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RESOLVING-FUNCTIONS METHOD FOR GAME PROBLEMS OF CONTROL WITH INTEGRAL CONSTRAINTS

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Abstract. *The method of resolving functions is investigated as applied to game problems of control with integral constraints. A scheme of the method is proposed that ensures the end of a game within a definite guaranteed time in the class of stroboscopic strategies. The results of comparison of the guaranteed times of this scheme of the resolving-functions method with those of the first direct Pontryagin method for integral constraints are shown.*

Keywords: *linear differential game, integral constraint, multivalued mapping, measurable selector, stroboscopic strategy.*

INTRODUCTION

An approach to solving game problems of control of dynamic processes under integral constraints [1–4] is investigated as applied to the general scheme of the resolving-functions method [5, 6]. According to the methodology presented in [7, 8], the concepts of upper and lower resolving functions of two types are introduced and sufficient conditions for the guaranteed result in a linear differential game with integral constraints are obtained in the case when the Nikol'skii condition [1] does not take place. Two schemes of the resolving-functions method are proposed, the corresponding control strategies are constructed, and guaranteed times are compared.

An important peculiarity of the general scheme of the resolving-functions method is the use of information on the behavior of the opponent in the past in constructing a control action. At the same time, this information is necessary only for determining some moment of switching separating active and passive intervals of evolving a game. On the intervals themselves, the pursuer applies the countercontrol determined by the Hajek stroboscopic strategy [9]. Conditions under which information on the prehistory of the opponent is not used are found in [6, 10].

One of main results of this work is that, to realize the guaranteed completion time of a linear differential game with integral constraints, one can restrict oneself only to countercontrol without additional conditions.

This work develops ideas of [1–11] and investigations [12–23] and also presents new possibilities of applying the resolving-functions method to the solution of game control problems.

PROBLEM STATEMENT AND THE SCHEME OF THE METHOD

Consider a conflict-controlled process whose evolution is described by the linear differential equation

$$\dot{z} = Az + Bu - Cv. \tag{1}$$

Here, $z \in R^n$, $u \in R^m$, and $v \in R^k$, $n \geq 1$, $m \geq 1$, $k \geq 1$; A, B, and C are constant rectangular matrices of orders $n \times n$,

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$n\times m$, and $n\times k$, respectively; u is the control parameter of the first player; υ is the control parameter of the second player. The parameters u and υ are chosen in the form of measurable functions $u=u(\cdot)$ and $\upsilon=\upsilon(\cdot)$ from a class $L_p[0,+\infty)$, $p>1$, and satisfy the constraints

$$\int_0^\infty \|u(\tau)\|^p\, d\tau \le \mu^p,\ \mu>0, \tag{2}$$

$$\int_0^\infty \|\upsilon(\tau)\|^p\, d\tau \le \nu^p,\ \nu>0. \tag{3}$$

Such controls are called admissible.

In addition to process (1), a terminal set M^* of cylindrical form

$$M^* = M_0 + M \tag{4}$$

is given, where M_0 is a linear subspace from R^n and M is a compact from the orthogonal complement L to the subspace M_0 in R^n.

The trajectory of process (1)–(3) from its initial position $z_0\in R^n$ can be continued to terminal set (4) at a moment $T=T(z_0)$ if, based on any admissible function $\upsilon(t)$, $t\in[0,T]$, the following admissible function:

$$u(t)=u(t,z_0,\upsilon_t(\cdot)),\ t\in[0,T], \tag{5}$$

where $\upsilon_t(\cdot)=\{\upsilon(\tau),\tau\in[0,t]\}$, or the following admissible countercontrol:

$$u(t)=u(t,z_0,\upsilon(\cdot)),\ t\in[0,T], \tag{6}$$

is constructed such that an absolutely continuous solution $z(t)$ of the Cauchy problem $\dot z = Az+Bu(t)-C\upsilon(t)$, $z(0)=z_0$, turns out to be in the terminal set M^* at the moment $T=T(z_0)$.

Let π be the operator of orthogonal projection from R^n onto the subspace L. Consider linear mappings $\pi e^{tA}B_1R^m\to L$ and $\pi e^{tA}CR^k\to L$, $t\ge 0$, where B_1 is a constant rectangular matrix of order $n\times m$.

Condition 1. Let there exist a continuous non-singular matrix $D(\cdot): R^k\to R^m$ that is a solution of the matrix equation $\pi e^{tA}B_1X=\pi e^{tA}C$, where X is the sought-for matrix and B_1 is a constant rectangular matrix of order $n\times m$.

Consider the function [1]

$$\chi^p(t)=\sup_{\int_0^t \|\omega(\tau)\|^p d\tau\le 1}\int_0^t \|D(t-\tau)\omega(\tau)\|^p\, d\tau,$$

where $\omega(\cdot)$ is an arbitrary function from a space $L_p^k[0,\infty)$ with the mentioned constraint and $D(\cdot): R^k\to R^m$ is some continuous non-singular matrix meeting Condition 1.

Using the function $\chi^p(t)$, we define the quantity $\mathrm{X}^p=\sup_{0\le t<\infty}\chi^p(t)$ [1].

Condition 2. The inequality $\hat\gamma=\mu^p-\nu^p\mathrm{X}^p>0$ holds.

Let Conditions 1 and 2 be fulfilled, and let $\gamma(\tau)$ be a measurable function from a class $L_p^m[0,+\infty)$, $p>1$, that satisfies the constraint $\int_0^\infty \|\gamma(\tau)\|^p\, d\tau\le\hat\gamma$. Following [7, 8], we call it a shift function. We fix some shift function $\gamma(\tau)$ and put

$$\xi(t)=\xi(t,z_0,\gamma(\cdot))=\pi e^{tA}z_0+\int_0^t \pi e^{(t-\tau)A}B\gamma(\tau)d\tau.$$

Denote

$$U(t,\tau,v,\alpha)=\left\{u\in R^m: ||u||\le (||D(t-\tau)v||^p+\alpha\hat{\gamma})^{\frac{1}{p}}\right\},$$

where $(t,\tau)\in\Delta=\{(t,\tau): 0\le\tau\le t<\infty\}$, $v\in R^k$, and $\alpha\ge 0$. The mapping $U(t,\tau,v,\alpha)$ is a convex-valued compact-valued multivalued mapping [24].

For $(t,\tau)\in\Delta$, $v\in R^k$, and $z_0\in R^n$, we consider a multivalued mapping

$$\mathfrak{A}(t,\tau,v,z_0)=\{\alpha\ge 0: [\pi e^{(t-\tau)A}B[U(t,\tau,v,\alpha)-\gamma(\tau)]-\pi e^{(t-\tau)A}Cv]\cap\alpha[M-\xi(t)]\ne\varnothing\}. \tag{7}$$

Condition 3. For some initial position $z_0\in R^n$ and a shift function $\gamma(\cdot)$, the multivalued mapping $\mathfrak{A}(t,\tau,v,z_0)$ assumes nonempty values on a set $\Delta\times R^k$.

If this condition is satisfied, then, by analogy with [7, 8], we introduce the following upper and lower resolving functions:

$$\alpha^*(t,\tau,v,z_0)=\sup\{\alpha:\alpha\in\mathfrak{A}(t,\tau,v,z_0)\} \text{ and } \alpha_*(t,\tau,v,z_0)=\inf\{\alpha:\alpha\in\mathfrak{A}(t,\tau,v,z_0)\},$$

where $\tau\in[0,t]$, $v\in R^k$, and $z_0\in R^n$. It may be proved [6] that the multivalued mapping $\mathrm{A}(t,\tau,v,z_0)$ is closed-valued and $L\otimes B$-measurable with respect to the totality of (τ,v), $\tau\in[0,t]$, $v\in R^k$, and that the upper and lower resolving functions are $L\otimes B$-measurable with respect to the totality of (τ,v), $\tau\in[0,t]$, $v\in R^k$. Therefore, they are superpositionally measurable [6], i.e., $\alpha^*(t,\tau,v(\tau),z_0)$ and $\alpha_*(t,\tau,v(\tau),z_0)$ are measurable with respect to τ, $\tau\in[0,t]$, with any admissible function $v(\cdot)$ for which constraint (3) is satisfied. Note also that the upper resolving function is upper semicontinuous and the lower one is lower semicontinuous with respect to the variable v.

