DOI 10.1007/s10559-018-0023-2
Cybernetics and Systems Analysis, Vol. 54, No. 2, March, 2018

COMBINATORIAL CONFIGURATIONS IN BALANCE LAYOUT OPTIMIZATION PROBLEMS

I. V. Grebennik,[1†] A. A. Kovalenko,[1‡] T. E. Romanova,[2] I. A. Urniaieva,[1††] and S. B. Shekhovtsov[3]

UDC 519.85

Abstract. *The balance layout optimization problem for a given set of 3D objects in a container divided by horizontal racks into subcontainers is considered. For analytical description of non-overlapping and containment constraints, the phi-function technique is used. Combinatorial configurations describing the combinatorial structure of the problem are defined. Based on the introduced configurations, a mathematical model is constructed that takes into account not only the placement constraints and mechanical properties of the system but also the combinatorial features of the problem associated with generation of partitions of the set of objects placed inside the subcontainers. A solution strategy is proposed. The results of numerical experiments are provided.*

Keywords: *balance layout, combinatorial configurations, 3D objects, phi-function method, mathematical model, optimization.*

INTRODUCTION

Balance layout problems are NP-hard arrangement problems [1] and are a subject of research in computing geometry [2], and methods of their solution are a new direction in the theory of operations research [3]. The essence of the problem is to find optimal arrangement of a given set of 3D objects in some bounded area (container) with regard for behavior constraints, which ensure balance of the system under study.

The necessity of taking into account behavior constraints in arrangement optimization problems occurs in various application domains of science and engineering, for example, in logistics problems (cargo packing for transportation or storage), in mechanical engineering (arrangement of aircraft, ships, submarines, as well as equipment, devices, and product details). Design in space-rocket engineering draws a special attention to this class of problems. At the initial stage of design (arrangement) of a spacecraft, it is necessary to consider a number of constraints on static and dynamic characteristics (center of mass, axial and centrifugal moments of inertia) [4].

The studies [5–9] consider problems of layout of cylinders in a cylindrical container with behavior constraints, which should be taken into account to counterbalance satellite system. These publications present mathematical models with different objective functions. To solve these problems, heuristic algorithms are proposed, which take into account special features of each of them.

Mathematical models of some balance layout optimization problems can be constructed with regard for special features of their discrete structure. For mathematical modeling and solution of such problems, it is necessary to use combinatorial configurations that have respective properties. The main approaches to mathematical modeling and solution of optimization problems on combinatorial configurations are described in [10, 11].

[1]Kharkiv National University of Radio Electronics, Kharkiv, Ukraine, [†]*igor.grebennik@nure.ua*; [‡]*hanna.kovalenko@nure.ua*; [††]*inna.urniaieva@nure.ua*. [2]A. Podgorny Institute for Mechanical Engineering Problems, Kharkiv, Ukraine, *sherom@kharkov.ua*. [3]Kharkiv National University of Home Affairs, Kharkiv, Ukraine, *tarom7@yahoo.com*. Translated from Kibernetika i Sistemnyi Analiz, No. 2, March–April, 2018, pp. 55–67. Original article submitted July 17, 2017.

The studies [12, 13] present mathematical models and solution techniques for balance layout problems (BLP) for sets of 3D objects in a given container, which is divided into subcontainers by circular racks. And partition of the set of objects into subsets according to the arrangement of objects inside the subcontainers is assumed to be specified.

In the present paper, we will consider the problem of balance layout of 3D objects (a full-sphere, a cylinder, a torus, a spherocylinder, a parallelepiped, and a regular prism) in a container (in the form of a parallelepiped, a cylinder, a paraboloid of revolution or a cone). Partition of the set of objects with respect to the membership in subcontainers is not specified.

The purpose of the present study is to construct and implement the mathematical model of the problem of optimal balance layout of a set of 3D objects, which takes into account not only arrangement constraints and mechanical properties of the system, but also combinatorial features of the problem related to generation of partitions of the set of objects placed inside subcontainers.

PROBLEM STATEMENT

Let Ω be a container of height H, in the form of a parallelepiped, cylinder, paraboloid of revolution or a cone. Container Ω is defined in the intrinsic fixed frame *Oxyz*, where *Oz* is longitudinal symmetry axis. Assume that Ω is divided by horizontal racks S_j into compartments Ω^j, $j \in J_m = \{1, \dots, m\}$. Denote the distance between racks S_j and S_{j+1} by t_j and $j \in J_m$, $\sum_{j=1}^{m} t_j = H$. The origin of the intrinsic frame *Oxyz* is centered at the bottom of the container.

We have the set $A = \{\mathbb{T}_i, i = 1, \dots, n\}$ of homogeneous 3D objects with given metric characteristics. Each object has height h_i and mass m_i, $i \in J_n = \{1, \dots, n\}$.

Object $\mathbb{T}_i$ is defined in the intrinsic frame $O_i x_i y_i z_i$. Position of object $\mathbb{T}_i$ inside the container Ω is defined by vector $u_i = (v_i, z_i, \theta_i)$, where (v_i, z_i) is the vector of translation of the object in fixed frame *Oxyz*, θ_i is the angle of rotation of object $\mathbb{T}_i$ in plane $O_i x_i y_i$, $v_i = (x_i, y_i)$, and the value of z_i, $i \in J_n$, is uniquely defined by subcontainer Ω^j, $j \in J_m$, in which the object $\mathbb{T}_i$ is located. The following constraints are imposed on the layout of object $\mathbb{T}_i$, $i \in J_n$, inside Ω^j:

$$z_i = \sum_{l=1}^{j} t_{l-1} + h_i,$$

where $j \in J_m$. We suppose that $t_0 = 0$ and $\forall i \in J_n$ there exists $j^* \in J_m$: $h_i \le t_{j^*}$.

Unlike the BLP problems considered in [12, 13], where objects are required a priori to be placed in specific subcontainers Ω^j, $j \in J_m$, in the present study we will formulate a balance layout problem that assumes generating and choosing the partition of set A into nonempty subsets A^j, $j \in J_m$. Here, A^j is a subset of objects that should be placed on rack S_j inside subcontainer Ω^j.

