\Phi'_{ij}(u_i, u_j) = \max\{\Psi_{ij}^s(u_i, u_j),\ s = 1, \ldots, \wp_{ij}\}$, feasible subdomain W_0 will have the form of a system of inequalities that is generated due to choosing, in each Φ-function $\Phi'_{ij}(u_i, u_j)$, one of the functions $\Psi_{ij}^{a_{ij}}(u_i, u_j)$, $a_{ij} \in \{1, \ldots, \wp_{ij}\}$, $i < j \in I$, such that $\Phi'_{ij}(u_i^\bullet, u_j^\bullet) = \Psi_{ij}^{a_{ij}}(u_i^\bullet, u_j^\bullet) = \chi_{ij}^\bullet$. Inequalities from $\Phi_i(u_i, u_\Omega) \ge 0$, $i \in I$, can be selected similarly. As a result, we obtain system of inequalities $\Upsilon^0(X) \ge 0$, which describes the subdomain W_0. Then we solve the problem $F(u_\Omega^{0*}) = \min\limits_{X \in W_0 \subset R^\vartheta} F(u_\Omega)$ and calculate the point of local minimum X^{0*} for the initial point $X^\bullet \in W_0$. Then we separate active inequalities $\varsigma_{j0}(\xi_j^{0*}) \ge 0$, $j \in \Gamma_0 = \{1, 2, \ldots, \mu_0\} \subset \Gamma = \{1, 2, \ldots, \mu\}$, in the system $\Upsilon^0(X^{0*}) \ge 0$. Let these inequalities belong to the subsystem of inequalities $\Psi_{ij}^a(u_i, u_j) \ge 0$, $i \in I_{0\eta_1} \subset I$, $j \in I_{0\eta_2} \subset I$. This makes it possible to choose functions $\widehat{\Phi}'_{ij}(u_i, u_j)$, which contain functions $\Psi_{ij}^a(u_i, u_j)$, $i \in I_{0\eta_1}$, $j \in I_{0\eta_2}$, and calculate their values at point X^{0*}. Let $\Phi'_{ij}(u_i^{0*}, u_j^{0*}) = \Psi_{ij}^c(u_i^{0*}, u_j^{0*}) = \chi_{ij}^0$, $i \in I_{0\eta_1}$, $j \in I_{0\eta_2}$. If $\chi_{ij}^0 > 0$, $i \in I_{0\eta_1}$, $j \in I_{0\eta_2}$, we replace subsystems $\Psi_{ij}^a(u_i, u_j) \ge 0$ with subsystems $\Psi_{ij}^c(u_i, u_j) \ge 0$, $i \in I_{0\eta_1}$, $j \in I_{0\eta_2}$. As a result, we obtain a new subsystem of inequalities that defines a new feasible subdomain $W_1 \subset W$. Obviously, $X^{0*} \in W_1$. For the initial point X^{0*}, we solve the problem $F(u_\Omega^{1*}) = \min\limits_{X \in W_1 \subset R^m} F(u_\Omega)$ and calculate a local minimum point X^{1*}. We repeat the computations ρ times until the condition $F(u_\Omega^{(\rho-1)*}) = F(u_\Omega^{\rho*})$ is satisfied.

Reducing the Number of Constraints. Systems of inequalities $\Upsilon^\tau(X) \ge 0$ that define subdomains W_τ, $\tau \in \{0, \ldots, \zeta\}$, also comprise a large number of inequalities. It is obvious that the number of constraints need to be significantly reduced for efficient solution of packing problems. To this end, the paper proposes a special decomposition method, which reduces problem (1), (2) to a sequence of optimization subproblems. For each such subproblem, an additional system of constraints for variables is introduced, which allows, at each stage of searching for local extremum, to consider only some constraints from the entire system $\Upsilon^{\tau}(X) \geq 0$. An advantage of this approach is that it substantially reduces the number of constraints that define feasible domains of subproblems.

