DOI 10.1007/s10559-017-9905-y
Cybernetics and Systems Analysis, Vol. 53, No. 1, January, 2017

NUMERICAL METHOD TO SOLVE THE CAUCHY PROBLEM WITH PREVIOUS HISTORY

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UDC 517.9+518.6

Abstract. *The paper analyzes the theoretical aspects of constructing a family of single-stage multi-step methods for solving the Cauchy problem with prehistory for ordinary differential equations. The authors consider general issues related to discretization, approximation, convergence, and stability. The problem of improving the accuracy of numerical solutions is analyzed in detail. The results presented in the paper are also applicable for the numerical solution of partial differential equations.*

Keywords: *ordinary differential equations, single-stage multi-step method, discretization, approximation, convergence, strong stability.*

INTRODUCTION

The authors investigated hydrodynamic modeling of atmospheric processes and have concluded that the advection operator appearing in all the equations of atmospheric models caused the greatest amount of problems in application of the numerical methods of solution of these equations [1–6]. Estimates [2] show that approximations of horizontal terms of advection are the main source of errors (almost 40% of the total error) in short-term numerical forecasting of the state of the atmosphere.

The spatial finite-difference approximation of the advection equation generates errors of false representation, description of phase velocity and computing variance, as well as effects of nonlinear instability [7]. When symmetric difference scheme of the second order of accuracy is used, it appears that, first, advection velocity is less than the exact advection velocity (a consequence of this error is total deceleration of advection process), and second, advection velocity varies depending on the wave number. Such parasitic variance, in particular, is rather considerable for short waves. If a system being a superposition of waves is transferred, then parasitic variance will cause strain of this system. This concerns first of all mesoscale systems such as originating fronts and cyclones, orographic and thermal perturbations, and other sets of waves representing superposition. For this reason, in numerical forecasting of the state of the atmosphere, these systems (if they are available in the original fields) start to be deformed very fast, becoming more smooth. Since these special features of fields are very important in mesoscale processes, the effect of computing variance is of interest.

It is obvious that schemes with noncentral space differences, which hinder the propagation of perturbations in the direction opposite to physical advection, are more preferable than schemes in which space derivative is approximated by means of central differences [8, 9]. However, rather strong smoothing effect and only first order of accuracy of noncentral difference operators considerably reduce these advantages as compared with difference operators of second order of accuracy.

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Translated from Kibernetika i Sistemnyi Analiz, No. 1, January–February, 2017, pp. 42–67. Original article submitted June 29, 2016.

The analysis concerned space approximation of the linear equation of advection. However, in atmosphere models, one has to solve nonlinear hydrodynamic equations, and their nonlinearity is caused by the presence of terms that describe advection. The nonlinearity of hydrodynamic equations causes nonlinear instability and "explosion" of the numerical solution if these unfavorable constraints cannot be prevented.

It is possible to overcome the main difficulties (in particular, errors of false representation, description of phase velocity, and computing variance) by representing the equations of model of the atmosphere in conservative form and by applying difference schemes of higher order of accuracy to approximate advective terms.

The studies [1, 3] propose an efficient method of the fourth order of accuracy to approximate partial derivatives of the first and second orders containing in the system of equations of the model of circulation of the atmosphere. We present the results that allow coordinating approximation orders on space and time networks. This can be attained by constructing a set of efficient, of enhanced order of accuracy, finite-difference multistep single-stage schemes of the solution of the Cauchy problem.

PROBLEM STATEMENT. AUXILIARY DEFINITIONS AND RESULTS

In practice of numerical solution of the Cauchy problem for a first-order differential equation, one-step methods are usually used [10–22]; their construction implies involving the information on the problem being solved only on a one-step interval $\tau = t_l - t_{l-1}$, $l = \overline{1, N}$. Along with computationally convenient special features, such methods have one significant shortcoming: a low accuracy. In the methods whose accuracy $O(\tau^p)$ corresponds to cases $p > 1$, such information at each process stage should be obtained over again, which makes respective computing rules much labor-consuming. Having discarded the single-step requirement, computing methods can be constructed so as to repeatedly use a part of the obtained information at several adjacent steps of the computing process. Such methods, which possess information about the problem under solution on an interval longer than one step, are the main target of research of the present study.

The paper is devoted to the theoretical and practical aspects of the numerical solution of the problem for an ordinary differential equation of first order

$$\frac{\partial y}{\partial t} = f(y, t), \tag{1}$$

$$y|_{t=t_0} = y_0, \tag{2}$$

where $y_0 \in \mathcal{R}$, $f \in C(\mathcal{R} \to \mathcal{R})$, and function f satisfies the Lipschitz constraint.