Let $J_n^j \subseteq J_n$ be the set of subscripts of objects placed in subcontainer Ω^j, $j \in J_m$, $\bigcup_{j=1}^{m} J_n^j = J_n$, $J_n^i \cap J_n^j = \varnothing$, $i \ne j \in J_m$; $k_j = |A^j|$ is the number of objects placed in subcontainer Ω^j, $k_j > 0$, $j \in J_m$, and

$$\sum_{j=1}^{m} k_j = n. \tag{1}$$

In some applications, additional constraints can be imposed on the total mass of objects placed in subcontainers Ω^j, $j \in J_m$, for example,

$$\sum_{i=1}^{k_1} m_i \ge \sum_{i=k_1+1}^{k_1+k_2} m_i \ge \dots \ge \sum_{i=k_1+k_2+\dots+k_{m-1}+1}^{n} m_i, \tag{2}$$

where $k_j = |A^j| \ge 1$, $j \in J_m$.

Moreover, the following conditions of the arrangement should be satisfied:

$$\text{int}\,\mathbb{T}_{i_1} \cap \text{int}\,\mathbb{T}_{i_2} = \varnothing,\ i_1 < i_2 \in J_n^j,\ j \in J_m, \tag{3}$$

$$\mathbb{T}_i \subset \Omega^j,\ i \in J_n^j,\ j \in J_m, \tag{4}$$

$$h^j \le t_j,\ h^j = \max\{h_i^j, i \in J_n^j\},\ j \in J_m. \tag{5}$$

Denote by Ω_A the system generated as a result of the arrangement of objects $\mathbb{T}_i$ of family A in container Ω and by O_sXYZ the frame Ω_A, where $O_s = (x_s(v), y_s(v), z_s(v))$ is center of mass Ω_A, and axes $O_sX||Ox, O_sY||Oy, O_sZ||Oz$,

$$x_s(v) = \frac{\sum_{i=1}^{n} m_i x_i}{M},\quad y_s(v) = \frac{\sum_{i=1}^{n} m_i y_i}{M},\quad z_s(v) = \frac{\sum_{i=1}^{n} m_i z_i}{M},$$

$M = \sum_{i=1}^{n} m_i$ is the mass of system Ω_A.

As the objective function, in the present study we consider the deviation of the center of mass O_s of system Ω_A from the given point (x_0, y_0, z_0).

Combinatorial Balance Layout Problem (CBLP). It is necessary to find a partition of set A into nonempty subsets A^j, $j \in J_m$, taking into account constraints (1)–(5) and layout parameters (x_i^*, y_i^*, z_i^*) of objects $\mathbb{T}_i$, $i \in J_n$, that minimizes the objective function.

We assume that the problem has at least one feasible solution.

Remark. Behavior constraints (constraints on the axial and centrifugal moments of the system) and constraints on feasible distances between objects can also be imposed on the layout of objects.

Variants of partition of set A into nonempty subsets A^j, $j \in J_m$, are defined by the number of elements in each subset and by the order of the latter. Consider subcontainers Ω^j and sets of objects corresponding to them A^j, $j \in J_m$. Then the tuple of natural numbers $(k_1, k_2, \ldots, k_m)$ such that $\sum_{j=1}^{m} k_j = n$, defines possible number k_j of objects in each subcontainer Ω^j. The quantity of all such tuples is equal to the quantity of compositions of number n of length m [14], which makes C_{n-1}^{m-1}.

Let us consider how many ways are there to decompose n different objects from set A into m subcontainers Ω^j, $j \in J_m$, provided that they contain, respectively, $k_1, k_2, \ldots, k_m$ objects, and the sets of objects A^j, $j \in J_m$, inside the corresponding subcontainers Ω^j, $j \in J_m$, are not ordered. Without loss of generality, we will distinguish objects with identical values of metric characteristics, height h_i and mass m_i (for example, assume that they differ in number).

Let us order the set of elements A. Let us associate each object with the number of subcontainer in which it will be placed. We will obtain a tuple consisting of n elements, which generate a permutation with repetitions out of m numbers $1, 2, \ldots, m$, in which the first element (number of the first subcontainer) is repeated k_1 times, the second one k_2 times, and the last one k_m times. Each such permutation defines its own layout technique, and their total number is

$$P(n, k_1, k_2, \ldots, k_m) = \frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_m!}.$$

Then the number of variants of distributing n different objects from set A among m subcontainers Ω^j provided that each subcontainer contains at least one object and the order of the arrangement of objects inside the subcontainer is unimportant, can be calculated as

$$\sum_{k_1+k_2+\ldots+k_m=n} P(n, k_1, k_2, \ldots, k_m) = \sum_{k_1+k_2+\ldots+k_m=n} \frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_m!}. \tag{6}$$

Note that the number of terms in the sum is equal to the number of compositions of n of length m, which is $N = |C_{n-1}^{m-1}|$.

To generate various variants of subsets A^j, $j \in J_m$, let us construct a combinatorial configuration as follows.

Denote by $Pt(n,m)$ the set of compositions of number n of length m (corresponds to the distribution n of different cylinders from set A among m subcontainers Ω^j, $j \in J_m$, provided that each subcontainer contains at least one object and the order of objects inside the subcontainer is unimportant). Here, $|Pt(n,m)| = N = |C_{n-1}^{m-1}|$.

Let $(k_1, k_2, \dots, k_m) \in Pt(n,m)$, $\sum_{j=1}^{m} k_j = n$, $k_i \geq 1$, $i \in J_m$. Let us introduce combinatorial set $\mathbb{Q}$, which is a composite image of combinatorial sets (k-set) $Pt(n,m)$, $C_n^{k_1}$, $C_{n_1}^{k_2}$, $C_{n_2}^{k_3}, \dots, C_{n_{m-1}}^{k_m}$, generated by sets I_{n_0}, I_{n_1}, $I_{n_2}, \dots, I_{n_{m-1}}$ [15], where $n_i = n - k_1 - \dots - k_i$, $i \in J_{m-1}$,

$$I_{n_0} = J_n,$$

$$I_{n_1} = I_{n_0} \setminus \{j_1^{n_0}, j_2^{n_0}, \dots, j_{k_1}^{n_0}\}, \ (j_1^{n_0}, j_2^{n_0}, \dots, j_{k_1}^{n_0}) \in C_n^{k_1},$$

