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AN EFFICIENT METHOD FOR STABILITY ANALYSIS OF HIGHLY NONLINEAR DYNAMIC SYSTEMS*

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Abstract. *A simple and quick method is proposed for estimating the asymptotic stability of highly nonlinear dynamic systems, in particular, high-dimensional systems for which Taylor series of the right sides of differential equations converge slowly and the sum of terms whose order of smallness is more than two can considerably exceed the value of any second-order term. In this case, the method of Lyapunov functions cannot guarantee a correct stability estimate. The new method is based on a procedure of maximizing the rate of the change in the metric of the perturbed state space. This metric can turn out to simultaneously be a Lyapunov function only in particular cases. The proposed new method is not aimed at estimating the stability of linear systems.*

Keywords: *motion stability, nonlinear dynamic system.*

INTRODUCTION

The basic foundations of the general stability theory of dynamic systems were laid in the thesis of A. M. Lyapunov in 1892 and were later replenished by N. G. Chetayev, N. N. Krasovskii, et al. Some partial stability investigation methods were also earlier developed, for example, by E. J. Routh, N. E. Zhukovsky, and other scientists, but a systematized general statement of motion stability problems was proposed and deeply worked out by A. M. Lyapunov. The Lyapunov method is still most general and allows to exactly estimate the stability of linear dynamic systems, but it is not always correct in the case of nonlinear systems [1–4].

The method of Lyapunov functions is simple in its ideological basis, but its practical use leads to large and quite often insolvable difficulties. The first drawback is conditioned by the fact that a sign-defined function (a quadratic form) $V(x)$ must be chosen so that its total time derivative $\dot{V} = \frac{\partial V}{\partial x}\frac{dx}{dt}$, where $\frac{dx}{dt} = f(x)$ is a vector differential motion equation, is a sign-defined function of the opposite sign with respect to the function V. However, the search for such a function can rather be attributed to mathematical intuition than to science. The second drawback is that a Lyapunov function $V(x)$ is only a quadratic form composed of second-order infinitesimals in expanding the right sides of differential equations into Taylor series. But a quadratic form in essentially high-dimensional problems ($n \geq 5$) with slowly convergent Taylor series does not take into account the partial sums of expansion in a Taylor series that in high-dimensional problems can exceed even the largest term of quadratic form. In this case, it cannot be the basis for estimating stability. For example, the sum of all the terms of third-order smallness in essentially nonlinear high-dimensional problems in which Taylor series converge slowly, as a rule, exceeds (due to a large number of third-order expansion terms when n is large) any of second-order terms from which a Lyapunov function is constructed. It follows from this that a Lyapunov function does

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not take into account complete information on the dynamics of a system and, hence, it cannot ensure a reliable sensitivity analysis of the system. This is the cause of errors in estimating stability by the method of Lyapunov functions and an essential mismatch of the stability domains obtained based on different Lyapunov functions in the same problem. The following question arises: how correct are the stability estimates obtained based on Lyapunov functions?

This work considers a "variation" method that has the following positive characteristics: the preliminary decomposition of right sides of differential equations into Taylor series is not required; a complicated and not always efficient search for a Lyapunov function is not required; the problem is reduced to a simple problem of searching for the maximum of a function of a finite number of variables. This method ensures necessary stability conditions unlike of the Lyapunov method that is based on sufficient conditions. The new method proposes to determine the maximum rate of changing the Euclidean metric $S(x)$ in the space X instead of a laborious search for Lyapunov functions. In particular cases when this metric turns out to be a Lyapunov function in a concrete problem being solved, the found stability conditions become necessary and sufficient.

PROBLEM STATEMENT

Let some process in an n-dimensional Euclidean space be described by the following vector ordinary differential equation:

$$\frac{dy}{dt} = Y(y(t), t), \tag{1}$$

where $Y(y(t), t)$ is a given vector function satisfying the requirements ensuring the existence of a solution to Eq. (1) and $y(t) = (y_1(t), \dots, y_n(t))$ is the vector function of phase coordinates $y_i(t)$, $i = 1, \dots, n$. The existence and continuity of second partial derivatives $\frac{\partial^2 Y_i}{\partial y_k^2}$, $i, k = 1, \dots, n$, are additionally supposed.