Let us consider the following set:

$$T(z_0,\gamma(\cdot))=\left\{t\ge 0: \inf_{\int_0^t\|v(\tau)\|^p d\tau\le\nu^p}\int_0^t\alpha^*(t,\tau,v(\tau),z_0)d\tau\ge 1,\ \sup_{\int_0^t\|v(\tau)\|^p d\tau\le\nu^p}\int_0^t\alpha_*(t,\tau,v(\tau),z_0)d\tau<1\right\}. \tag{8}$$

If, for some $t>0$, the upper resolving function $\alpha^*(t,\tau,v,z_0)\equiv+\infty$ for $\tau\in[0,t]$ and $v\in R^k$, then it is natural to put the value of the corresponding integral in the braces of relationship (8) equal to $+\infty$ and $t\in T(z_0,\gamma(\cdot))$ if, for this t, the other inequality in the braces of relationship (8) holds. In the case when the inequalities in relationship (8) are not satisfied for all $t>0$, we put $T(z_0,\gamma(\cdot))=\varnothing$.

THEOREM 1. Let conditions 1–3 be fulfilled, and let, for the corresponding shift function $\gamma(\cdot)$, the set $T(z_0,\gamma(\cdot))$ is not empty and $T\in T(z_0,\gamma(\cdot))$. Then the trajectory of process (1)–(3) from its initial position $z_0\in R^n$ can be continued to terminal set (4) at the moment T using an admissible control of the form (5).

Proof. Let $v(\cdot)$ be an arbitrary admissible control of the evader for which constraint (3) is satisfied.

We first consider the case when $\xi(T,z_0,\gamma(\cdot))\notin M$ and introduce the following control function:

$$h(t)=1-\int_0^t\alpha^*(T,\tau,v(\tau),z_0)d\tau-\int_t^T\alpha_*(T,\tau,v(\tau),z_0)d\tau,\ t\in[0,T].$$

The functions $\alpha^*(T,\tau,v,z_0)$ and $\alpha_*(T,\tau,v,z_0)$ are $L\otimes B$-measurable with respect to the totality of (τ,v), $\tau\in[0,T]$, $v\in R^k$, and, therefore, they are superpositionally measurable, i.e., the functions $\alpha^*(T,\tau,v(\tau),z_0)$ and $\alpha_*(T,\tau,v(\tau),z_0)$ are measurable with respect to τ, $\tau\in[0,T]$.

By the definition of T, we have

$$h(0) = 1 - \int_0^T \alpha_*(T, \tau, v(\tau), z_0) d\tau \geq 1 - \sup_{\int_0^T \| v(\tau) \|^p d\tau \leq \nu^p} \int_0^T \alpha_*(T, \tau, v(\tau), z_0) d\tau > 0,$$

$$h(T) = 1 - \int_0^T \alpha^*(T, \tau, v(\tau), z_0) d\tau \leq 1 - \inf_{\int_0^T \| v(\tau) \|^p d\tau \leq \nu^p} \int_0^T \alpha^*(T, \tau, v(\tau), z_0) d\tau \leq 0.$$

Therefore, by virtue of the continuity of the function $h(t)$, there is a time instant t_*, $t_* \in (0, T]$, such that $h(t_*) = 0$. Note that the moment of switching t_* depends on the prehistory of control of the second player $v_{t_*}(\cdot) = \{v(s) : s \in [0, t_*]\}$.

Let us consider a multivalued mapping when $v \in R^k$ and $\tau \in [0, T]$,

$$U(\tau, v) = \{u \in U(T, \tau, v, \alpha(T, \tau, v, z_0)) :$$
$$\pi e^{(T-\tau)A} B[u - \gamma(\tau)] - \pi e^{(T-\tau)A} Cv \in \alpha(T, \tau, v, z_0)[M - \xi(T)]\}, \tag{9}$$

where

$$\alpha(T, \tau, v, z_0) = \begin{cases} \alpha^*(T, \tau, v, z_0), & 0 \leq \tau \leq t_*, \\ \alpha_*(T, \tau, v, z_0), & t_* < \tau \leq T. \end{cases}$$

By virtue of the properties of the parameters of process (1) and the upper $\alpha^*(T, \tau, v, z_0)$ and lower $\alpha_*(T, \tau, v, z_0)$ resolving functions, the mapping $U(\tau, v)$ is $L \otimes B$-measurable [6] and compact-valued when $v \in R^k$ and $\tau \in [0, T]$.

In [25], A. F. Filippov first introduced the concept of a lexicographic maximum with respect to an orthogonal basis $e_1, \ldots, e_n$ of a compact $A \in K(R^n)$ using the following formula:

$$\operatorname*{lex\,max}_{e_1, \ldots, e_n} A = \bigcap_{k=0}^{n} A_k,$$

where $A_0 = A$, $A_k = \{x \in A_{k-1} : (x, e_k) = c(A_{k-1}, \psi)\}$, and $c(A_{k-1}, \psi)$ is the support function of a set A_{k-1} [24], $k = 1, \ldots, n$. The set $\operatorname*{lex\,max}_{e_1, \ldots, e_n} A$ consists of one point belonging to the set of extreme points of the convex hull of the set A.

In this case, if a multivalued mapping $U(\tau, v)$ and an orthogonal basis such that $e_1 = \psi$, $\psi \in R^m$, and $\psi \neq 0$ are chosen, then the equality $(\operatorname*{lex\,max}_{e_1, \ldots, e_m} U(\tau, v), \psi) = c(U(\tau, v), \psi)$ holds [26]. Therefore, by virtue of the theorem on a support function [27], the multivalued mapping $U(\tau, v)$ contains an $L \otimes B$-measurable selector $u(\tau, v) = \operatorname*{lex\,max}_{e_1, \ldots, e_m} U(\tau, v)$ that is a superpositionally measurable function [6] and $\| u(\tau, v) \| = (\| D(T - \tau) v \|^p + \alpha(T, \tau, v, z_0) \hat{\gamma})^{\frac{1}{p}}$, $\tau \in [0, T]$, $v \in R^k$. Assume that the control of the first player is as follows: $u(\tau) = u(\tau, v(\tau))$, $\tau \in [0, T]$.

From the Cauchy formula for process (1) with the chosen controls, we have

$$\pi z(T) = \xi(T, z_0, \gamma(\cdot)) + \int_0^T \pi e^{(T-\tau)A} [B[u(\tau) - \gamma(\tau)] - Cv(\tau)] d\tau.$$

Then, considering relationship (9), we obtain

$$\pi z(T) \in \xi(T) + \int_0^{t_*} \alpha^*(T,\tau,\upsilon(\tau),z_0)[M-\xi(T)]d\tau + \int_{t_*}^{T} \alpha_*(T,\tau,\upsilon(\tau),z_0)[M-\xi(T)]d\tau$$

$$= \xi(T)\left[1 - \int_0^{t_*} \alpha^*(T,\tau,\upsilon(\tau),z_0)d\tau - \int_{t_*}^{T} \alpha_*(T,\tau,\upsilon(\tau),z_0)d\tau\right]$$

$$+ \int_0^{t_*} \alpha^*(T,\tau,\upsilon(\tau),z_0)M d\tau + \int_{t_*}^{T} \alpha_*(T,\tau,\upsilon(\tau),z_0)M\, d\tau$$

$$= \left[\int_0^{t_*} \alpha^*(T,\tau,\upsilon(\tau),z_0)d\tau + \int_{t_*}^{T} \alpha_*(T,\tau,\upsilon(\tau),z_0)d\tau\right] M = M.$$

Thus, $z(T) \in M^*$ and it remains to show the admissibility of the control $u(\tau) = u(\tau, \upsilon(\tau))$, $\tau \in [0,T]$. By construction, the following relationships hold:

$$\int_0^T \|u(\tau)\|^p\, d\tau = \int_0^T \|D(T-\tau)\upsilon(\tau)\|^p\, d\tau + \hat{\gamma}\left[\int_0^{t_*} \alpha^*(T,\tau,\upsilon(\tau),z_0)d\tau\right.$$

$$\left. + \int_{t_*}^{T} \alpha_*(T,\tau,\upsilon(\tau),z_0)d\tau\right] \le \nu^p \mathrm{X}^p + \hat{\gamma} = \mu^p.$$