$$I_{n_2} = I_{n_1} \setminus \{j_1^{n_1}, j_2^{n_1}, \dots, j_{k_2}^{n_1}\}, \ (j_1^{n_1}, j_2^{n_1}, \dots, j_{k_2}^{n_1}) \in C_{n_1}^{k_2},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$I_{n_{m-1}} = I_{n_{m-2}} \setminus \{j_1^{n_{m-2}}, j_2^{n_{m-2}}, \dots, j_{k_{m-1}}^{n_{m-2}}\}, \ (j_1^{n_{m-2}}, j_2^{n_{m-2}}, \dots, j_{k_{m-1}}^{n_{m-2}}) \in C_{n_{m-2}}^{k_{m-1}},$$

$$I_{n_{m-1}} = \{j_1^{n_{m-1}}, j_2^{n_{m-1}}, \dots, j_{k_m}^{n_{m-1}}\}, \ (j_1^{n_{m-1}}, j_2^{n_{m-1}}, \dots, j_{k_m}^{n_{m-1}}) \in C_{n_{m-1}}^{k_m}.$$

Note that

$$I_{n_0} \cup I_{n_1} \cup \dots \cup I_{n_{m-1}} = J_n = \{1, 2, \dots, n\},$$

$$I_{n_s} \cap I_{n_t} = \varnothing, \ s \neq t \in J_{m-1}^0 = \{0, 1, \dots, m-1\}.$$

Element $q = (q_1, \dots, q_n) \in \mathbb{Q}$ can be described as follows:

$$q = (q_1, \dots, q_{k_1}, q_{k_1+1}, \dots, q_{k_1+k_2}, \dots, q_{k_1+\dots+k_{m-1}}, \dots, q_{k_{m-1}+k_m}),$$

where $(q_1, \dots, q_{k_1}) = (j_1^{n_0}, j_2^{n_0}, \dots, j_{k_1}^{n_0}) \in C_n^{k_1}$,

$$(q_{k_1+1}, \dots, q_{k_1+k_2}) = (j_1^{n_1}, j_2^{n_1}, \dots, j_{k_2}^{n_1}) \in C_{n_1}^{k_2},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(q_{k_1+\dots+k_{m-1}}, \dots, q_{k_{m-1}+k_m}) = (j_1^{n_{m-1}}, j_2^{n_{m-1}}, \dots, j_{k_m}^{n_{m-1}}) \in C_{n_{m-1}}^{k_m}.$$

The cardinality of set $\mathbb{Q}$ is defined by formula (6).

In what follows, we will call element $q = (q_1, \dots, q_n)$ of set $\mathbb{Q}$ a tuple of partition of the set of objects A into subsets A^j, $j \in J_m$.

Let us define the vector of basic variables of CBLP: $u = (v, z, \theta)$, where $v = (v_1, \dots, v_n) \in \mathbf{R}^{2n}$, $\theta = (\theta_1, \dots, \theta_n) \in \mathbf{R}^n$, $v_i = (x_i, y_i) \in \mathbf{R}^2$, x_i, y_i, θ_i are continuous variables, $z = (z_1, \dots, z_n) \in \mathbf{R}^n$, z_i are discrete variables.

The values of variables z_i, $i=1,2,..,n$, are found in the order specified by elements $q=(q_1,\dots,q_n)$ of the combinatorial set $\mathbb{Q}$ as follows:

$$z_{q_i}=\sum_{l=1}^{s} t_{l-1}+h_{q_i}, \tag{7}$$

where

$$s=\begin{cases}1 & \text{if } i\le k_1,\\ 2 & \text{if } k_1<i\le k_1+k_2,\\ \dots & \\ m & \text{if } k_1+k_2+\ldots+k_{m-1}<i\le k_1+k_2+\ldots+k_m,\end{cases}$$

$$i=1,2,\dots,n,\ q_i\in\{1,2,..,n\},\ q=(q_1,\dots,q_n)\in\mathbb{Q}.$$

Constraints on the layout of objects of set A in container Ω of the form (3), (4) are described by the system of inequalities $\Upsilon_1(u,\tau)\ge 0$, $\Upsilon_2^*(u)\ge 0$, where $\Upsilon_1(u,\tau)\ge 0$ is a constraint that describes nonintersection of 3D objects, $\Upsilon_2^*(u)\ge 0$ is constraint that describes inclusion of 3D objects in container Ω. Here,

$$\Upsilon_1(u,\tau)=\min\{\Upsilon_1^j(u,\tau),\, j\in J_m\},$$

$$\Upsilon_1^j(u,\tau)=\min\{\Upsilon_{q_1q_2}^j(u_{q_1},u_{q_2},u_{q_1q_2}),\, q_1<q_2\in J_n^j\}, \tag{8}$$

$$\Upsilon_2^*(u)=\min\{\Upsilon_2^{*j}(u),\, j\in J_m\},\ \Upsilon_2^{*j}(u)=\min\{\Upsilon_{q_i}^*(u_{q_i}),\, q_i\in J_n^j\}, \tag{9}$$

$\Upsilon_{q_1q_2}^j(u_{q_1},u_{q_2},u_{q_1q_2})$ is function that describes the condition of nonintersection of objects $\mathbb{T}_{q_1}$ and $\mathbb{T}_{q_2}$, $u_{q_1}=(x_{q_1},y_{q_1},z_{q_1},\theta_{q_1})$, $u_{q_2}=(x_{q_2},y_{q_2},z_{q_2},\theta_{q_2})$, $\Upsilon_{q_i}^*(u_{q_i})$ is function that describes the condition of nonintersection of objects $\mathbb{T}_{q_i}$ and $\Omega^{*j}=\mathbf{R}^3/\operatorname{int}\Omega^j$.

Let $u_1=(v_1,z_1,\theta_1)\in\mathbf{R}^3$, $u_2=(v_2,z_2,\theta_2)\in\mathbf{R}^3$, $v_1=(x_1,y_1)$, $v_2=(x_2,y_2)$, x_1,y_1,θ_1, x_2,y_2,θ_2 be continuous variables, z_1 and z_2 be discrete variables, and u_{12} be vector of additional variables.