Let $z(t)$ be some solution to Eq. (1) concerning which it is required to establish whether it is stable against small disturbances $x(t)$. These disturbances can be specified in the form

$$x(t) = y(t) - z(t).$$

We substitute this equality into Eq. (1) and rewrite it in coordinates

$$\frac{dz_i}{dt} + \frac{dx_i}{dt} = Y_i(z_1 + x_1, \dots, z_n + x_n, t), \quad i = 1, \dots, n. \tag{1a}$$

We will investigate the stability based on these equations or by previously decomposing the right sides of Eqs. (1a) in Taylor series in the neighborhood of the solution $z(t)$ as follows:

$$\frac{dz_i}{dt} + \frac{dx_i}{dt} = Y_i(z, t) + \left(\sum_{k-1}^{n} \frac{\partial Y_i}{\partial x_k} \right)_z x_k + \Delta Y_i, \quad i = 1, \dots, n,$$

where ΔY_i is the sum of decomposition terms with higher than the first-order of smallness with respect to x.

Since the solution $z(t)$ that is examined for stability satisfies Eq. (1), i.e., the equation

$$\frac{dz}{dt} = Y(z, t), \tag{2}$$

we obtain the equations of perturbed motion in the form

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} \left(\frac{\partial Y_i}{\partial x_k} \right)_z x_k + \Delta Y_i, \quad i = 1, \dots, n. \tag{3}$$

The stability of the given solution $z(t)$ is determined by the character of tending the solution $x(t)$ of Eqs. (3) or (1a) to zero. The use of only the linear part of Eq. (3) is rarely resultative, and the analysis of this equation with allowance for its nonlinear terms is usually carried out with the help of Lyapunov functions $V(x)$.

If, as the Lyapunov function, we manage to select a definite positive quadratic form $V(x)$ such that its total time derivative $\dot{V} = \left(\operatorname{grad} V(x)\, \frac{dx}{dt}\right) < 0$ used taking into account differential equations (3) of perturbed motion turns out to be definite negative, then the solution $z(t)$ to Eq. (2) according to the main Lyapunov theorem [1, p. 39] is asymptotically stable. However, as a rule, the search for such a function $V(x)$ is laborious and not always efficient; therefore, the search for the asymptotic stability corresponding to the pair $(V > 0,\ \dot{V} < 0)$ is replaced with the search for the pseudo-stability corresponding to the pair $(V > 0,\ \dot{V} \equiv 0)$ [1–4].

DESCRIPTION OF THE VARIATION METHOD

We will consider an approach to the problem of investigating the stability of dynamic systems that is different from the classical methods [1–4] and is based on the use of calculus of variations [5]. Let, in Euclidean space, a metric (for practical computations, it is more convenient to use the “semimetric” $S = \frac{1}{2}\sum_{k=1}^{n} x_k^2$) and some small number $\varepsilon > 0$ be given. We will analyze the solutions to Eqs. (3) or (1a) in a small neighborhood of zero of the space X, i.e., in a domain

$$S = \frac{1}{2}\sum_{k=1}^{n} x_k^2 < \varepsilon. \tag{4}$$

Definition 1. Some solution to Eq. (1) is called ε-stable if a small number $\varepsilon > 0$ and a moment $0 < t_1 < \infty$ can be found such that, for all $t > t_1$, the trajectory of motion $x(t)$ will not leave domain (4). Assume that motion is monotone asymptotically stable if, for any arbitrarily small number $\varepsilon > 0$, the trajectory of $x(t)$ monotonically tends to zero in domain (4) and reaches zero when $t \le \infty$ and, hence, the trajectory of motion in the space X meets the condition

$$\dot{S} = (\operatorname{grad} S\, \dot{x}) \le 0.$$

Assume that, in the problem being considered, asymptotic stability takes place. As follows from Definition 1, an object moves in the space X into sphere (4) and, hence, the function $\dot{S}$ is nonpositive in domain (4) and its maximum is reached (when $t > t_1$) at some point x of this domain, in particular, when $x = 0$. It is obvious that the global maximum in this domain, if any, is achieved at zero of the space X, which signifies monotone asymptotic stability, and any local maxima in domain (4) that are reached at interior points of this domain with the exception of the point $x = 0$ can determine no more than ε-stability according to Definition 1.