For the case when $\xi(T, z_0, \gamma(\cdot)) \in M$, the control of the first player on the whole interval $[0,T]$ is chosen in the form of a measurable function $u(\tau) = u_*(\tau, \upsilon(\tau))$, $\tau \in [0,T]$, where $u_*(\tau,\upsilon) = \operatorname{lex\,max}_{e_1,\dots,e_m} U(\tau,\upsilon)$ is an $L \otimes B$-measurable selector of the mapping $U(\tau,\upsilon)$ of relationship (9) with the resolving function $\alpha(T,\tau,\upsilon,z_0) = \alpha_*(T,\tau,\upsilon,z_0)$ on the whole interval $[0,T]$. Then, considering the apparatus of support functions [24], relationships (9) give

$$\pi z(T) \in \xi(T) + \int_0^T \alpha_*(T,\tau,\upsilon(\tau),z_0)[M-\xi(T)]d\tau \subset M$$

since, by the assumption, $c(M - \xi(T)) \ge 0$ and

$$\int_0^T \alpha_*(T,\tau,\upsilon(\tau),z_0)d\tau \le \sup_{\int_0^T \|\upsilon(\tau)\|^p d\tau \le \nu^p} \int_0^T \alpha_*(T,\tau,\upsilon(\tau),z_0)d\tau < 1.$$

Therefore, $\pi z(T) \in M$ and, hence, $z(T) \in M^*$.

Let us show the admissibility of the corresponding control of the pursuer. In view of relationships (8) and (9), we have

$$\int_0^T \|u(\tau)\|^p\, d\tau = \int_0^T \|D(T-\tau)\upsilon(\tau)\|^p\, d\tau + \hat{\gamma}\int_0^T \alpha_*(T,\tau,\upsilon(\tau),z_0)d\tau$$

$$\le \nu^p \mathrm{X}^p + \hat{\gamma} \sup_{\int_0^t \|\upsilon(\tau)\|^p d\tau \le \nu^p} \int_0^T \alpha_*(T,\tau,\upsilon(\tau),z_0)d\tau < \nu^p \mathrm{X}^p + \hat{\gamma} = \mu^p.$$

The theorem is completely proved.

It is clear from the proof of Theorem 1 that, at a moment t, the pursuer uses information about $v_t(\cdot)$, and this information is necessary only for determining the moment of switching t_* that separates the active and passive intervals. On the intervals themselves, the pursuer applies countercontrol which is determined by a stroboscopic strategy. It is shown in Theorem 2 presented below that, for ensuring the guaranteed time of Theorem 1, one can restrict oneself to countercontrol.

We put $\alpha^*(t,\tau,z_0)=\inf\limits_{v\in R^k}\alpha^*(t,\tau,v,z_0)$ and $\alpha_*(t,\tau,z_0)=\sup\limits_{v\in R^k}\alpha_*(t,\tau,v,z_0)$, $\tau\in[0,t]$, $z_0\in R^n$.

LEMMA 1. The functions $\alpha^*(t,\tau,z_0)$ and $\alpha_*(t,\tau,z_0)$ are measurable with respect to τ, $\tau\in[0,t]$.

Proof. For each $\varepsilon\in R^1$ and $\tau\in[0,t]$, consider a multivalued mapping $G_\varepsilon^*(\tau)=\{v\in R^k:\alpha^*(t,\tau,v,z_0)<\varepsilon\}$ that has open values by virtue of upper semicontinuity with respect to $v\in R^k$ of the function $\alpha^*(t,\tau,v,z_0)$ for any $\tau\in[0,t]$. The graph of this mapping $\operatorname{gr}G_\varepsilon^*(\tau)=\{(\tau,v):v\in G_\varepsilon^*(\tau),\tau\in[0,t]\}$ is $L\otimes B$-measurable with respect to the totality of (τ,v), $v\in R^k$, $\tau\in[0,t]$, for any t since the function $\alpha^*(t,\tau,v,z_0)$ is $L\otimes B$-measurable with respect to the totality of (τ,v) [6]. Then, by the theorem on the measurability of projection [27], the set $\{\tau\in[0,t]:(\tau,v)\in\operatorname{gr}G_\varepsilon^*(\tau)$ for some $v\in G_\varepsilon^*(\tau)\}$ will be L-measurable.

Let Q be a countable dense set in R^k [28]. For any $\varepsilon\in R^1$, we have

$$\{\tau\in[0,t]:\alpha^*(t,\tau,z_0)<\varepsilon\}=\{\tau\in[0,t]:G_\varepsilon^*(\tau)\neq\varnothing\}$$

$$=\{\tau\in[0,t]:(\tau,v)\in\operatorname{gr}G_\varepsilon^*(\tau)\ \text{for some}\ v\in G_\varepsilon^*(\tau)\}$$

$$=\{\tau\in[0,t]:(\tau,q)\in\operatorname{gr}G_\varepsilon^*(\tau)\ \text{for some}\ q\in G_\varepsilon^*(\tau)\cap Q\}$$

$$=\bigcup_{q\in G_\varepsilon^*(\tau)\cap Q}\{\tau\in[0,t]:(\tau,q)\in\operatorname{gr}G_\varepsilon^*(\tau)\}.$$

However, as has been noted earlier, the set $\{\tau\in[0,t]:(\tau,q)\in\operatorname{gr}G_\varepsilon^*(\tau)$ for some $q\in G_\varepsilon^*(\tau)\cap Q\}$ is measurable, and, therefore, the function $\alpha^*(t,\tau,z_0)=\inf\limits_{v\in R^k}\alpha^*(t,\tau,v,z_0)$ is a function measurable with respect to τ. Similarly, it is possible to prove the measurability of the function $\alpha_*(t,\tau,z_0)=\sup\limits_{v\in R^k}\alpha_*(t,\tau,v,z_0)$.

Consider now the set

$$\widehat{T}(z_0,\gamma(\cdot))=\left\{t\geq 0:\int_0^t\alpha^*(t,\tau,z_0)d\tau\geq 1,\quad\int_0^t\alpha_*(t,\tau,z_0)d\tau<1\right\}.\tag{10}$$

By virtue of Lemma 1, the functions $\alpha^*(t,\tau,z_0)$ and $\alpha_*(t,\tau,z_0)$ are measurable with respect to τ, $\tau\in[0,t]$. If, for some $t>0$, the function $\alpha^*(t,\tau,z_0)\equiv+\infty$ for $\tau\in[0,t]$, then it is natural to put the value of the corresponding integral in the braces of relationship (10) equal to $+\infty$ and $t\in\widehat{T}(z_0,\gamma(\cdot))$ if, for this t, the other inequality in the braces of relationship (10) holds. In the case when the inequalities in relationship (10) do not hold for all $t>0$, we put $\widehat{T}(z_0,\gamma(\cdot))=\varnothing$.

THEOREM 2. Let conditions 1–3 be satisfied, let the set $\widehat{T}(z_0,\gamma(\cdot))$ be not empty for the corresponding shift function $\gamma(\cdot)$, and let $\widehat{T}\in\widehat{T}(z_0,\gamma(\cdot))$. Then the trajectory of process (1)–(3) from its initial position $z_0\in R^n$ can be continued to terminal set (4) at a moment $\widehat{T}$ using an admissible control of the form (6).

Proof. Let $v(\cdot)$ be an arbitrary admissible control of the evader for which constraint (3) is satisfied.

Put $\alpha^*(\hat{T},\tau,z_0)=\inf_{v\in R^k}\alpha^*(\hat{T},\tau,v,z_0)$ and $\alpha_*(\hat{T},\tau,z_0)=\sup_{v\in R^k}\alpha_*(\hat{T},\tau,v,z_0)$, $\tau\in[0,\hat{T}]$, $z_0\in R^n$, and first consider the case when $\xi(\hat{T},z_0,\gamma(\cdot))\notin M$.

We introduce the following control function:

$$h(t)=1-\int_0^t\alpha^*(\hat{T},\tau,z_0)d\tau-\int_t^{\hat{T}}\alpha_*(\hat{T},\tau,z_0)d\tau,\ t\in[0,\hat{T}].$$

By the definition of $\hat{T}$, we have

$$h(0)=1-\int_0^{\hat{T}}\alpha_*(\hat{T},\tau,z_0)d\tau>0,\ h(\hat{T})=1-\int_0^{\hat{T}}\alpha^*(\hat{T},\tau,z_0)d\tau\le 0.$$

Therefore, by virtue of the continuity of the function $h(t)$, there is a time instant t_*, $t_*\in(0,\hat{T}]$, such that $h(t_*)=0$. Note that the switching moment t_* is independent of the prehistory of control of the second player $v_{t_*}(\cdot)=\{v(s):s\in[0,t_*]\}$.