Let us consider the decomposition method in more detail. First of all, we specify $\sigma > 0$ and among the inequalities $\Phi_{ij}(u_i, u_j) \geq 0$, $i< j \in I$, and $\Phi_i(u_i, u_\Omega) \geq 0$ separate the inequalities $\Phi_{ij}(u_i, u_j) \geq 0$ such that

$$||v_i^{(\tau-1)*} - v_j^{(\tau-1)*}|| - (r_i^0 + r_j^0) \leq \sigma,\ i< j \in I, \tag{9}$$

and the inequalities $\Phi_i(u_i, u_\Omega) \geq 0$ such that at least one of the conditions is satisfied:

$$\begin{gathered} x_i^{(\tau-1)*} - r_i^0 - l_1^{(\tau-1)*} \leq \sigma,\ y_i^{(\tau-1)*} - r_i^0 - w_1^{(\tau-1)*} \leq \sigma, \\ z_i^{(\tau-1)*} - r_i^0 - h_1^{(\tau-1)*} \leq \sigma,\quad l_2^{(\tau-1)*} - x_i^{(\tau-1)*} - r_i^0 \leq \sigma, \\ w_2^{(\tau-1)*} - y_i^{(\tau-1)*} - r_i^0 \leq \sigma,\ h_2^{(\tau-1)*} - z_i^{(\tau-1)*} - r_i^0 \leq \sigma,\ i \in I. \end{gathered} \tag{10}$$

In the system $\Upsilon^{\tau}(X) \geq 0$, let us separate inequalities of the form $\varsigma_{t\tau}(\xi_t) \geq 0$, which are parts of the inequalities $\Phi'_{ij}(u_i, u_j) \geq 0$, $i< j \in I$, and $\Phi_i(u_i, u_\Omega) \geq 0$, $i \in I$. These inequalities form the subsystem $\Upsilon^{\tau 0}(X) \geq 0$, which consists of t_{μ_0} inequalities and define the subset $W_{\tau 0} \subset W_\tau$. Obviously, $\Upsilon^{\tau 0}(X^{(\tau-1)*}) \geq 0$ and $t_{\mu_0} < \mu_\tau$.

Subsystem $\Upsilon^{\tau 0}(X) \geq 0$ describes the subset of points $X^{\tau 0} \in W_{\tau 0} \subset R^m$. For the initial point $X^{(\tau-1)0*} \in W_{\tau 0}$, let us solve the problem

$$F(u_\Omega^{\tau 0*}) = \min_{X_{\tau 0} \in W_{\tau 0} \subset R^{\vartheta}} F(u_\Omega), \tag{11}$$

$$W_{\tau 0} = \left\{ X^{\tau 0} \in R^m : \Upsilon^{\tau 0}(X^{\tau 0}) \geq 0,\ \frac{1}{2}\sigma - ||v_i^{(\tau-1)*} - v_i|| \geq 0, i \in I \right\}. \tag{12}$$

Then we generate a new system of inequalities $\Upsilon^{\tau 1}(X) \geq 0$ for the point $X^{\tau 0*}$. With regard for this point, we generate point $X^{\tau 1} \in W_{\tau 1}$ as initial one and solve the problem $F(u_\Omega^{\tau 1*}) = \min\limits_{X^{\tau 1} \in W_{\tau 1} \subset R^m} F(u_\Omega)$. Thus, the sequence of subproblems $F(u_\Omega^{\tau k*}) = \min\limits_{X^{\tau k} \in W_{\tau k} \subset R^m} F(u_\Omega)$, $k = 1, 2, 3,$ is solved until the condition $F(u_\Omega^{\tau(k-1)*}) = F(u_\Omega^{\tau k*})$ is satisfied.

It can be easily seen that the less σ, the less the number of inequalities that define $W_{\tau k}$ and the more k subproblems need to be solved to find a local minimum point. On the other hand, the greater the σ, the more inequalities are contained in the system that defines $W_{\tau k}$, and the more time is required to solve the subproblems. As numerical experiments showed, the values of σ can be chosen equal to the average radius of balls that cover the polyhedra.