In fact, this statement is the proof of the following theorem.

THEOREM 1. In order that some solution of dynamic system (1) be ε-stable in relation to small disturbances (4), it is necessary that the total time derivative $\dot{S}$ computed with allowance for differential equations (1a) or (3) of perturbed motion achieve its maximum in domain (4), and, in the case of monotone asymptotic stability, the maximum be reached at zero of the space X.

COROLLARY 1. In order that the asymptotic stability of the zero solution of Eq. (1a) or Eq. (3) with respect to coordinates x_i take place, it is necessary that, in an arbitrarily small neighborhood of the point $x = 0$, the following relations be satisfied [5, pp. 35 and 36]:

$$\frac{\partial^2 \dot{S}}{\partial x_i^2} \le 0, \quad \frac{\partial \dot{S}}{\partial x_i} = 0 \quad (i = 1, 2, \ldots, n).$$

COROLLARY 2. In order that, in open domain (4), the asymptotic stability of the zero solution of Eq. (1a) or Eq. (3) take place, it is necessary that, in an arbitrarily small neighborhood of the point $x = 0$, the following relations be satisfied [5, pp. 35 and 36]:

$$\frac{\partial^2 \dot{S}}{\partial x_i^2} \le 0, \quad \frac{\partial \dot{S}}{\partial x_i} = 0, \quad i = 1, 2, \dots, n.$$

Sufficiency takes place only if the function S in a concrete problem also turns out to be a Lyapunov function.

It is expedient to use Theorem 1, which allows to quickly and simply estimate the stability of complex essentially nonlinear dynamic systems, as a preliminary tool for estimating stability before passing to complex Lyapunov methods [1–4].

Comment 1. If a problem possesses asymptotic stability specified by Definition 1 and Theorem 1, then the necessary conditions of Corollaries 1 and 2 are satisfied. However, their satisfaction does not necessarily imply asymptotic stability. Hence, some confirmation of the obtained result is required using the theory of Lyapunov functions $V(x)$ that gives sufficient stability conditions. We have such a confirmation, for example, when the function S turns out to be a Lyapunov function in the problem being considered. Due to its simplicity, the proposed method can be used for obtaining a fast preliminary estimate for stability, and Theorem 1 does not imply that the function S must be a Lyapunov function. For Example 2 considered below, it was impossible in [1] to find the Lyapunov function guaranteeing the asymptotic stability with respect to the coordinates (x_1, x_2), and it was found only with respect to the variables (x_1, x_2^2). It is easy to find the complete domain of stability from the coordinates (x_1, x_2) with the help of Theorem 1.

Comment 2. In the classical theory [1–4] based on different Lyapunov functions found for the same concrete problem, as a rule, different stability conditions are obtained. Then the following question arises before the researcher (engineer): which of such Lyapunov functions is correct? If we presume that all functions are correct, then how many functions should be "thought up" to reveal all the stability domains in the problem being considered with allowance for the fact that the search for even one such a function is quite often unsuccessful? Thus, the estimation of the stability of essentially nonlinear dynamic systems should not be based on the method of Lyapunov functions. At the same time, the Lyapunov method is flawless with respect to linear systems and is probably irreplaceable for them.

Consider now the technique of using the proposed variation method (which is based on necessary optimality conditions and is simple and efficient in comparison with Lyapunov methods) for investigating the stability of essentially nonlinear dynamic systems.

EXAMPLES OF APPLICATION OF THE NEW VARIATION METHOD

Example 1 [1, p. 46]. To estimate the stability of the essentially nonlinear dynamic system

$$\dot{x}_1 = -\frac{2x_1}{(1+x_1^2)} + 2x_2, \quad \dot{x}_2 = -\frac{2x_1}{(1+x_1^2)} - \frac{2x_2}{(1+x_1^2)},$$

the Lyapunov function

$$V = \frac{x_1^2}{(1+x_1^2)} + x_2^2$$

is found in [1, p. 46] that has confirmed the asymptotic stability of this dynamic system.