Let us consider the following multivalued mapping when $v\in R^k$ and $\tau\in[0,\hat{T}]$:

$$\hat{U}(\tau,v)=\{u\in U(\hat{T},\tau,v,\alpha(\hat{T},\tau,z_0)):$$
$$\pi e^{(\hat{T}-\tau)A}B[u-\gamma(\tau)]-\pi e^{(\hat{T}-\tau)A}Cv\in\alpha(\hat{T},\tau,z_0)[M-\xi(\hat{T})]\},\tag{11}$$

where

$$\alpha(\hat{T},\tau,z_0)=\begin{cases}\alpha^*(\hat{T},\tau,z_0), & 0\le\tau\le t_*,\\ \alpha_*(\hat{T},\tau,z_0), & t_*<\tau\le\hat{T}.\end{cases}$$

By virtue of the properties of the parameters of process (1) and the upper $\alpha^*(\hat{T},\tau,z_0)$ and lower $\alpha_*(\hat{T},\tau,z_0)$ resolving functions, the mapping $\hat{U}(\tau,v)$ is $L\otimes B$-measurable [6] when $v\in R^k$ and $\tau\in[0,\hat{T}]$. By virtue of the theorem on a support function [27], the multivalued mapping $\hat{U}(\tau,v)$ contains an $L\otimes B$-measurable selector $\hat{u}(\tau,v)=\underset{e_1,\dots,e_m}{\text{lex max}}\,\hat{U}(\tau,v)$, which is a superpositionally measurable function [6] and $\|\hat{u}(\tau,v)\|=\left(\|D(\hat{T}-\tau)v\|^p+\alpha(\hat{T},\tau,v,z_0)\hat{\gamma}\right)^{\frac{1}{p}}$, $\tau\in[0,\hat{T}]$, $v\in R^k$. Assume that the control of the first player is $\hat{u}(\tau)=\hat{u}(\tau,v(\tau))$, $\tau\in[0,\hat{T}]$.

From the Cauchy formula for process (1) with the chosen controls, we have

$$\pi z(\hat{T})=\xi(\hat{T},z_0,\gamma(\cdot))+\int_0^{\hat{T}}\pi e^{(\hat{T}-\tau)A}[B[\hat{u}(\tau)-\gamma(\tau)]-Cv(\tau)]d\tau.$$

Then, taking into account relationship (11), we obtain

$$\pi z(\hat{T})\in\xi(\hat{T})+\int_0^{t_*}\alpha^*(\hat{T},\tau,z_0)[M-\xi(\hat{T})]d\tau+\int_{t_*}^{\hat{T}}\alpha_*(\hat{T},\tau,z_0)[M-\xi(\hat{T})]d\tau$$

$$=\xi(\hat{T})\left[1-\int_0^{t_*}\alpha^*(\hat{T},\tau,z_0)d\tau-\int_{t_*}^{\hat{T}}\alpha_*(\hat{T},\tau,z_0)d\tau\right]+\int_0^{t_*}\alpha^*(\hat{T},\tau,z_0)Md\tau$$
$$+\int_{t_*}^{\hat{T}}\alpha_*(\hat{T},\tau,z_0)Md\tau=\left[\int_0^{t_*}\alpha^*(\hat{T},\tau,z_0)d\tau+\int_{t_*}^{\hat{T}}\alpha_*(\hat{T},\tau,z_0)d\tau\right]M=M.$$

Thus, $z(\widehat{T}) \in M^*$, and it remains to show the admissibility of the control $\widehat{u}(\tau) = \widehat{u}(\tau, \upsilon(\tau))$, $\tau \in [0, \widehat{T}]$. By construction, the following relationships hold true:

$$\int_0^{\widehat{T}} || \widehat{u}(\tau) ||^p \, d\tau = \int_0^{\widehat{T}} || D(\widehat{T} - \tau) \upsilon(\tau) ||^p \, d\tau$$

$$+ \hat{\gamma} \left[\int_0^{t_*} \alpha^* (\widehat{T}, \tau, z_0) d\tau + \int_{t_*}^{\widehat{T}} \alpha_* (\widehat{T}, \tau, z_0) d\tau \right] \le \nu^p \mathrm{X}^p + \hat{\gamma} = \mu^p .$$

For the case when $\xi(\widehat{T}, z_0, \gamma(\cdot)) \in M$, the control of the first player on the entire interval $[0, \widehat{T}]$ is chosen in the form of a measurable function $\widehat{u}(\tau) = \widehat{u}_* (\tau, \upsilon(\tau))$, $\tau \in [0, \widehat{T}]$, where $\widehat{u}_* (\tau, \upsilon) = \underset{e_1, \dots, e_n}{\text{lex max}} \, \widehat{U}(\tau, \upsilon)$ is an $L \otimes B$-measurable selector of the mapping $\widehat{U}(\tau, \upsilon)$ of relationship (11) with the resolving function $\alpha(\widehat{T}, \tau, z_0) = \alpha_* (\widehat{T}, \tau, z_0)$ on the entire interval $[0, \widehat{T}]$. Then relationships (11) give

$$\pi z(\widehat{T}) \in \xi(\widehat{T}) + \int_0^{\widehat{T}} \alpha_* (\widehat{T}, \tau, z_0) [M - \xi(\widehat{T})] d\tau .$$

Since, by assumption, $c(M - \xi(\widehat{T}), \psi) \ge 0$, and, by the definition of the moment $\widehat{T}$, $\int_0^{\widehat{T}} \alpha_* (\widehat{T}, \tau, z_0) d\tau < 1$, applying the apparatus of support functions, we obtain $\xi(\widehat{T}) + \int_0^{\widehat{T}} \alpha_* (\widehat{T}, \tau, z_0) [M - \xi(\widehat{T})] d\tau \subset M$. Therefore, $\pi z(\widehat{T}) \in M$ and, hence, $z(\widehat{T}) \in M^*$.

We now show the admissibility of the corresponding control of the pursuer. In view of relationships (10) and (11), we have

$$\int_0^{\widehat{T}} || u(\tau) ||^p \, d\tau = \int_0^{\widehat{T}} || D(\widehat{T} - \tau) \upsilon(\tau) ||^p \, d\tau + \hat{\gamma} \int_0^{\widehat{T}} \alpha_* (\widehat{T}, \tau, z_0) d\tau < \nu^p \mathrm{X}^p + \hat{\gamma} = \mu^p ,$$

which completes the proof of the theorem.

Let us consider a multivalued mapping

$$\mathfrak{A}(t, \tau, z_0) = \bigcap_{\upsilon \in R^k} \mathfrak{A}(t, \tau, \upsilon, z_0), \; (t, \tau) \in \Delta, \; z_0 \in R^n .$$

Condition 4. For some initial position $z_0 \in R^n$ and a shift function $\gamma(\cdot)$, the multivalued mapping $\mathfrak{A}(t, \tau, z_0)$ assumes nonempty values on the set Δ.

If this condition is satisfied, then, following [7, 8], we introduce the following upper and lower resolving functions of the second type:

$$\beta^* (t, \tau, z_0) = \sup \{ \beta : \beta \in \mathfrak{A}(t, \tau, z_0) \} ,$$

$$\beta_* (t, \tau, z_0) = \inf \{ \beta : \beta \in \mathfrak{A}(t, \tau, z_0) \}, \; (t, \tau) \in \Delta, \; z_0 \in R^n .$$

It can be shown [6] that the multivalued mapping $\mathfrak{A}(t, \tau, z_0)$ is closed-valued and measurable with respect to τ and that the upper and lower resolving functions are measurable with respect to the variable τ when t is fixed.

Consider the following set:

$$\Theta(z_0,\gamma(\cdot))=\left\{ t\ge 0: \int_0^t \beta^*(t,\tau,z_0)d\tau \ge 1,\ \int_0^t \beta_*(t,\tau,z_0)d\tau<1 \right\}.$$

If, for some $t>0$, the upper resolving function $\beta^*(t,\tau,z_0)\equiv+\infty$ for $\tau\in[0,t]$, then it is natural to put the value of the corresponding integral in the braces equal to $+\infty$, and $t\in\Theta(z_0,\gamma(\cdot))$ if the other inequality in the braces holds for this t. In the case when the inequalities in the braces are not satisfied for all $t>0$, we put $\Theta(z_0,\gamma(\cdot))=\varnothing$.