Note that search for a local minimum of problem (1), (2) can be divided into two stages: optimization with a system of linear constraints and nonlinear programming. Optimization stage can be implemented by fixing the rotation angles $\theta_i^0 = \text{const}$ of the polyhedra P_i, $i \in I_n$, at the initial point $(u^0, u_\Omega^0) \in W$. Fixation of the rotation angles allows significant reduction of the dimension of problem (1), (2), passage to the linear constraints that form the domain of feasible solutions W, and modification of the decomposition algorithm in order to reduce the number of constraints. This significantly reduces the computing costs and allows faster finding of an approximation to the local minimum. At this stage, the mathematical model of the problem is

$$H(u_\Omega^*) = \min_{X \in W} H(u_\Omega),$$

$$W = \{X = (v, u_\Omega) \in R^{3n+6} : \Phi_{ij}(v_i, v_j) \geq 0, i< j \in I,\ \Phi_i(v_i, u_\Omega) \geq 0, i \in I, F(u_\Omega) \geq 0\}.$$

Since the rotation angles are fixed, the systems of inequalities $W_{\tau k}$ (12) are linear.

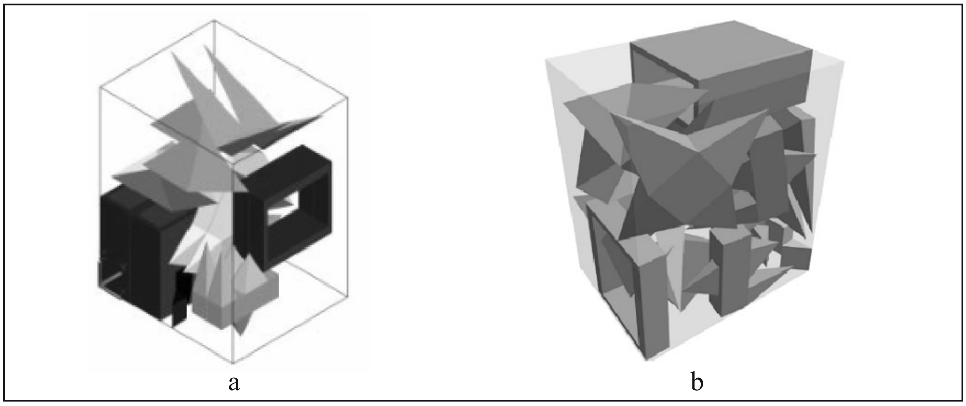

Fig. 2. Packing 20 polyhedra: (a) method HAPE3D (volume 32,550; time 26,202 sec); (b) the developed approach (volume 28,500; time 6,656 sec).

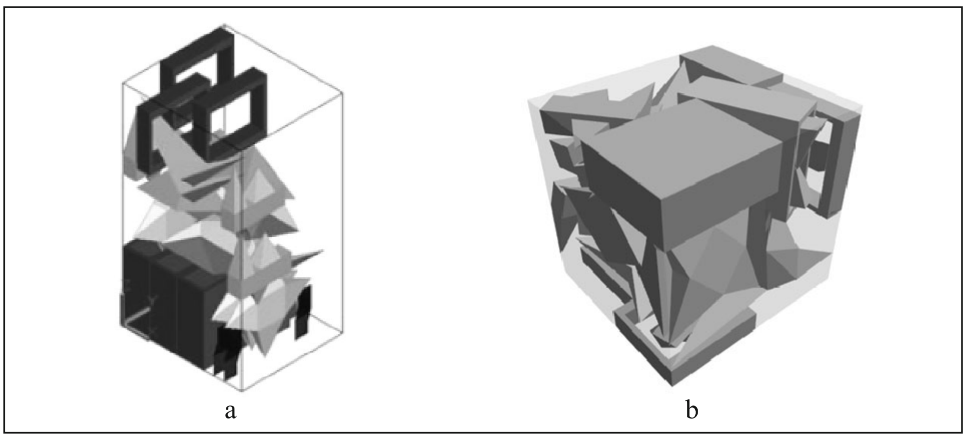

Fig. 3. Packing 30 polyhedra: (a) method HAPE3D (volume 48,300; time 53,741 sec); (b) the developed approach (volume 42,450; time 9,543 sec).