Note that asymptotic stability in this problem is determined using the proposed method without any difficulties. Formulating the function $\dot{S} = x_1\dot{x}_1 + x_2\dot{x}_2$ and computing its second partial derivatives, we obtain

$$\frac{\partial^2 \dot{S}}{\partial x_i^2} = -4 < 0, \quad i = 1, 2,$$

which implies asymptotic stability according to Corollary 2. Note that, in this case, the function S turns out to be one more Lyapunov function.

Example 2. Let the nonlinear differential equations of perturbed motion be of the form [1, p. 54, 55]

$$\dot{x}_1 = ax_1 + bx_2^2, \;\; \dot{x}_2 = cx_1x_2 + ex_2^3. \tag{5}$$

It is required to find constraints on the parameters (a, b, c, e) of the dynamic system being considered that ensure its asymptotic resistance to small disturbances (x_1, x_2).

In [1, pp. 54 and 55], the Lyapunov function for system (5) is searched for in the form

$$V = \frac{1}{2}(\lambda x_1^2 + 2\mu x_1 x_2 + x_2^2), \tag{6}$$

where λ and μ are selected proceeding from the condition that the function V is positively defined and $\dot{V}$ negatively defined. For the positiveness of the function V according to the Sylvester criterion [1, p. 32], it is necessary and sufficient that the principal diagonal minors of the matrix $\begin{pmatrix} \lambda & \mu \\ \mu & 1 \end{pmatrix}$ be positive, whence the inequalities $\lambda > 0$ and $\lambda > \mu^2$ follow. By virtue of Eqs. (5), the function $\dot{V}$ is of the form

$$\dot{V} = \lambda a x_1^2 + (\lambda b + c)x_1 x_2^2 + e x_2^4 + \mu(a x_1 x_2 + b x_2^3 + c x_1^2 x_2 + e x_1 x_2^3).$$

When $\mu \neq 0$, this function is sign-variable (the proof of this fact is nontrivial). Putting $\mu = 0$, we obtain the following quadratic form $\dot{V}$ with respect to the variables x_1 and $\bar{x}_2 = x_2^2$:

$$\dot{V} = \lambda a x_1^2 + (\lambda b + c)x_1 x_2^2 + e x_2^4 = \lambda a x_1^2 + (\lambda b + c)x_1 \bar{x}_2 + e\bar{x}_2^2$$

for which the Sylvester criterion (only with respect to the variables x_1 and x_2^2)

$$\begin{pmatrix} \lambda \alpha & \frac{1}{2}(\lambda b + c) \\ \frac{1}{2}(\lambda b + c) & e \end{pmatrix}$$

implies the inequalities

$$\lambda a < 0, \;\; 4\lambda a e - (\lambda b + c)^2 > 0. \tag{7}$$

From this, taking into account the computation of the roots of the quadratic equation

$$4\lambda a e - (\lambda b + c)^2 = 0,$$

it follows that

$$a < 0, \;\; e < 0, \;\; bc < ae, \;\; \lambda_1 < \lambda < \lambda_2, \tag{8}$$

where λ_1 and λ_2 are real positive roots of the trinomial in (7).

Under conditions (8), the function $V(x)$ becomes definitely positive, the function $\dot{V}(x)$ becomes definitely negative, and, by the Lyapunov theorem, asymptotic stability takes place not with respect to the variables x_1 and x_2 but only with respect to x_1 and $\bar{x}_2 = x_2^2$.

Note that if we put $\lambda = 1$ and $\mu = 0$ in function (6), then, as is easily seen, the obtained function (we will call it S instead of V) remains positively defined and $\dot{S}$ remains negatively definite. But, in this case, taking into account inequalities (7) that assume another form, it turns out that asymptotic stability takes place under other constraints on the parameters of the equations of perturbed motion (5),

$$a < 0, \;\; e < 0, \;\; (b + c)^2 < 4ae. \tag{9}$$

It is obvious that the parametric domains of asymptotic stability (8) and (9) for two different Lyapunov functions are essentially different, and it turned out to be impossible in [1, p. 54, 55] to find a Lyapunov function ensuring the asymptotic stability with respect to the pair (x_1, x_2) since it is absent for problem (5), and the function S for this pair of coordinates is, in fact, not a Lyapunov function (it is such a function only for the pair (x_1, x_2^2)).