THEOREM 3. Let Conditions 1, 2, and 4 be satisfied, let, for the corresponding shift function $\gamma(\cdot)$, the set $\Theta(z_0,\gamma(\cdot))$ be not empty, and let $\Theta\in\Theta(z_0,\gamma(\cdot))$. Then the trajectory of process (1)–(3) from its initial position $z_0\in R^n$ can be continued to terminal set (4) at the moment Θ using an admissible control of the form (6).

Proof. Let $v(\cdot)$ be an arbitrary admissible control of the evader under constraint (3).

We put $\beta(\Theta,\tau,z_0)=\dfrac{1}{\int_0^\Theta \beta^*(\Theta,\tau,z_0)d\tau}\beta^*(\Theta,\tau,z_0)$, $\tau\in[0,\Theta]$, $z_0\in R^n$, and first consider the case when $\xi(\Theta,z_0,\gamma(\cdot))\notin M$.

Since $\int_0^\Theta \beta^*(\Theta,\tau,z_0)d\tau\ge 1$, we have $\beta(\Theta,\tau,z_0)\le\beta^*(\Theta,\tau,z_0)$, $\tau\in[0,\Theta]$, $z_0\in R^n$.

By virtue of the condition $\int_0^\Theta \beta_*(\Theta,\tau,z_0)d\tau<1$, we obtain $\beta(\Theta,\tau,z_0)>\beta_*(\Theta,\tau,z_0)$, $\tau\in[0,\Theta]$, $z_0\in R^n$. Indeed, suppose that $\beta(\Theta,\tau,z_0)\le\beta^*(\Theta,\tau,z_0)$. Then we have $1=\int_0^\Theta \beta(\Theta,\tau,z_0)d\tau\le\int_0^\Theta \beta_*(\Theta,\tau,z_0)d\tau$, contrary to the definition of the time instant Θ.

Thus, $\beta(\Theta,\tau,z_0)\in\mathfrak{A}(\Theta,\tau,z_0)$, $\tau\in[0,\Theta]$, $z_0\in R^n$.

Consider the following multivalued mapping of $v\in R^k$ and $\tau\in[0,\Theta]$:

$$\begin{aligned}\overset{\frown}{U}(\tau,v)=\{u\in U(\Theta,\tau,v,\beta(\Theta,\tau,z_0)):\\ \pi e^{(\Theta-\tau)A}B[u-\gamma(\tau)]-\pi e^{(\Theta-\tau)A}Cv\in\beta(\Theta,\tau,z_0)[M-\xi(\widehat{T})]\}.\end{aligned}\tag{12}$$

As before, the mapping $\overset{\frown}{U}(\tau,v)$ is compact-valued and $L\otimes B$-measurable [6] when $v\in R^k$ and $\tau\in[0,\Theta]$. By virtue of the theorem on the support function [27], the multivalued mapping $\overset{\frown}{U}(\tau,v)$ contains an $L\otimes B$-measurable selector $\overset{\frown}{u}(\tau,v)=\operatorname{lex\,max}_{e_1,\dots,e_m}\overset{\frown}{U}(\tau,v)$, which is a superpositionally measurable function [6] and

$$\|\overset{\frown}{u}(\tau,v)\|=\left(\|D(\Theta-\tau)v\|^p+\beta(\Theta,\tau,v,z_0)\hat{\gamma}\right)^{\frac{1}{p}},\ \tau\in[0,\Theta],\ v\in R^k.$$

For the first player, we set the control equal to $\overset{\frown}{u}(\tau)=\overset{\frown}{u}(\tau,v(\tau))$, $\tau\in[0,\Theta]$.

From the Cauchy formula for process (1) with the chosen controls, we have

$$\pi z(\Theta)=\xi(\Theta,z_0,\gamma(\cdot))+\int_0^\Theta \pi e^{(\Theta-\tau)A}[B[\overset{\frown}{u}(\tau)-\gamma(\tau)]-Cv(\tau)]d\tau.$$

Then, considering relationship (12), we obtain

$$\pi z(\Theta) \in \xi(\Theta, z_0, \gamma(\cdot)) \left[1 - \int_0^\Theta \beta(\Theta, \tau, z_0) d\tau\right] + \int_0^\Theta \beta(\Theta, \tau, z_0) M d\tau.$$

Since M is a convex compact and $\beta(\Theta, \tau, z_0) \geq 0$, $\tau \in [0, \Theta]$, and $\int_0^\Theta \beta(\Theta, \tau, z_0) d\tau = 1$, we have $\int_0^\Theta \beta(\Theta, \tau, z_0) M d\tau = M$. Therefore, $\pi z(\Theta) \in M$ and $z(\Theta) \in M^*$, and it remains to show the admissibility of the control $\underset{\frown}{u}(\tau) = \underset{\frown}{u}(\tau, v(\tau))$, $\tau \in [0, \Theta]$.

By construction, the following relationships hold:

$$\int_0^\Theta \| \underset{\frown}{u}(\tau) \|^p \, d\tau = \int_0^\Theta \| D(\Theta - \tau) v(\tau) \|^p \, d\tau + \hat{\gamma} \int_0^\Theta \beta(\Theta, \tau, z_0) d\tau \leq \nu^p \mathrm{X}^p + \hat{\gamma} = \mu^p.$$

For the case when $\xi(\Theta, z_0, \gamma(\cdot)) \in M$, the control of the first player on the entire interval $[0, \Theta]$ is chosen in the form of a measurable function $\underset{\frown}{u}(\tau) = \underset{\frown}{u}_*(\tau, v(\tau))$, $\tau \in [0, \Theta]$, where $\underset{\frown}{u}_*(\tau, v) = \operatorname{lex\,max}_{e_1, \ldots, e_m} \underset{\frown}{U}(\tau, v)$ is an $L \otimes B$-measurable selector of the mapping $\underset{\frown}{U}(\tau, v)$ of relationship (12) with a resolving function $\beta(\Theta, \tau, z_0) = \beta_*(\Theta, \tau, z_0)$ on the entire interval $[0, \Theta]$. Then relationships (12) give

$$\pi z(\Theta) \in \xi(\Theta, z_0, \gamma(\cdot)) + \int_0^\Theta \beta_*(\Theta, \tau, z_0)[M - \xi(\Theta, z_0, \gamma(\cdot))] d\tau.$$

Since, by assumption, $c(M - \xi(\Theta), \psi) \geq 0$, and, by the definition of the moment Θ, $\int_0^\Theta \beta_*(\Theta, \tau, z_0) d\tau < 1$, applying the apparatus of support functions, we obtain $\xi(\Theta) + \int_0^\Theta \beta_*(\Theta, \tau, z_0)[M - \xi(\Theta)] d\tau \subset M$. Therefore, $\pi z(\Theta) \in M$, and, consequently, $z(\Theta) \in M^*$.

Let us show the admissibility of the corresponding control of the pursuer. We have

$$\int_0^\Theta \| u(\tau) \|^p \, d\tau = \int_0^\Theta \| D(\Theta - \tau) v(\tau) \|^p \, d\tau + \hat{\gamma} \int_0^\Theta \beta_*(\Theta, \tau, z_0) d\tau < \nu^p \mathrm{X}^p + \hat{\gamma} = \mu^p,$$

which completes the proof of the theorem.

Note that if Conditions 1, 2, and 4 are fulfilled, then, for some initial position $z_0 \in R^n$, the multivalued mapping $\mathfrak{A}(t, \tau, z_0)$ assumes nonempty convex values on the set Δ and $\alpha_*(t, \tau, z_0) = \beta_*(t, \tau, z_0)$. If, moreover, for some $t > 0$, $\xi(\Theta, z_0, \gamma(\cdot)) \notin M$, then $\alpha^*(t, \tau, z_0) = \beta^*(t, \tau, z_0)$. Therefore, if the set $\widehat{T}(z_0, \gamma(\cdot))$ is not empty, then the set $\Theta(z_0, \gamma(\cdot))$ is not empty, and they coincide.