The exact solution of the original problem formulated in such a way exists and is unique.

The purpose of the present study is to construct and analyze the fundamental structural properties of the efficient finite-difference *l*-step single-stage scheme of the solution of problem (1), (2). Let us define difference time mesh on which we will solve problem (1), (2) numerically. For simplicity, in what follows we will assume that $t_0 = 0$, which does not restrict the generality.

Definition 1. A mesh on $[0, T]$ is a given finite set of nodes $t_n \in [0, T]$, $n = \overline{0, N}$, such that

$$t_0 = 0,\ t_n > t_{n-1},\ n = \overline{0, N}. \tag{3}$$

The quantities $\tau_n = t_n - t_{n-1} > 0$, $n = \overline{1, N}$, are called steps of mesh $\omega_\tau = \{\tau_n, n = \overline{1, N}\}$, and $\gamma = t_{n-1} / t_n$, $n = \overline{1, N}$, is the ratio of mesh steps. Thus, it is possible to define mesh $t_n \in [0, T]$, $n = \overline{0, N}$, as well as the set of its steps $\omega_\tau = \{\tau_n, n = \overline{1, N}\}$, where $\tau_n = t_n - t_{n-1} > 0$, $n = \overline{1, N}$.

Definition 2. A mesh $\overline{\omega}_\tau = \{\tau = T / N\}$ is called uniform if

$$t_n = n\tau,\ n = \overline{1, N},\ \gamma = 1. \tag{4}$$

The structure of the available general l-step methods is rather difficult [10]; therefore, we will use only a special class of single-stage l-step methods, whose step procedure composes a linear combination of values of y_i and f_i, $i = \overline{n, n+l}$. No repeated substitutions of values of f (which are the essence of multi-step methods) are admitted.

The most popular in l-step single-stage scheme of solution of problem (1), (2) is step procedure in the form

$$\sum_{i=0}^{l} a_i y_{n+i} - \tau_l \sum_{i=0}^{l} b_i f_{n+i} = 0, \; n = \overline{0, N}. \tag{5}$$

The term τ_l before the second sum is separated to make coefficients b_i, as well as a_i, dimensionless. The $2 \times (N+1)$ matrix

$$\mathbf{M} = \begin{pmatrix} a_0 \dots a_N \\ b_0 \dots b_N \end{pmatrix}$$

is the generating matrix of scheme (5). If $b_l = 0$, then computing methods (5) are usually called explicit, if $b_l \neq 0$, they are implicit.

The formula for evaluation of (5) should be exact if $y(t)$ is a constant ($\partial y / \partial t = f \equiv 0$). Then it will give the equalities

$$a_l \neq 0, \; \sum_{i=0}^{l} a_i = 0. \tag{6}$$

Let us make the replacement $q^i = y_i$ and $r^i = f_i$, $i = \overline{0, l}$, and associate the linear l-step procedure (5) and its generating matrix with the "characteristic" polynomials

$$\Phi_a(q) = \sum_{i=0}^{l} c q^i, \; \Phi_b(r) = \sum_{i=0}^{l} b_i r^i. \tag{7}$$

If c is some number, then the number $\Phi_a(c) = \sum_{i=0}^{l} a_i c^i$, obtained by replacing in expression $\Phi_a(q)$ the unknown q with the number c followed by performing all the specified operations is called the value of the polynomial $\Phi_a(q)$ for $q = c$. If $\Phi_a(c) = 0$, i.e., polynomial $\Phi_a(q)$ becomes zero after the number c is substituted in it for the unknown, then c is called the root of the polynomial $\Phi_a(q)$ (or of the equation $\Phi_a(q) = 0$).

Definition 3. Number c is a root of the polynomial $\Phi_a(q)$ if and only if polynomial Φ_a is equal to zero for $q = c$.

Since one is a zero of the characteristic polynomial Φ_a, it cannot be a zero of the characteristic polynomial Φ_b. Therefore, in each class of procedures there is a unique linear l-step scheme and a unique generating matrix corresponding to it, satisfying the condition

$$\sum_{i=0}^{l} b_i = 1. \tag{8}$$

In what follows, we will call the finite-difference scheme (5) sutisfying conditions (6) and (8) normalized.