Let us find the asymptotic stability with respect to the pair (x_1, x_2) with the help of the variation method. We first define the function $\dot{S}$ based on metric (4) as follows:

$$\dot{S} = x_1\dot{x}_1 + x_2\dot{x}_2 = ax_1^2 + x_1x_2^2(b+c) + ex_2^4 \tag{10}$$

and obtain the first partial derivatives of function (10) that determine the extremals

$$\frac{\partial \dot{S}}{\partial x_1} = 2ax_1 + (b+c)x_2^2 = 0, \tag{11}$$

$$\frac{\partial \dot{S}}{\partial x_2} = 2x_1x_2(b+c) = 4ex_2^3 = 0. \tag{12}$$

Owing to the assumption on the existence of a maximum of the function $\dot{S}$, its second partial derivatives must be nonpositive,

$$\frac{\partial^2 \dot{S}}{\partial x_1^2} = 2a \le 0, \tag{13}$$

$$\frac{\partial^2 \dot{S}}{\partial x_2^2} = 2[x_1(b+c) + 6ex_2^2] \le 0. \tag{14}$$

From inequality (13), we obtain $a \le 0$, and the substitution of extremal (11) into inequality (14) signifies that, for any arbitrarily small x_1 and x_2, the following relations hold:

$$\frac{\partial^2 \dot{S}}{\partial x_2^2} = x_2^2\left[6e - \frac{(b+c)^2}{2a}\right] \le 0, \tag{15}$$

whence we obtain

$$12ae \ge (b+c)^2. \tag{16}$$

After substituting extremal (12) in inequality (14), we obtain $8ex_2^2 \le 0$, whence it follows that $e \le 0$. Note that extremals (11) and (12) are compatible if $4ae = (b+c)^2$.

Thus, based on the variation approach, we determine that the asymptotic stability of motion (with respect to the variables (x_1, x_2)) takes place in the problem being considered under the conditions $12ae > (b+c)^2$, $a < 0$, and $e < 0$.

Example 3. In this example as well as in Example 2, it was succeeded to find asymptotic stability in [1, p. 41] only with respect to the variables x_1^2 and x_2. Using the variation method, we will determine the stability of the following equations of perturbed motion with respect to the variables x_1 and x_2:

$$\dot{x}_1 = -x_2 + x_1x_2 - x_1^3 - \frac{1}{2}x_1x_2^2,\ \dot{x}_2 = -3x_2 + x_1x_2 + x_1^2x_2 - \frac{1}{2}x_1x_2^2. \tag{17}$$

In this case, the total time derivative of metric (4) is of the form

$$\dot{S} = -x_1x_2 + x_1^2x_2 - x_1^4 - \frac{1}{2}x_1^2x_2^2 + 3x_2^2 + x_1x_2^2 + x_1^2x_2^2 - \frac{1}{2}x_1x_2^2. \tag{18}$$

If asymptotic stability holds in problem (17), then the second partial derivatives of function (18) must be nonpositive,

$$\frac{\partial^2 \dot{S}}{\partial x_1^2} = 2x_2 - 12x_1^2 + x_2^2 \le 0, \tag{19}$$

$$\frac{\partial^2 \dot{S}}{\partial x_2^2} = x_1^2 - 6 + 2x_1 - 3x_1 x_2 \le 0. \tag{20}$$

Since, as follows from inequality (20), the inequality $\frac{\partial^2 \dot{S}}{\partial x_2^2} = -6 < 0$ holds in any small neighborhood of zero, asymptotic tending to zero takes place with respect to the coordinate x_2. It follows from inequality (19) that, for an arbitrarily small $x_2 \to 0$, it is transformed into the inequality $\frac{\partial^2 \dot{S}}{\partial x_1^2} = -12x_1^2 \le 0$, whence implies also asymptotic tending to zero with respect to the coordinate x_1.