REGULARIZATION OF RESOLVING FUNCTIONS

Denote by $\operatorname{epi} \alpha^*(t, \tau, v, z_0) = \{(\tau, \mu) \in [0, t] \times R^1 : \alpha^*(t, \tau, v, z_0) \leq \mu\}$ the epigraph of a function $\alpha^*(t, \tau, v, z_0)$ [28]. A function $\overline{\operatorname{co}}\, \alpha^*(t, \tau, v, z_0)$ is called the closure of the convexity of the function $\alpha^*(t, \tau, v, z_0)$ if $\operatorname{epi} \overline{\operatorname{co}}\, \alpha^*(t, \tau, v, z_0)$

$= \overline{\text{co}}\, \text{epi}\, \alpha^*(t, \tau, v, z_0)$ [24, 27, 28]. In this case, the inequality $\alpha^*(t, \tau, v, z_0) \geq \overline{\text{co}}\, \alpha^*(t, \tau, v, z_0)$ holds for all $t \geq \tau \geq 0$ and $v \in V$. Note that since the function $\alpha^*(t, \tau, v, z_0)$ for each $\tau \in [0, t]$ is upper semicontinuous [5] with respect to $v \in R^k$, the function $\overline{\text{co}}\, \alpha^*(t, \tau, v, z_0)$ will be the same. By definition, the function $\overline{\text{co}}\, \alpha^*(t, \tau, v, z_0)$ for each $\tau \in [0, t]$ is lower semicontinuous [28] with respect to $v \in R^k$ and, therefore, the function $\overline{\text{co}}\, \alpha^*(t, \tau, v, z_0)$ will be continuous with respect to $v \in R^k$. Such functions are called Caratheodory functions [27], and they have a number of useful properties. Thus, some Caratheodory function can be related with each resolving function, and the corresponding calculations of the resolving-functions method can be carried out with it. In this case, as the following lemma shows, the guaranteed completion time of a game will not change.

The Banach space of Lebesgue measurable mappings $v(\cdot)$ of a segment $[0, t]$ in R^k for which the integral $\int_0^t \| v(\tau) \|^p \, d\tau$ is finite when $1 \leq p < \infty$ is denoted by $L_p^k[0, t]$. Put $\delta = \inf_{v(\cdot) \in L_p^k[0, t]} \int_0^t \alpha^*(t, \tau, v(\tau), z_0) d\tau$.

LEMMA 2. If $\int_0^t \alpha^*(t, \tau, v_1(\tau), z_0) d\tau < +\infty$ for at least one function $v_1(\cdot) \in L_p^k[0, t]$, then the following equality holds:

$$\delta = \inf_{v(\cdot) \in L_p^k[0, t]} \int_0^t \overline{\text{co}}\, \alpha^*(t, \tau, v(\tau), z_0) d\tau. \tag{13}$$

Proof. It is clear that the following inequality takes place:

$$\delta \geq \inf_{v(\cdot) \in L_p^k[0, t]} \int_0^t \overline{\text{co}}\, \alpha^*(t, \tau, v(\tau), z_0) d\tau. \tag{14}$$

Let relationship (14) be the strict inequality. Then there is a summable function $\delta_0(\tau)$ such that

$$\delta > \int_0^t \delta_0(\tau) d\tau \tag{15}$$

and, for some function $v_0(\cdot)$, $v_0(\cdot) \in L_p^k[0, t]$, for almost all $\tau \in [0, t]$, we have

$$\delta_0(\tau) > \overline{\text{co}}\, \alpha^*(t, \tau, v_0(\tau), z_0). \tag{16}$$

Note that

$$\overline{\text{co}}\, \alpha^*(t, \tau, v_0(\tau), z_0) \geq \inf_{v \in R^k} \overline{\text{co}}\, \alpha^*(t, \tau, v, z_0) = \inf_{v \in R^k} \alpha^*(t, \tau, v, z_0). \tag{17}$$

Therefore, for almost all $\tau \in [0, t]$, relationships (16) and (17) give

$$\delta_0(\tau) > \inf_{v \in R^k} \alpha^*(t, \tau, v, z_0). \tag{18}$$

Consider a multivalued mapping

$$G^*_{\delta_0(\tau)}(t, \tau) = \{ v \in R^k : \alpha^*(t, \tau, v, z_0) < \delta_0(\tau) \}.$$

Relationship (18) guarantees that $G^*_{\delta_0(\tau)}(t, \tau) \neq \varnothing$. The multivalued mapping $G^*_{\delta_0(\tau)}(t, \tau) = \{ v \in R^k : \alpha^*(t, \tau, v, z_0) < \varepsilon \}$, $\tau \in [0, t]$, has open values by virtue of its upper semicontinuity of the function $\alpha^*(t, \tau, v, z_0)$ with respect to $v \in R^k$ for any $\tau \in [0, t]$. The graph of this mapping $\text{gr}\, G^*_{\delta_0(\tau)}(t, \tau)$ will be $L \otimes B$-measurable with respect to the totality of (τ, v),

$\upsilon \in R^k$, $\tau \in [0, t]$, for any t since the function $\alpha^*(t, \tau, \upsilon, z_0)$ is $L \otimes B$-measurable with respect to the totality of (τ, υ) [6]. Then, by the theorem on the measurability of projection [27], for each $\upsilon \in R^k$, the set $\{\tau \in [0, t] : (\tau, \upsilon) \in \operatorname{gr} G^*_{\delta_0(\tau)}(t, \tau)\}$ will be L-measurable. By virtue of Statement 4 [28, Sec. 8.1], the multivalued mapping $G^*_{\delta_0(\tau)}(t, \tau)$ is measurable and, by the theorem on the measurable choice [28, Sec. 8.1], has a measurable selector. Let $\upsilon_2(\tau)$ be an arbitrary measurable selector of the multivalued mapping $G^*_{\delta_0(\tau)}(t, \tau)$. Then we obtain

$$\alpha^*(t, \tau, \upsilon_2(\tau), z_0) < \delta_0(\tau), \ \tau \in [0, t]. \tag{19}$$

For any $\varepsilon > 0$, a number $N > 0$ can be chosen so large that the measure of the set $T_N = \{\tau \in [0, t] : \|\upsilon_2(\tau)\| > N\}$ is less than ε. Put

$$\rho(\tau) = \begin{cases} \delta_0(\tau) & \text{if } \ \tau \in [0, t], \tau \notin T_N, \\ \max\{\delta_0(\tau), \alpha^*(t, \tau, \upsilon_1(\tau), z_0)\} & \text{if } \ \tau \in T_N. \end{cases}$$

By virtue of inequality (15), the summable function $\rho(\tau)$ satisfies the inequality

$$\delta > \int_0^t \rho(\tau) d\tau. \tag{20}$$

Putting

$$\upsilon(\tau) = \begin{cases} \upsilon_2(\tau) & \text{if } \ \tau \in [0, t], \ \tau \notin T_N, \\ \upsilon_1(\tau) & \text{if } \ \tau \in T_N, \end{cases}$$

we have $\upsilon(\cdot) \in L_p^k[0, t]$ and, according to inequality (19),

$$\int_0^t \alpha^*(t, \tau, \upsilon(\tau), z_0) d\tau < \int_0^t \rho(\tau) d\tau. \tag{21}$$

Inequalities (20) and (21) give

$$\delta > \int_0^t \rho(\tau) d\tau > \int_0^t \alpha^*(t, \tau, \upsilon(\tau), z_0) d\tau \geq \delta.$$

The obtained contradiction shows that there is some equality in relationship (14).

COROLLARY 1. If $\int_0^t \alpha^*(t, \tau, \upsilon_1(\tau), z_0) d\tau < +\infty$ for at least one function $\upsilon_1(\cdot) \in L_p^k[0, t]$, then the following equality holds:

$$\delta = \int_0^t \inf_{\upsilon \in R^k} \alpha^*(t, \tau, \upsilon, z_0) d\tau.$$

Proof. Since $\overline{\operatorname{co}}\, \alpha^*(t, \tau, \upsilon, z_0)$ is a Caratheodory function, its epigraph $\operatorname{epi} \overline{\operatorname{co}}\, \alpha^*(t, \tau, \upsilon, z_0)$ is a measurable closed-valued multivalued mapping and, therefore, by virtue of Statement 2 [28, Sec. 8.3], the following equality holds:

$$\int_0^t \inf_{\upsilon \in R^k} \overline{\operatorname{co}}\, \alpha^*(t, \tau, \upsilon, z_0) d\tau = \inf_{\upsilon(\cdot) \in L_p^k[0, t]} \int_0^t \overline{\operatorname{co}}\, \alpha^*(t, \tau, \upsilon(\tau), z_0) d\tau.$$

Now, taking into account the relationship $\inf_{\upsilon \in R^k} \overline{\operatorname{co}}\, \alpha^*(t, \tau, \upsilon, z_0) = \inf_{\upsilon \in R^k} \alpha^*(t, \tau, \upsilon, z_0)$, equality (13) yields the required result.