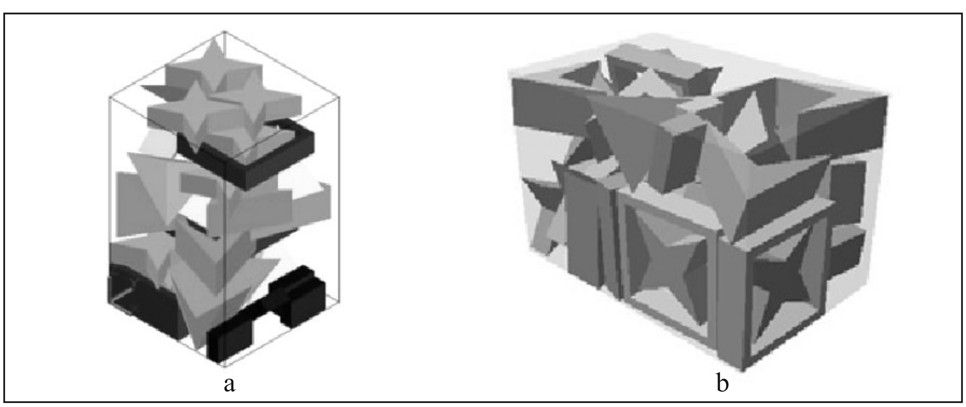

Fig. 4. Packing 36 polyhedra: (a) method HAPE3D (volume 12,480; time 9,637 sec); (b) the developed approach (volume 10,720; time 4,789 sec).

COMPUTING EXPERIMENTS

The experiments were conducted with a C# software. The software is developed according to the modularity principle and consists of the following modules: input of initial data; calculation of Φ-functions; generation of

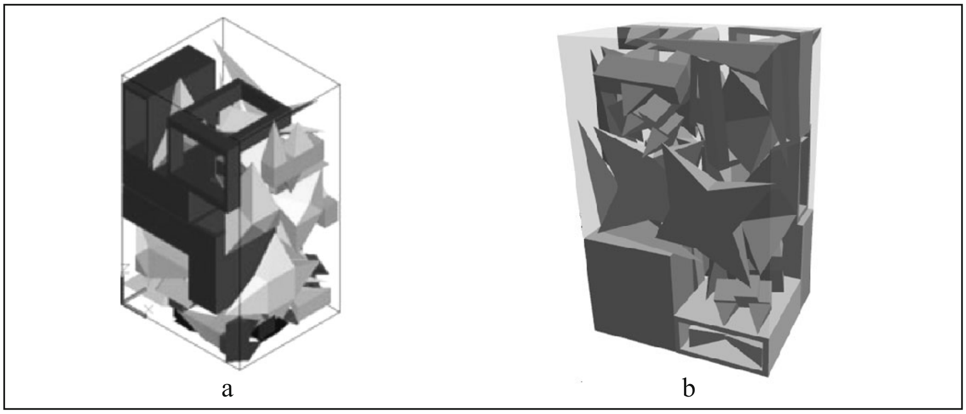

Fig. 5. Packing 40 polyhedra: (a) method HAPE3D (volume 6,150; time 99,952 sec); (b) the developed approach (volume 56,012; time 24,543 sec).