Definition 1. Function $\Upsilon_{12}(u_1,u_2)$ is called *D-phi*-function for 3D objects $\mathbb{T}_1$ and $\mathbb{T}_2$ if for fixed values of $z_1=z_1^0$ and $z_2=z_2^0$ function $\Upsilon_{12}(v_1,z_1^0,\theta_1,v_2,z_2^0,\theta_2)$ is a *phi*-function $\Phi_{12}(v_1,z_1^0,\theta_1,v_2,z_2^0,\theta_2)$ for objects $\mathbb{T}_1$ and $\mathbb{T}_2$.

Definition 2. Function $\Upsilon'_{12}(u_1,u_2,u_{12})$ is called quasi-*D-phi*-function for 3D objects $\mathbb{T}_1$ and $\mathbb{T}_2$ if for fixed values of $z_1=z_1^0$ and $z_2=z_2^0$ function $\Upsilon'_{12}(v_1,z_1^0,\theta_1,v_2,z_2^0,\theta_2,u_{12})$ is a quasi-*phi*-function $\Phi'_{12}(v_1,z_1^0,\theta_1,v_2,z_2^0,\theta_2,u_{12})$ for objects $\mathbb{T}_1$ and $\mathbb{T}_2$.

Thus, in relations (8) and (9) for fixed values of z_{q_1} and z_{q_2} we get $\Upsilon_{q_1q_2}^j(u_{q_1},u_{q_2},u_{q_1q_2})\equiv\Phi_{q_1q_2}^{\mathbb{TT}}(u_{q_1},u_{q_2})$ is *phi*-function [2] for objects $\mathbb{T}_{q_1}$ and $\mathbb{T}_{q_2}$ or $\Upsilon_{q_1q_2}^j(u_{q_1},u_{q_2},u_{q_1q_2})\equiv\Phi'^{\mathbb{TT}}_{q_1q_2}(u_{q_1},u_{q_2},u_{q_1q_2})$ is quasi-*phi*-function [16] for objects $\mathbb{T}_{q_1}$ and $\mathbb{T}_{q_2}$; $\Upsilon_{q_i}^*(u_{q_i})\equiv\Phi_{q_i}^{\mathbb{T}\Omega^{*j}}(u_{q_i})$ is *phi*-function for objects $\mathbb{T}_{q_i}$ and Ω^{*j}.

If minimum admissible distances between objects are specified, pseudo-normalized *phi*-functions (quasi-*phi*-functions) for respective pairs of objects [2, 15] are used.

THE MATHEMATICAL MODEL

We can define the mathematical model of CBLP problem as follows:

$$F(u^*,\tau)=\min F(u,\tau)\ \text{s.t.}\ (u,\tau)\in W, \tag{10}$$

$$W = \{(u,\tau) \in \mathbf{R}^{\sigma} : \Upsilon_1(u,\tau) \ge 0,\ \Upsilon_2^*(u) \ge 0,\ \mu(u) \ge 0\}, \tag{11}$$

where

$$F(u) = d = (x_s(v,z))^2 + (y_s(v,z))^2 + (z_s - z_0)^2,$$

$u = (v, z, \theta)$, $v = (v_1, \dots, v_n)$, $\theta = (\theta_1, \dots, \theta_n)$, $v_i = (x_i, y_i)$, $i \in I_n$, $z = (z_1, \dots, z_n)$, function $\Upsilon_1(u,\tau)$ is described by relation (8) for $\Xi = \bigcup_{j=1}^{m} \Xi^j$, $\Xi^j = \{(q_1, q_2) : q_1 < q_2 \in J_n^j\}$, $\tau = (\tau_1, \dots, \tau_s)$ is the vector of auxiliary variables for constructing the quasi-*phi*-functions, $s = |\Xi|$, function $\Upsilon_2^*(u)$ is defined by formula (9), elements of vector z are defined by relation (7), and $\mu(u) \ge 0$ are behavior constraints.

For example, mathematical model (10), (11) for CBLP problem of the layout of cylinders in a cylindrical container becomes

$$\min d, \text{ s.t. } u = (v, z) \in W,$$

where $v = (x_1, y_1, \dots, x_n, y_n)$, $z = (z_1, \dots, z_n)$,

$$d = \left[\sum_{i=1}^{n} m_i' x_i\right]^2 + \left[\sum_{i=1}^{n} m_i' y_i\right]^2 + \left[\sum_{i=1}^{n} m_i' z_i\right]^2,$$

and the domain W can be described by the system of inequalities

$$\begin{cases} (x_{q_2} - x_{q_1})^2 + (y_{q_2} - y_{q_1})^2 - (r_{q_2} + r_{q_1})^2 \ge 0, \\ q_1, q_2 \in \Xi^j, \quad j \in J_m, \\ -x_{q_i}^2 - y_{q_i}^2 + (R_{q_i}^z - r_{q_i})^2 \ge 0, \\ q_i \in \Xi^j, \quad j \in J_m. \end{cases}$$

Note that $m_i' = \frac{m_i}{M} = \text{const}$ and $M = \sum_{i=1}^{n} m_i = \text{const}$.

The mathematical model of CBLP can be represented as a mixed integer programming problem (MIP) with the use of Boolean variables. However, unlike (10), (11), such approach substantially increases the number of discrete variables of the model, and hence increases the CBLP dimension.

SOLUTION STRATEGY

To solve CBLP, the following strategy is used.

Step 1. Randomly generate the set $\{q\}$ of partition tuples $q = (q_1, \dots, q_n) \in \mathbb{Q}$ with the use, for example, of the algorithm presented in [17].

Step 2. Check conditions (2) and (5) for each of the tuples $q \in \{q\}$. Construct the subset $\{q'\} \subseteq \{q\}$ whose elements satisfy conditions (1)–(5). If $\{q'\} = \varnothing$, then go back to Step 1.

Step 3. Construct the set of feasible starting points $\{u_0'\}$ for each tuple from set $\{q'\}$.

Step 4. Find local extremum of problem (10), (11) for each starting point $u_0' \in W$ for a fixed tuple q'.

Step 5. Take the best of the obtained local extrema for all the tuples of set $\{q'\}$ and feasible starting points of set $\{u_0'\}$ as a locally optimal solution of the CBLP.

The proposed strategy uses reasonable choice of feasible starting points and NLP-solver to find locally optimal solutions of the NP-hard conditional optimization problem (10), (11).