Corollary 1 of Lemma 2 shows that the sets $T(z_0, \gamma(\cdot))$ and $\hat{T}(z_0, \gamma(\cdot))$ coincide. Therefore, by virtue of Theorem 2, it is possible to strengthen Theorem 1 and to apply stroboscopic strategy (6) instead of quasistrategy (5).

COMPARISON OF GUARANTEED TIMES

Consider now linear mappings $\pi e^{tA} BR^m \to L$ and $\pi e^{tA} CR^k \to L$, $t \ge 0$.

Condition 5. Let there be a continuous non-singular matrix $\Phi(\cdot): R^k \to R^m$ that is a solution of the matrix equation $\pi e^{tA} BX = \pi e^{tA} C$, where X is the sought-for matrix.

As before, we consider the function [1]

$$\chi^p(t) = \sup_{\int_0^t \|\omega(\tau)\|^p d\tau \le 1} \int_0^t \| \Phi(t-\tau)\omega(\tau) \|^p \, d\tau,$$

where $\omega(\cdot)$ is an arbitrary function from the space $L_p^k[0, t)$ with the mentioned constraint, and determine the value of $\mathrm{X}^p = \sup_{0 \le t < \infty} \chi^p(t)$.

Condition 6. The inequality $\hat{\gamma} = \mu^p - \nu^p \mathrm{X}^p \ge 0$ holds.

Let conditions 5 and 6 be fulfilled. Consider a multivalued mapping $W(t) = \left\{ u \in R^m : \|u\| \le \left(\frac{1}{t} \hat{\gamma} \right)^{\frac{1}{p}} \right\}$, $t > 0$.

Denote by $\Gamma_t = \{\gamma(\cdot) : \gamma(t) \in W(t)\}$ the totality of measurable selectors of the mapping $W(t)$. If $\gamma(\cdot) \in \Gamma_t$, then $\gamma(\cdot)$ is a measurable function from a class $L_p^m[0, t)$, $p > 1$, that satisfies the constraint $\int_0^t \|\gamma(\tau)\|^p \, d\tau \le \hat{\gamma}$, $t > 0$.

It is easily verified that the following inclusion takes place for $\gamma(\cdot) \in \Gamma_t$:

$$0 \in \pi e^{(t-\tau)A} BU\left(t, \tau, \upsilon, \frac{1}{t} \right) - \pi e^{(t-\tau)A} C\upsilon - \pi e^{(t-\tau)A} B\gamma(\tau), \tag{22}$$

where $U\left(t, \tau, \upsilon, \frac{1}{t} \right) = \left\{ u \in R^m : \| u \| \le \left(\| \Phi(t-\tau)\upsilon \|^p + \frac{1}{t} \hat{\gamma} \right)^{\frac{1}{p}} \right\}$, $(t, \tau) \in \Delta = \{(t, \tau) : 0 \le \tau \le t < \infty\}$, $\upsilon \in R^k$.

Consider the following set:

$$\mathrm{P}(z_0) = \left\{ t \ge 0 : e^{tA} z_0 \in M - \int_0^t \pi e^{(t-\tau)A} BW(\tau) d\tau \right\}. \tag{23}$$

If the inclusion in the braces does not take place for any $t \ge 0$, then put $\mathrm{P}(z_0) = \varnothing$.

THEOREM 4. Let conditions 5 and 6 be satisfied, let the set $\mathrm{P}(z_0)$ be nonempty, and let $\mathrm{P} \in \mathrm{P}(z_0)$. Then the trajectory of process (1)–(3) from its initial position $z_0 \in R^n$ can be continued to terminal set (4) at a moment P using an admissible control of the form (6).

Proof. Let $\upsilon(\cdot)$ be an arbitrary admissible control of the evader under constraint (3). The condition of the theorem and relationship (23) imply that $e^{\mathrm{P}A} z_0 \in M - \int_0^{\mathrm{P}} \pi e^{(\mathrm{P}-\tau)A} BW(\tau) d\tau$. Then there is a point $m \in M$ and a measurable selector $\gamma(\cdot) \in \Gamma_{\mathrm{P}}$ such that

$$e^{\mathrm{P}A} z_0 = m - \int_0^{\mathrm{P}} \pi e^{(\mathrm{P}-\tau)A} B\gamma(\tau) d\tau. \tag{24}$$

With allowance for inclusion (22), we consider a multivalued mapping

$$U_0(\tau, v) = \left\{ u \in U\left(\mathrm{P}, \tau, v, \frac{1}{\mathrm{P}}\right) : \pi e^{(\mathrm{P}-\tau)A} Bu - \pi e^{(\mathrm{P}-\tau)A} Cv \right.$$

$$\left. - \pi e^{(\mathrm{P}-\tau)A} B\gamma(\tau) = 0 \right\}, \ \tau \in [0, \mathrm{P}], \ v \in R^k.$$

By virtue of the theorem on a support function [27], the multivalued mapping $U_0(\tau, v)$ contains an $L \otimes B$-measurable selector $u_0(\tau, v) = \underset{e_1, \ldots, e_m}{\text{lex max}} U_0(\tau, v)$ that is a superpositionally measurable function [6] and $\|u_0(\tau, v)\| = \left(\|\Phi(\mathrm{P}-\tau)v\|^{\mathrm{P}} + \frac{1}{\mathrm{P}} \hat{\gamma} \right)^{\frac{1}{p}}$, $\tau \in [0, \mathrm{P}]$, $v \in R^k$. We use the following control of the first player: $u_0(\tau) = u_0(\tau, v(\tau))$, $\tau \in [0, \mathrm{P}]$. From the Cauchy formula for process (1), the chosen controls, and allowance for formula (24), we have

$$\pi z(\mathrm{P}) = e^{\mathrm{P}A} z_0 + \int_0^{\mathrm{P}} \pi e^{(\mathrm{P}-\tau)A} [Bu_0(\tau) - Cv(\tau)] d\tau = m \in M$$

and, hence, $z(\mathrm{P}) \in M^*$.

Let us show the admissibility of the corresponding control of the pursuer. We have

$$\int_0^{\mathrm{P}} \|u_0(\tau)\|^p \, d\tau = \int_0^{\mathrm{P}} \|\Phi(\mathrm{P}-\tau) v(\tau)\|^p \, d\tau + \frac{1}{P} \hat{\gamma} \int_0^{\mathrm{P}} d\tau \le \nu^p \mathrm{X}^p + \hat{\gamma} = \mu^p,$$

which completes the proof of the theorem.

The presented scheme can be considered as an analog of the first direct Pontryagin method [13, 14] for linear differential games with integral constraints. The proof of Theorem 4 uses the conditions of M. S. Nikol'skii [1] and is a generalization of his results.

Let us consider the following sets:

$$\mathrm{P}_*(z_0, \gamma(\cdot)) = \left\{ t \ge 0 : \xi(t, z_0, \gamma(\cdot)) \in M, \quad \sup_{\int_0^t \|v(\tau)\|^p d\tau \le \nu^p} \int_0^t \alpha_*(t, \tau, v(\tau), z_0) d\tau < 1 \right\}, \tag{25}$$

$$\widehat{\mathrm{P}}_*(z_0, \gamma(\cdot)) = \left\{ t \ge 0 : \xi(t, z_0, \gamma(\cdot)) \in M, \int_0^t \alpha_*(t, \tau, z_0) d\tau < 1 \right\},$$

$$\underset{\frown}{\mathrm{P}}_*(z_0, \gamma(\cdot)) = \left\{ t \ge 0 : \xi(t, z_0, \gamma(\cdot)) \in M, \ \int_0^t \beta_*(t, \tau, z_0) d\tau < 1 \right\}.$$

If the relationships in braces are not satisfied for any $t \ge 0$, then put $\mathrm{P}_*(z_0, \gamma(\cdot)) = \varnothing$, $\widehat{\mathrm{P}}_*(z_0, \gamma(\cdot)) = \varnothing$, and $\underset{\frown}{\mathrm{P}}_*(z_0, \gamma(\cdot)) = \varnothing$, respectively.