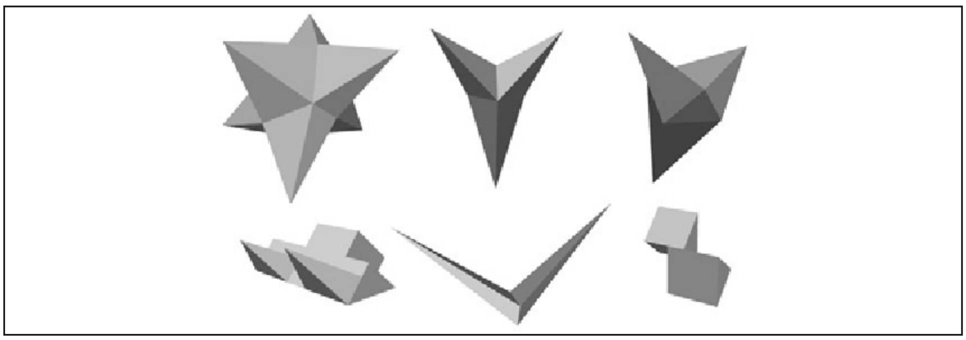

Fig. 6. Forms of polyhedral bodies.

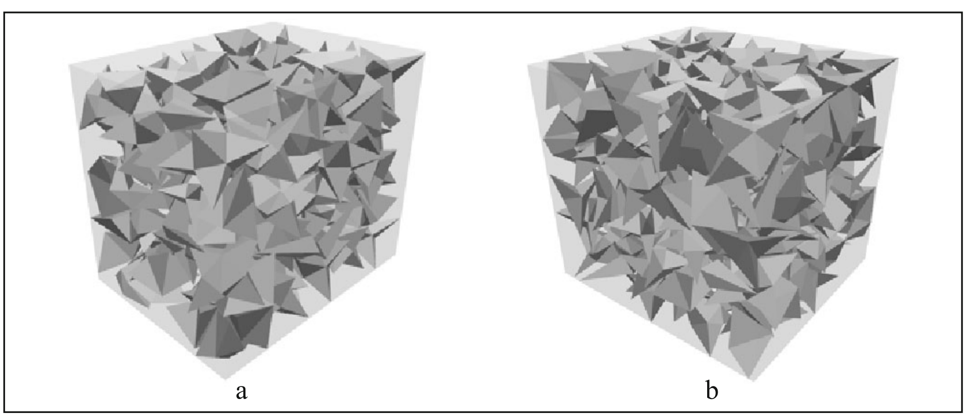

Fig. 7. Packing nonconvex polyhedral bodies: (a) 150 polyhedral bodies; (b) 200 polyhedral bodies.

subproblems; generation of initial points; local optimization; global optimization. Software modularity has allowed decomposition of the algorithm and application of the parallel computing technology, which has reduced the solution time. To find local extrema, a modern solver for nonlinear programming problems as a free library IPOPT v. 3.9.1 (Interior Point OPTimizer) [18] was used. Note that the IPOPT library is efficient due to the method of internal points. Parameters of the computer used for the computing experiments are Intel Core I5-750 processor, 2.5 GHz, 6 Gb RAM.

To test the developed approach for efficiency, a number of test examples from [12] were solved using the HAPE3D method. The results of polyhedra packing are presented in Figs. 2–5.

As can be seen from the figures, all the test examples show improvements by both the result and solution time. Thus, the computing experiments confirm the efficiency and reliability of the developed methods.

The packing problem was solved for sets of 150 and of 200 polyhedral bodies shown in Fig. 6. The corresponding results are shown in Fig. 7. The problem time was 32 and 41 hours, respectively.

CONCLUSIONS

In the paper, we have constructed an exact mathematical model for the problem of optimal packing of nonconvex unoriented polyhedra. We applied the method of Φ-functions, which allows the use of modern methods of nonlinear programming at all stages of solution of problem (1), (2), including construction of initial points, calculation of local minima, and search for "approximations" to the global minimum.

Due to the method of clustering of nonconvex unoriented polyhedral three-dimensional bodies, construction of initial points reduces the problem of packing a twice smaller number of convex bodies of much simpler spatial form. Thus, the time for generating the initial points significantly reduces. Noteworthy is that the computing costs are reduced due to division of search for a local extremum into two stages: the stage of solving the linear problem by fixing the rotation angles and the stage of solving the nonlinear programming problem (1), (2).

The iterative processes used to solve the problem can be easily parallelized. The results have shown the efficiency of the proposed approach to the problem solution.

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