Coefficients a_i and b_i can be chosen in different ways so that (5) would approximate $\partial y / \partial t - f = 0$. The studies [11, 13, 23] present the techniques to construct and analyze the error, approximation, stability, and convergence of some special classes of efficient linear l-step schemes. Systematization of the theoretical results obtained therein shows that the basic stability conditions of single-stage multistep schemes mainly concern the coefficients a_i of the characteristic polynomial Φ_a and can be formulated by the following Definitions.

Definition 4. Characteristic polynomial Φ_a satisfies the rooted criterion if its roots lie in a unit closed circle, and it has no multiple zero on a unit circle.

Definition 5. A homogeneous linear l-step scheme is called strongly stable if its characteristic polynomial Φ_a satisfies the root criterion.

Elementary and at the same time most used schemes (5) can be obtained on the basis of quadrature formulas and the method of undetermined coefficients. However, not all schemes constructed in such a way are stable. For example, the explicit scheme

$$y_{n+l} + 4y_{n+l-1} - 5y_{n+l-2} - \frac{\tau}{6}(4f_{n+l-1} + 2f_{n+l-2}) = 0$$

satisfying the normalization conditions (6), (8) is not stable and cannot converge since its characteristic polynomial $\Phi_a(q) = q^2 + 4q - 5 = (q-1)(q+5)$ has a root outside the limits of one.

In what follows, as the basis for the method of constructing a family of multistep single-stage higher-order schemes, we will use the special relation between coefficients of the characteristic polynomial Φ_a, according to which real coefficients of a linear l-step scheme a priori satisfy the root criterion in narrow sense, i.e., characteristic polynomial Φ_a has only real roots, which are inside a unit interval of real axis, and there are no multiple zeroes in one.

Beforehand, let us consider some main properties of divisibility of polynomials, which we will use below. Generally, when dividing a polynomial with real coefficients

$$\eta(q) = a_l q^l + a_{l-1} q^{l-1} + \ldots + a_1 q + a_0 = \sum_{i=0}^{l} a_i q^i \tag{9}$$

by a linear polynomial

$$\varphi(q) = q - c, \tag{10}$$

the obtained remainder is some number ε (zero or a polynomial of zero degree). The theorem below allows finding this remainder without executing the division.

THEOREM 1. The remainder of division of polynomial $\eta(q)$ by linear polynomial $\varphi(q)$ is equal to the value $\eta(c)$ of polynomial $\eta(q)$ for $q = c$.

Proof. As is known from elementary algebra, for two polynomials with real coefficients $\eta(q)$ and $\varphi(q) = q - c$, it is possible to find polynomials $\psi(q)$ and ε such that

$$\eta(q) = (q - c)\psi(q) + \varepsilon. \tag{11}$$

Then it is proved that polynomials $\psi(q)$ and ε satisfying this condition are defined uniquely.

Selecting the values of both parts of Eq. (11) for $q = c$, we obtain $\eta(c) = (c-c)\psi(c) + \varepsilon = \varepsilon$, which proves the theorem.

COROLLARY. If $c = 1$ and condition (6) is satisfied, the remainder ε is equal to zero.

Indeed, substituting $q = 1$ into (9), we obtain

$$\eta(q) = a_l + a_{l-1} + \ldots + a_1 + a_0 = \sum_{i=0}^{l} a_i = 0.$$

The theorem below follows herefrom.

THEOREM 2. Polynomial $\varphi(q)$ is a divider of polynomial $\eta(q)$ if and only if there exists a polynomial $\psi(q)$ satisfying the equality

$$\eta(q) = \varphi(q)\psi(q). \tag{12}$$

Polynomial $\psi(q)$ satisfying this condition is defined uniquely.

Proof. Let there also exist a polynomial $\overline{\psi}(q)$ satisfying the equality

$$\eta(q) = \varphi(q)\overline{\psi}(q). \tag{13}$$

Equating the right-hand sides of equalities (12) and (13), we find $\varphi(q)\{\psi(q) - \overline{\psi}(q)\} = 0$. Since in the obtained expression $\varphi(q) \neq 0$, the expression $\psi(q) - \overline{\psi}(q) = 0$ should hold, i.e., $\psi(q) = \overline{\psi}(q)$, as was to be shown. Then it is obvious that the degree of $\varphi(q)$ is no greater than the degree of $\eta(q)$.

THEOREM 3. The number $c = d/a$, $a \neq 0$, is a root of the polynomial $\eta(q)$ if and only if $\eta(q)$ is divisible by the linear polynomial $aq - d$.