To reduce computational cost (time and memory), modification of the LOFRT algorithm proposed in [16] is used. This algorithm reduces high-dimensional problem (10), (11) with a great number of inequalities to a sequence of subproblems with much smaller number of variables and inequalities.

To solve nonlinear programming problems, IPOPT is used (https://projects.coin-or.org/Ipopt), which implements the interior point method [18].

Algorithm of Finding Feasible Starting Points. The algorithm includes the following steps for the given tuple q'.

Step 1. Randomly generate set of points $v_i^0 = (x_i^0, y_i^0)$, $i \in I_n$, which belong to respective cuts of the container. Form vector $v^0 = (x_1^0, y_1^0, \ldots, x_n^0, y_n^0)$. Fix rotation angles $\theta_i = \theta_i^0 = 0$, $i \in I_n$.

Step 2. Let $\lambda = \lambda_i$ be coefficient of homothety for objects A_i, $i \in I_n$. Using obvious geometrical constructions, find the vector of additional variables u'^0 of dimension τ, such that each *phi*-function or quasi-*phi*-function in (11) attains its maximum value with respect to additional variables u'^0 at point (u_λ^0, u'^0), where $u_\lambda^0 = (v^0, \theta^0, \lambda^0)$, $\lambda^0 = 0$, $v^0 = (v_1^0, \ldots, v_n^0)$, $\theta^0 = (\theta_1^0, \ldots, \theta_n^0)$.

Step 3. Calculate $\alpha^0 = \min\{\Upsilon_1(u_\lambda^0, u'^0), \Upsilon_2(u_\lambda^0)\}$. If $\alpha^0 < 0$, go to Step 4; otherwise form point $u_\alpha^* = (u_\lambda^0, u'^0, \alpha^0)$ and go to Step 5.

Step 4. Specify $\lambda = 0$, $\theta_i = \theta_i^0 = 0$, $i \in I_n$, and use $u_\alpha^0 = (u_\lambda^0, u'^0, \alpha^0)$ as a starting point for the solution of the following auxiliary nonlinear programming problem:

$$\alpha^* = \max \alpha, \text{ s.t. } u_\alpha \in W_\alpha, \tag{12}$$

$$W_\alpha = \{u_\alpha \in \mathbb{R}^{3n+\tau+1} : \Upsilon_1(u_\lambda, u') - \alpha \geq 0, \Upsilon_2(u_\lambda) - \alpha \geq 0, -\alpha \geq 0\}, \tag{13}$$

where $u_\alpha = (u_\lambda, u', \alpha)$.

If $\alpha^* = 0$, then point $u_\alpha^* = (u_\lambda^*, u'^*, \alpha^*)$ of global maximum of problem (12), (13) has been found; go to Step 5. If $\alpha^* < 0$, then it is impossible to find feasible starting point for problem (12), (13) since layout constraints are not satisfied for $\lambda = 0$. In this case, go back to Step 1.

Step 5. Assume that parameters θ_i, $i \in I_n$, are variable. Randomly generate starting values of rotation angles $\theta_i^* \in [0, 2\pi)$, $i \in I_n$.

Step 6. Form a feasible starting point (u_λ^*, u'^*) using u_α^* and solve the following auxiliary nonlinear programming problem:

$$\lambda^* = \max \lambda, \text{ s.t. } (u_\lambda, u') \in W_\lambda, \tag{14}$$

$$W_\lambda = \{(u_\lambda, u') \in \mathbb{R}^{3n+\tau+1} : \Upsilon_1(u_\lambda, u') \geq 0, \ \Upsilon_2(u_\lambda) \geq 0, \ 1 - \lambda \geq 0, \ \lambda \geq 0\}. \tag{15}$$

If $\lambda^* = 1$, then point $(u_\lambda^*, u'^*) = (v^*, \theta^*, \lambda^*, u'^*)$ of global maximum (14), (15) has been found; go to Step 7. If $\lambda^* < 1$, then go back to Step 1.

Step 7. Calculate $\mu(v^*, \theta^*)$. If $\mu(v^*, \theta^*) < 0$, then go to Step 8; otherwise go to Step 9.

Step 8. Starting from point $u_\beta^0 = (v^*, \theta^*, u'^*, \beta^0 = \mu(u^*))$, solve the auxiliary problem

$$\beta^* = \max \beta, \text{ s.t. } u_\beta \in W_\beta, \tag{16}$$

$$W_\beta = \{u_\beta \in \mathbb{R}^{3n+\tau+1} : \Upsilon_1(u, u') \geq 0, \ \Upsilon_2(u) \geq 0, \ \mu(u) - \beta \geq 0, -\beta \geq 0\}, \tag{17}$$

where β is residual, $u_\beta = (u, u', \beta)$, $u = (v, \theta)$.

TABLE 1. Initial Information about Cylinders

Parameters of a Cylinder	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$	$\mathbb{C}_4$	$\mathbb{C}_5$	$\mathbb{C}_6$	$\mathbb{C}_7$	$\mathbb{C}_8$
m_i	4	2	2	1	3	3	5	5
r_i	1	0.7	0.6	0.45	0.8	0.85	0.9	1
h_i	1.27	1.3	1.77	1.57	1.49	1.32	1.96	1.59

If $\beta^* = 0$, then point $u_\beta^* = (v^*, \theta^*, u'^*, \beta^*)$ of global maximum of problem (16), (17) has been found; go to Step 9. If $\beta^* < 0$, then go back to Step 1.

Step 9. Form a feasible starting point $u^0 = (v^*, \theta^*, u'^*) \in W$ for the CBLP.

RESULTS OF NUMERICAL EXPERIMENTS

Problem 1. Consider the problem of balance layout of cylinders $\mathbb{C}_i$, $i = 1, \dots, 8$, in a cylindrical container divided by two circular racks into subcontainers in order to minimize the deviation of the center of mass of the system Ω_A from point (x_0, y_0, z_0).

Let $n = 8$, $m = 3$, $H = 6$, $R = 2.5$, $t_1 = t_2 = 2$, and $(x_0, y_0, z_0) = (0, 0, 3)$. The masses and metric characteristics (radii and heights) of cylinders $\mathbb{C}_i$, $i = 1, \dots, 8$, are given in Table 1.