Example 4 [1, p. 45]. Consider now the following differential equations of perturbed motion:

$$\dot{x}_1 = -x_1 + 3x_2^2, \ \dot{x}_2 = -x_1 x_2 - x_2^3. \tag{21}$$

Consider the function $\dot{S} = x_1\dot{x}_1 + x_2\dot{x}_2 = -(x_1 - x_2^2)^2$ and find its extremals

$$\frac{\partial \dot{S}}{\partial x_1} = -2(x_1 - x_2^2) = 0, \ \frac{\partial \dot{S}}{\partial x_2} = 4x_2(x_1 - x_2^2) = 0.$$

There is the following common extremal in this problem:

$$x_1 = x_2^2 \tag{22}$$

that, as is easily verified, is not a solution to the system of equations (21) and, hence, any trajectories of this system in domain (4) intersect this extremal. Let us find the second partial derivatives

$$\frac{\partial^2 \dot{S}}{\partial x_1^2} = -2, \tag{23}$$

$$\frac{\partial^2 \dot{S}}{\partial x_2^2} = 4(x_1 - x_2^2) - 8x_2^2. \tag{24}$$

It follows from Eq. (23) that the trajectory asymptotically tends to zero along the coordinate x_1. Since all trajectories of system (21) intersect extremal (22), by substituting it in Eq. (24), we obtain that, in any arbitrarily small neighborhood of zero, the inequality $\frac{\partial^2 \dot{S}}{\partial x_2^2} = -8x_2^2 < 0$ holds, which indicates the asymptotic tending of the trajectory to zero also with respect to the second coordinate x_2.

Example 5 [1, p. 52]. Let the equations of perturbed motion be of the form

$$\dot{x}_1 = x_1^2 + 2x_2^5, \ \dot{x}_2 = x_1 x_2^2. \tag{25}$$

For this system, we obtain

$$\dot{S} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1^3 + 2x_1 x x_2^5 + x_1 x_2^3.$$

Computing the second derivatives, we have

$$\frac{\partial^2 \dot{S}}{\partial x_1^2} = 6x_1, \ \frac{\partial^2 \dot{S}}{\partial x_2^2} = 40x_1 x_2^3 + 6x_1 x_2.$$

As is obvious, in any neighborhood (4) of zero, these partial derivatives can have any sign, and, hence, the system of equations (25) is unstable. The problem of stability or instability of system (25) is solved simply without much difficulty using the proposed variation method. At the same time, the solution based on the theory of Lyapunov functions presented in [1, p. 52] has turned out to be a complicated problem that required "to think up" a suitable Lyapunov function and to use not only Lyapunov theorems but also a rather complicated theorem of N. N. Krasovskii [1, pp. 51 and 52].

The next example is interesting in that all well-known theorems did not allow to make any single-valued conclusion about the stability or instability of motion.

Example 6 [1, pp. 10, 20, 21, and 105]. Let the equations of perturbed motion be of the form

$$\dot{x}_1 = \alpha(-x_2 + x_1\sqrt{x_1^2 + x_2^2}), \; \dot{x}_2 = \alpha(x_1 + x_2\sqrt{x_1^2 + x_2^2}). \tag{26}$$

With allowance for these equations, we obtain the function

$$\dot{S} = x_1\dot{x}_1 + x_2\dot{x}_2 = \alpha(x_1^2 + x_2^2)\sqrt{x_1^2 + x_2^2}$$

whose extremals are of the form

$$\frac{\partial \dot{S}}{\partial x_1} = 3\alpha x_1\sqrt{x_1^2 + x_2^2} = 0, \; \frac{\partial \dot{S}}{\partial x_2} = 3\alpha x_2\sqrt{x_1^2 + x_2^2} = 0.$$

From this, as is easily seen, we obtain that the extremals $x_1 = x_2 = 0$ satisfy equations (26) of perturbed motion, which, in fact, makes them useless for investigating stability. At the same time, the second partial derivatives of the function $\dot{S}$ also become identically zeros, which does not allow to make any single-valued conclusion about the stability or instability of system (26).

CONCLUSIONS

The considered variation method for investigating the stability of complex nonlinear dynamic systems considerably differs from the Lyapunov methods presented in [1–4] and is simple in implementation. It allows to considerably simplify and to many times speed up the search for domains of motion stability since the need does not arise to search for Lyapunov functions and to use the complicated and hardly computed criteria of J. J. Sylvester, L. Hurwicz, E. J. Routh, etc. in the case of large dimensions. Simple examples show that it is expedient to use the variation method for the preliminary stability analysis of essentially nonlinear dynamic system. Using this method, it is possible to find asymptotic stability in several minutes or hours if it exists in the problem being considered. At the same time, this method does not solve all stability problems and, in particular, cannot be used to estimate the stability of linear systems.

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