Proof. Indeed, let $\eta(q) = (aq - d)\psi(q) = a(q - c)\psi(q)$. Selecting the values of both parts of this equality for $q = c$, we obtain $\eta(c) = a(c - c)\psi(c) = 0$, which proves the theorem. Hence, finding the roots of polynomial $\eta(q)$ is equivalent to finding its linear dividers.

The Fundamental Theorem of Algebra. Any polynomial with real coefficients of lth degree

$$\eta(q) = a_l q^l + a_{l-1} q^{l-1} + \ldots + a_1 q + a_0 = \sum_{i=0}^{l} a_i q^i \tag{14}$$

has exactly $l+1$ real roots (provided that each root is considered the number of times equal to its multiplicity) is representable as follows:

$$\eta(q) = (a_0 q - 1)(a_1 q - 1)\ldots(a_l q - 1). \tag{15}$$

Expansion (15) is unique for polynomial (14) to within the order of multiplicands.

Proof. The theorems proved earlier allow us to state that for $\eta(q)$ the root $q_0 = a_0^{-1}$ exists. Therefore, polynomial $\eta(q)$ has the expansion $\eta(q) = (q - q_0)\psi(q)$. The coefficients of the polynomial $\psi(q)$ are real again; therefore, $\psi(q)$ has a root $q_1 = a_1^{-1}$, whence $\eta(q) = (q - q_0)(q - q_1)\xi(q)$.

Following the same line of reasoning, we obtain expansion (15) in a finite number of steps.

To prove the uniqueness of expansion (15), we assume that there also exists an expansion

$$\eta(q) = (q - \zeta_0)(q - \zeta_1)\ldots(q - \zeta_l). \tag{16}$$

From (15) and (16) the equality follows

$$(q - q_0)(q - q_1)\ldots(q - q_l) = (q - \zeta_0)(q - \zeta_1)\ldots(q - \zeta_l). \tag{17}$$

If a root q_l of multiplicity $s = 1$ would differ from all ζ_i, $i = 0, 1, \ldots, l$, then having substituted q_l into (17) instead of the unknown q, we would obtain zero on the left and some nonzero number on the right, i.e., obtain a contradiction. Thus, any root q_i of multiplicity $s = 1$ is equal to some root ζ_i and vice versa.

Then let the expansion contain $s > 1$ roots q_0 and among roots ζ_i, $i = 0, 1, \ldots, l$, there be m equal to the root q_0. It is required to prove that $s = m$. If we suppose, for example, that $s > m$, then reducing both parts of Eq. (17) by the factor $(q - q_0)^m$, we arrive at the equality whose left-hand side still contains factor $q - q_0$, and the right-hand side does not. We have mentioned above that this leads to a contradiction. Thus, we have proved the uniqueness of expansion (15) for polynomial (17).

Though the fundamental theorem of algebra [23] and the majority of previous results are true for complex coefficients a_i $(i = 0, 1, \ldots, l)$, in what follows we will consider them real.

In numerical computer implementation of hydrodynamic models of circulation of atmosphere, rather stringent requirements are imposed on the efficiency of computing methods according to which the method should:

— be identical to the explicit one and ensure fast computations to find the values of the problem at each time step;

— have favorable properties of strong stability in order to eliminate excessive propagation of error for the problem with decreasing solution components;

— make it possible to control the value of time step on the whole interval of integration of the equations of the problem, and step variation should not complicate the algorithm of the method implementation.

Despite the mutual contradiction of these requirements, it appears possible to construct a family of very simple linear l-step single-stage methods satisfying all the above requirements.

Based on the methodical necessity, we will start constructing the family of multistep single-stage methods on a uniform mesh $\overline{\omega}_\tau$. Then we will propagate the obtained result to mesh ω_τ with a variable step.

FAMILY OF STRONGLY STABLE LINEAR *l*-STEP FINITE-DIFFERENCE SCHEMES ON UNIFORM MESH $\overline{\omega}_\tau$

Let us obtain some corollaries from the theory of algebra of complex numbers, concerning polynomials with real coefficients. In essence, it is due to these corollaries the main theorem considered below is so important.

Let a polynomial with real coefficients $\eta(q) = a_l q^l + a_{l-1} q^{l-1} + \ldots + a_1 q + a_0$ have a complex root α, i.e., $a_l \alpha^l + a_{l-1}\alpha^{l-1} + \ldots + a_1\alpha + a_0 = 0$.