The results of numerical experiments for Problem 1 are as follows: the value of the objective function $d^* = 0.3851$ for $q^1 = (1, 4, 7, | 3, 5, 8, | 2, 6)$; the value of the objective function $d^* = 0.8847$ for $q^2 = (3, 7, 8, | 1, 5, 6, | 2, 4)$; the value of the objective function $d^* = 1.3938$ for $q^3 = (1, 5, 6, 8, | 2, 7, | 3, 4)$; and the value of the objective function $d^* = 1.8847$ for $q^4 = (1, 3, 5, 6, | 7, 8, | 2, 4)$. The best result $d^* = 0.3851$ is obtained for Problem 1 for $q^1 = (1, 4, 7, | 3, 5, 8, | 2, 6)$.

Figure 1 shows locally optimal layouts of cylinders in subcontainers, corresponding to the tuples q^1, q^2, q^3, and q^4.

Problem 2. Consider the problem of balance layout of a family of 3D objects (full-spheres $\mathbb{S}_i$, right circular cylinders $\mathbb{C}_i$, tori $\mathbb{Q}_i$, spherocylinders $\mathbb{S}_{\mathbb{C}_i}$, right rectangular parallelepipeds $\mathbb{P}_i$, and regular prisms $\mathbb{K}_i$) in a cylindrical container with regard for behavior constraints (constraints imposed on axial and centrifugal moments of the system) and constraints on the minimum admissible distances between objects, as well as between objects and container's boundary in order to minimize the deviation of the center of mass of system Ω_A from point (x_0, y_0, z_0).

Let $n = 20$, $m = 3$, $H = 1$, $R = 0.45$, $t_1 = t_2 = 0.35$, $A = \{\mathbb{S}_i, i = 1, \dots, 4, \mathbb{C}_i, i = 5, \dots, 8, \mathbb{Q}_i, i = 9, \dots, 12, \mathbb{S}_{\mathbb{C}_i}, i = 13, \dots, 16, \mathbb{P}_i, i = 17, 18, 19, \mathbb{K}_{20}\}$, $(x_0, y_0, z_0) = (0, 0, 0.5)$,

$$\{m_i,\ i = 1, \dots, 20\} = \{20.944, 15.2681, 27.8764, 34.5575, 63.7115, 41.8146, 30.4106, 28.4245,$$

$$49.9649, 24.8714, 38.6888, 26.2637, 20.7764, 17.2159, 16.8756, 52.8, 52.8, 52.8, 23.1489\};$$

$r_1 = 0.1$, $r_2 = 0.09$, $r_3 = 0.11$, $r_4 = 0.11$ be radii of full spheres $\mathbb{S}_i$, $i = 1, \dots, 4$;

$r_5 = 0.1$, $h_5 = 0.11$, $r_6 = 0.13$, $h_6 = 0.12$, $r_7 = 0.11$, $h_7 = 0.11$, $r_8 = 0.11$, $h_8 = 0.08$ be radii and semiheights of cylinders $\mathbb{C}_i$, $i = 5, \dots, 8$;

$r_9 = 0.08$, $h_9 = 0.07$, $r_{10} = 0.09$, $h_{10} = 0.075$, $r_{11} = 0.07$, $h_{11} = 0.06$, $r_{12} = 0.08$, $h_{12} = 0.07$ be distances from the centers of generating circles to spin axes and semiheight of tori $\mathbb{Q}_i$, $i = 9, \dots, 12$;

$r_{13} = 0.1$, $h_{13} = 0.05$, $l_{13} = 0.07$, $r_{14} = 0.05$, $h_{14} = 0.05$, $l_{14} = 0.08$, $r_{15} = 0.08$, $h_{15} = 0.05$, $l_{15} = 0.06$, $r_{16} = 0.08$, $h_{16} = 0.04$, $l_{16} = 0.07$ be radii, semiheights of cylinders, and heights of spherical segments for spherocylinders $\mathbb{S}_{\mathbb{C}_i}$, $i = 13, \dots, 16$;

$w_{17} = 0.11$, $l_{17} = 0.1$, $h_{17} = 0.12$, $w_{18} = 0.11$, $l_{18} = 0.1$, $h_{18} = 0.12$, $w_{19} = 0.11$, $l_{19} = 0.1$, $h_{19} = 0.12$ be the half-widths, semilengths, and semiheights of the parallelepipeds $\mathbb{P}_i$, $i = 16, 18, 19$;

$r_{20} = 0.09$, $h_{20} = 0.11$ be the length of the base and semiheight of the right hexagonal prism.