THEOREM 5. Let conditions 1–3 be satisfied, let the set $\mathrm{P}_*(z_0,\gamma(\cdot))$ be nonempty, and let $\mathrm{P}_* \in \mathrm{P}_*(z_0,\gamma(\cdot))$. Then $\mathrm{P}_*(z_0,\gamma(\cdot)) = \widehat{\mathrm{P}}_*(z_0,\gamma(\cdot)) = \underset{\frown}{\mathrm{P}}_*(z_0,\gamma(\cdot))$, and the trajectory of process (1)–(3) from its initial position $z_0 \in R^n$ can be continued to terminal set (4) at the moment P_* using an admissible control of the form (6).

Proof. It can be easily shown that if Conditions 1–3 are satisfied, then Condition 4 is fulfilled. In this case, the following relationships hold true for $(t,\tau) \in \Delta$ and $z_0 \in R^n$:

$$\sup_{\int_0^t \|v(\tau)\|^p d\tau \le \nu^p} \int_0^t \alpha_*(t,\tau,v(\tau),z_0)d\tau = \int_0^t \alpha_*(t,\tau,z_0)d\tau, \ \alpha_*(t,\tau,z_0) = \beta_*(t,\tau,z_0).$$

Therefore, the equalities $\mathrm{P}_*(z_0,\gamma(\cdot)) = \widehat{\mathrm{P}}_*(z_0,\gamma(\cdot)) = \underset{\frown}{\mathrm{P}}_*(z_0,\gamma(\cdot))$ hold.

Let $v(\cdot)$ be an arbitrary admissible control of the evader under constraint (3).

Consider the following multivalued mapping when $v \in R^k$ and $\tau \in [0, \mathrm{P}_*]$:

$$U_*(\tau,v) = \{u \in U(\mathrm{P}_*,\tau,v,\alpha_*(\mathrm{P}_*,\tau,v,z_0)):$$

$$\pi e^{(\mathrm{P}_*-\tau)A} B[u-\gamma(\tau)] - \pi e^{(\mathrm{P}_*-\tau)A} Cv \in \alpha_*(\mathrm{P}_*,\tau,v,z_0)[M-\xi(\mathrm{P}_*,z_0,\gamma(\cdot))]\}. \tag{26}$$

By virtue of the theorem on a support function [27], the multivalued mapping $U_*(\tau,v)$ contains an $L \otimes B$-measurable selector $u_*(\tau,v) = \operatorname{lex\,max}_{e_1,\dots,e_n} U_*(\tau,v)$ that is a superpositionally measurable function [6] and $\|u_*(\tau,v)\| = \left(\|\Phi(\mathrm{P}_*-\tau)v\|^p + \alpha_*(\mathrm{P}_*,\tau,v,z_0)\hat{\gamma}\right)^{\frac{1}{p}}$, $\tau \in [0,\mathrm{P}_*]$, $v \in R^k$. We put the following control of the first player: $u_*(\tau) = u_*(\tau, v(\tau))$, $\tau \in [0,\mathrm{P}_*]$. Then, considering the apparatus of support functions [24], relations (25) and (26) give

$$\pi z(\mathrm{P}_*) \in \xi(\mathrm{P}_*) + \int_0^{\mathrm{P}_*} \alpha_*(\mathrm{P}_*,\tau,v(\tau),z_0)[M-\xi(\mathrm{P}_*)]d\tau \subset M$$

since, by condition, $c(M-\xi(\mathrm{P}_*),\psi) \ge 0$ and

$$\int_0^{\mathrm{P}_*} \alpha_*(\mathrm{P}_*,\tau,v(\tau),z_0)d\tau \le \sup_{\int_0^{\mathrm{P}_*} \|v(\tau)\|^p d\tau \le \nu^p} \int_0^{\mathrm{P}_*} \alpha_*(\mathrm{P}_*,\tau,v(\tau),z_0)d\tau < 1.$$

Therefore, $\pi z(\mathrm{P}_*) \in M$ and, consequently, $z(\mathrm{P}_*) \in M^*$. We now show the admissibility of the corresponding control of the pursuer. In view of relationships (25) and (26), we have

$$\int_0^{\mathrm{P}_*} \|u(\tau)\|^p d\tau = \int_0^{\mathrm{P}_*} \|\Phi(\mathrm{P}_*-\tau)v(\tau)\|^p d\tau + \hat{\gamma} \int_0^{\mathrm{P}_*} \alpha_*(\mathrm{P}_*,\tau,v(\tau),z_0)d\tau$$

$$\le \nu^p \mathrm{X}^p + \hat{\gamma} \sup_{\int_0^{\mathrm{P}_*} \|v(\tau)\|^p d\tau \le \nu^p} \int_0^{\mathrm{P}_*} \alpha_*(\mathrm{P}_*,\tau,v(\tau),z_0)d\tau < \nu^p \mathrm{X}^p + \hat{\gamma} = \mu^p.$$

COROLLARY 1. Let conditions 5 and 6 be satisfied, let the set $\mathrm{P}(z_0)$ be nonempty, and let $\mathrm{P} \in \mathrm{P}(z_0)$. Then $\mathrm{P}_*(z_0, \gamma(\cdot)) = \widehat{\mathrm{P}}_*(z_0, \gamma(\cdot)) = \underset{\frown}{\mathrm{P}}_*(z_0, \gamma(\cdot)) = \mathrm{P}(z_0)$, and the trajectory of process (1)–(3) from its initial position $z_0 \in R^n$ can be continued to terminal set (4) at the moment P using an admissible control of the form (6).

Proof. If conditions 5 and 6 are fulfilled, then, taking into account relationship (22), we obtain $0 \in \mathfrak{A}(\mathrm{P}, \tau, v, z_0)$, $\tau \in [0, \mathrm{P}]$, $v \in R^k$. Therefore, $0 \in \mathfrak{A}(\mathrm{P}, \tau, z_0)$, conditions 3 and 4 are fulfilled, and, at the same time, for $\tau \in [0, \mathrm{P}]$ and $v \in R^k$, we have

$$\alpha_*(\mathrm{P}, \tau, v, z_0) = \inf\{\alpha : \alpha \in \mathfrak{A}(\mathrm{P}, \tau, v, z_0)\} = 0,$$

$$\sup_{\int_0^{\mathrm{P}} \|v(\tau)\|^p d\tau \le \nu^p} \int_0^{\mathrm{P}} \alpha_*(\mathrm{P}, \tau, v(\tau), z_0) d\tau = \int_0^{\mathrm{P}} \alpha_*(\mathrm{P}, \tau, z_0) d\tau = \int_0^{\mathrm{P}} \beta_*(\mathrm{P}, \tau, z_0) d\tau = 0.$$

THEOREM 6. Let conditions 5 and 6 be fulfilled, let the set $T(z_0, \gamma(\cdot))$ be nonempty, and let $T \in T(z_0, \gamma(\cdot))$. Then $T(z_0, \gamma(\cdot)) = \widehat{T}(z_0, \gamma(\cdot)) = \Theta(z_0, \gamma(\cdot))$, and the trajectory of process (1)–(3) from its initial position $z_0 \in R^n$ can be continued to terminal set (4) at the moment T using an admissible control of the form (6).

The proof directly follows from the constructions of the corresponding definitions and theorems.

CONCLUSIONS

This work considers linear differential games with integral constraints. Sufficient conditions for the completion of a game during a finite guaranteed time are formulated in the case when the condition of M. S. Nikol'skii [1] is not fulfilled. Two schemes of the resolving-functions method are proposed that ensure the completion of a game within a finite guaranteed time in the class of Hajek stroboscopic strategies [9]. It is shown that, without additional assumptions, this time coincides with the guaranteed time in the class of quasistrategies. A scheme of regularization of resolving functions is proposed that transforms them into Caratheodory functions [27] and considerably simplifies the general scheme of the resolving-functions method without changing the guaranteed time of completion of a game. The guaranteed times are compared.

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