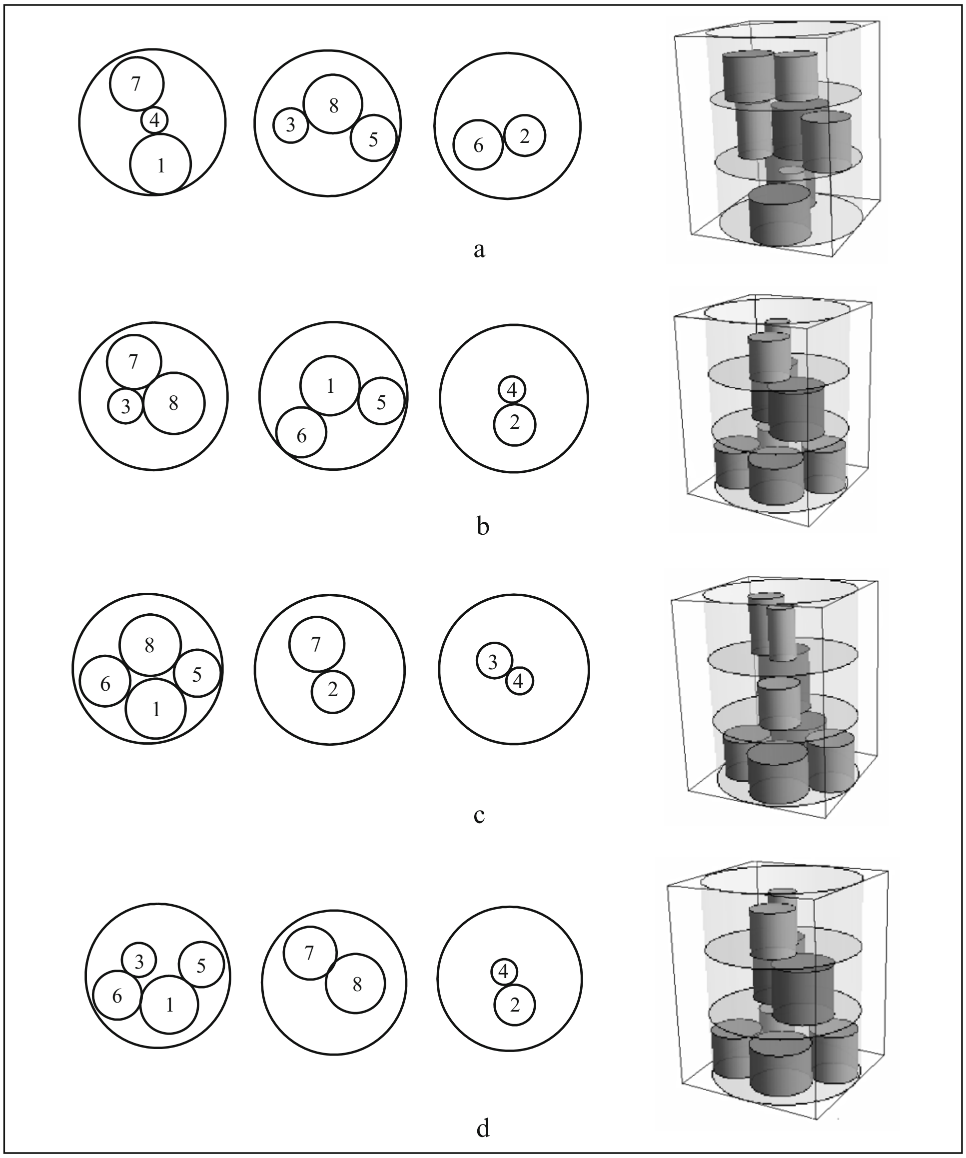

Fig. 1. Locally optimal layouts of cylinders in subcontainers Ω^1, Ω^2, Ω^3 for Problem 1, corresponding to tuple q^1 (a), tuple q^2 (b), tuple q^3 (c), and tuple q^4 (d).

Let the minimum admissible distances between objects, as well as between objects and container's boundary be $\rho_{\bar{i}} = \rho_{\bar{ij}} = 0.02$, $i < j = 1, \dots, 20$; feasible values of the axial and centrifugal moments of inertia be $\Delta J_X = \Delta J_Y = 170$, $\Delta J_Z = 150$, and $\Delta J_{XY} = \Delta J_{YZ} = \Delta J_{XZ} = 0$.

The results of numerical experiments for Problem 2 are as follows: the value of the objective function $d^* = 0.0019$ for $q^1 = (1, 5, 6, 9, 13, 14, 17, | 2, 3, 7, 10, 15, 18, 20, | 4, 8, 11, 12, 16, 19)$; the value of the objective function $d^* = 0.0056$ for $q^2 = (3, 5, 6, 7, 10, 13, 17, | 2, 4, 11, 12, 14, 18, 19, |1, 8, 9, 15, 16, 20)$; the value of the objective function $d^* = 0.0018$ for $q^3 = (1, 5, 7, 11, 13, 15, 19, 20, | 2, 4, 8, 10, 12, 14, 17, | 3, 6, 9, 16, 18)$; the value of the objective function $d^* = 0.0030$ fo $q^4 = (2, 3, 7, 9, 13, 16, 18, 19, | 5, 6, 8, 11, 12, 14, 20, | 1, 4, 10, 15, 17)$. The best result $d^* = 0.0018$ was obtained for Problem 2 for $q^3 = (1, 5, 7, 11, 13, 15, 19, 20, |2, 4, 8, 10, 12, 14, 17, |3, 6, 9, 16, 18)$.

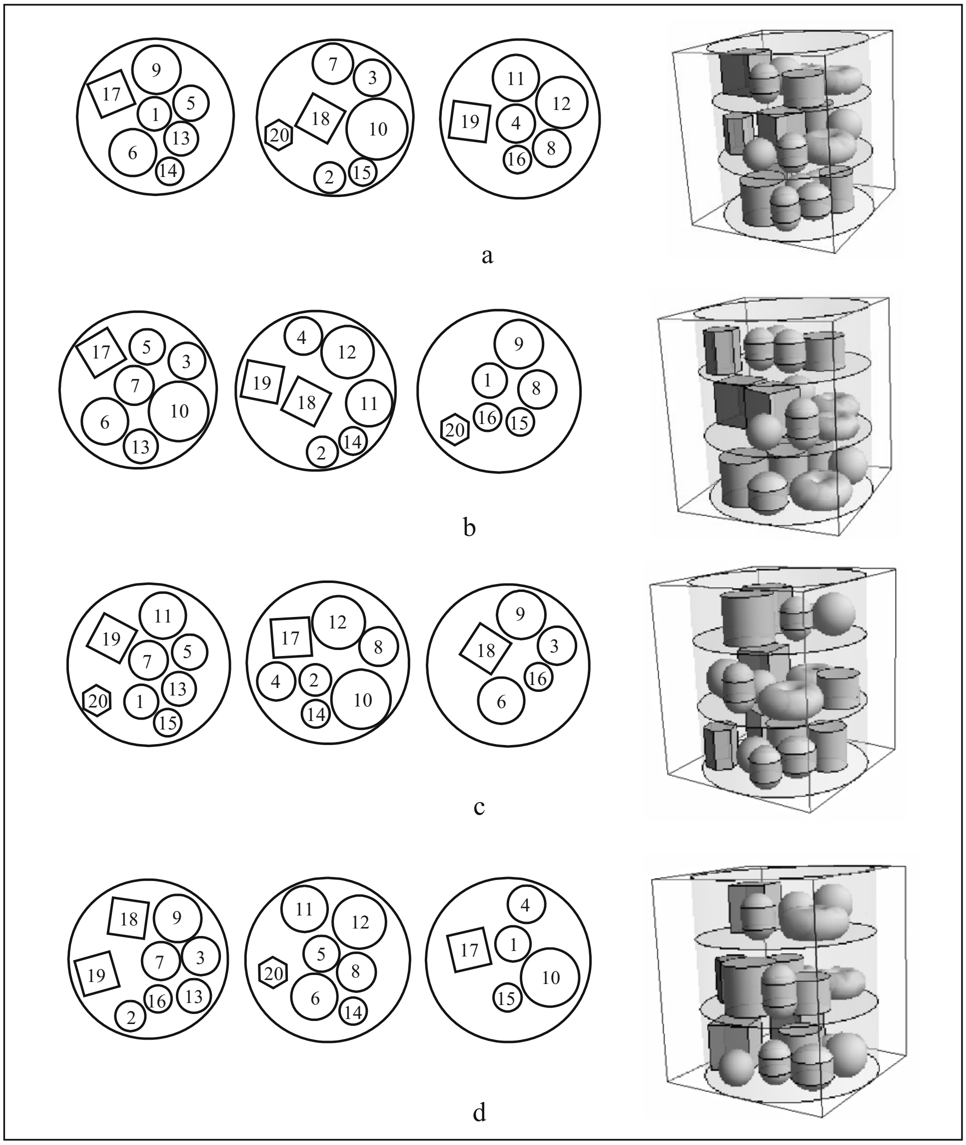

Fig. 2. Locally optimal layouts of objects in subcontainers Ω^1, Ω^2, Ω^3 for Problem 2, corresponding to tuple q^1 (a), tuple q^2 (b), tuple q^3 (c) and tuple q^4 (d).

Figure 2 shows locally optimal layouts of objects in subcontainers Ω^1, Ω^2, and Ω^3, corresponding to the tuples q^1, q^2, q^3, and q^4.

CONCLUSIONS

We have considered the problem of balance layout of 3D objects in a container divided by horizontal racks into subcontainers. We have constructed a mathematical model, which takes into account not only behavior and layout constraints but also the combinatorial features of the problem, related to the necessity of constructing partitions of the set of cylinders being arranged into subcontainers. We have proposed a solution strategy that includes procedures of generating partition tuples, constructing starting points from the domain of feasible solutions, and local optimization. This approach uses the principle of “multistart” for finding “good” feasible solutions. The results of numerical experiments have shown the efficiency of the proposed approach for the considered class of balance layout problems.

REFERENCES

1. B. Chazelle, H. Edelsbrunner, and L. J. Guibas, “The complexity of cutting complexes,” Discrete & Computational Geometry, Vol. 4, No. 2, 139–181 (1989).
2. N. Chernov, Y. Stoyan, T. Romanova, and A. Pankratov, “*Phi*-functions for 2D objects formed by line segments and circular arcs,” Advances in Operations Research, DOI:10.1155/2012/346358 (2012).
3. N. Chernov, Y. Stoyan, and T. Romanova, “Mathematical model and efficient algorithms for object packing problem,” Computational Geometry: Theory and Applications, Vol. 43, No. 5, 535–553 (2010).
4. G. Fasano and J. D. Pintér (eds.), Modeling and Optimization in Space Engineering, Springer, Optimization and Its Applications, Vol. 73 (2013).
5. C. Che, Y. Wang, and H. Teng, “Test problems for quasi-satellite packing: Cylinders packing with behavior constraints and all the optimal solutions known,” URL: http://www.optimizationonline.org/DB_HTML/2008/09/2093.html.
6. G. Fasano and J. D. Pintér (eds.), Space Engineering. Modeling and Optimization with Case Studies, Springer Optimization and its Applications, Vol. 114 (2016).
7. Z. Sun and H. Teng, “Optimal layout design of a satellite module,” Engineering Optimization, Vol. 35, No. 5, 513–530 (2003).
8. K. Lei, “Constrained layout optimization based on adaptive particle swarm optimizer,” in: Zhihua Cai, Zhenhua Li, Zhuo Kang, and Yong Liu (eds.), Proc. ISICA 2009, Advances in Computation and Intelligence, LNCS, Vol. 5821, 434–442 (2009).
9. Yu. Stoyan and T. Romanova, “Mathematical models of placement optimization: Two- and three-dimensional problems and applications,” in: G. Fasano and J. D. Pintér (eds.), Modeling and Optimization in Space Engineering, Springer Optimization and its Applications, Vol. 73, Chap. 15, 363–388 (2013).
10. I. V. Sergienko and V. P. Shilo, Discrete Optimization Problems: Challenges, Methods, Solutions, and Analysis [in Russian], Naukova Dumka, Kyiv (2003).
11. N. V. Semenova and L. M. Kolechkina, Vector Discrete Optimization Problems on Combinatorial Sets: Methods of the Analysis and Solution [in Ukrainian], Naukova Dumka, (2009).
12. A. A. Kovalenko, T. E. Romanova, and P. I. Stetsyuk, “Balance layout problem for 3D-objects: Mathematical model and solution methods,” Cybern. Syst. Analysis, Vol. 51, No. 4, 556–565 (2015).
13. Yu. Stoyan, T. Romanova, A. Pankratov, A. Kovalenko, and P. Stetsyuk, “Modeling and optimization of balance layout problems,” in: G. Fasano and J. D. Pintér (eds.), Space Engineering. Modeling and Optimization with Case Studies, Springer Optimization and its Applications Vol. 114, 177–208 (2016).
14. E. Reingold, J. Nievergelt, and N. Deo, Combinatorial Algorithms: Theory and Practice, Pearson Education (1977).
15. Yu. G. Stoyan and I. V. Grebennik, “Description and generation of combinatorial sets having special characteristics,” Intern. J. of Biomedical Soft Computing and Human Sciences, Special Volume “Bilevel Programming, Optimization Methods, and Applications to Economics,” Vol. 18, No. 1, 83–88 (2013).
16. Yu. Stoyan, A. Pankratov, and T. Romanova, “Quasi-*phi*-functions and optimal packing of ellipses,” J. of Global Optimization, Vol. 65, No. 2, 283–307 (2015).
17. I. Grebennik and O. Lytvynenko, “Random generation of combinatorial sets with special properties,” An Intern. Quarterly J. on Economics of Technology and Modelling Processes (ECONTECHMOD), Vol. 5, No. 4, 43–48 (2016).
18. A. Wachter and L. T. Biegler, “On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming,” Mathematical Programming, Vol. 106, No. 1, 25–57 (2006).