The last equality will not be violated if we replace all numbers in it with conjugate ones. However, all the coefficients a_i, $i = 0, 1, \ldots, l$, as well as the number 0, which are on the right in the polynomial $\eta(q)$, being real, remain the same after the replacement. Therefore, we will arrive at the equality $\eta(\overline{\alpha}) = a_l \overline{\alpha}^l + a_{l-1}\overline{\alpha}^{l-1} + \ldots + a_1\overline{\alpha} + a_0 = 0$.

Thus, we can formulate the following definition.

Definition 6. If a complex number α is a root of polynomial $\eta(q)$ with real coefficients, then the conjugate number $\overline{\alpha}$ will also be a root for $\eta(q)$.

Hence, polynomial $\eta(q)$ is multiple of the square trinomial $\psi(q) = (q-\alpha)(q-\overline{\alpha}) = q^2 - (\alpha + \overline{\alpha})q + \alpha\overline{\alpha}$, whose coefficients (as is known from the theory of algebra of complex numbers) are real.

Let us prove that the roots α and $\overline{\alpha}$ are of identical multiplicities in the polynomial $\eta(q)$.

By contradiction, let the roots in the polynomial $\eta(q)$ have the multiplicities k_1 and k_2, respectively, and let, for example, $k_1 > k_2$. Then $\eta(q)$ is multiple of the k_2th degree of the polynomial $\psi(q)$, $\eta(q) = \psi^{k_2}(q)\xi(q)$.

Polynomial $\xi(q)$, as a quotient of two polynomials with real coefficients, also has real coefficients; however, on the contrary to that proved above, the number α is its $(k_1 - k_2)$-multiple root, while the number $\overline{\alpha}$ is not a root for it. Herefrom, it follows that $k_1 = k_2$. Thus, we arrive at the fact (well-known from the course of algebra) that complex roots of any polynomial with real coefficients are pairwise conjugate. From here and from the proved uniqueness of expansions of the form (15), the final result follows.

Corollary of the Fundamental Theorem of Algebra. Any polynomial $\Phi_a(q)$ with real coefficients is uniquely (to within the order of factors) representable as the product of its higher coefficient a_l and several linear polynomials with real coefficients $q - \alpha$, corresponding to its real roots, and quadratic polynomials (18), corresponding to the pairs of complex conjugate roots.

The obtained result allows us to state the following.

THEOREM 4. For strongly stable homogeneous linear *l*-step finite-difference scheme (5) with real coefficients a_i and b_i $(i = \overline{0, l})$, which satisfy the normalization conditions (6) and (8), characteristic polynomial $\Phi_a(q)$ has only real roots, which satisfy strong root criterion (except for one simple root $q_0 = 1$, its other roots q_i $(i = \overline{1, l})$ are within an open unit interval) and is defined uniquely:

$$\Phi_a(q) = (q-1)\prod_{i=1}^{l}(2q-1)^i = 0. \tag{18}$$

Proof. First, we will consider polynomial (18) with the coefficients $\overline{a}_i$ $(i = \overline{1, l})$ such that all the roots $\overline{q}_i$ of the polynomial

$$\overline{\Phi}_a(q) = \sum_{i=1}^{l} \overline{a}_i q^i = 0$$

are inside an open unit interval. Then we will consider polynomial (15), which has only a simple root q_0 in unity, and will specify the relation between coefficients a_i and $\overline{a}_i$ $(i = \overline{1, l})$ by the equality

$$\Phi_a(q) = (q - q_0)\overline{\Phi}_a(q). \tag{19}$$

We will express coefficients of the polynomial $\overline{\Phi}_a(q)$ in terms of its roots by means of the Vieta formulas:

$$\bar{a}_0 = (-1)^{i-1},$$

$$\bar{a}_1 = (-1)^{i-2}(\alpha_1 + \alpha_2 + \ldots + \alpha_l),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{20}$$

$$\bar{a}_{l-1} = \alpha_1\alpha_2\ldots\alpha_l + \alpha_1\alpha_2\ldots\alpha_{l-2}\alpha_l + \ldots + \alpha_2\alpha_3\ldots\alpha_l,$$

$$\bar{a}_l = \alpha_1\alpha_2\ldots\alpha_l.$$

Then with regard for Eq. (19), coefficients of the polynomial $\Phi_a(q)$ become

$$a_0 = (-1)^l,$$

$$a_1 = (-1)^{l-1}(\alpha_1 + \alpha_2 + \ldots + \alpha_l + 1),$$

$$a_2 = (-1)^{l-2}(\alpha_1 + \alpha_2 + \ldots + \alpha_l + \alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots + \alpha_{l-1}\alpha_l),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{21}$$

$$a_{l-1} = -(\alpha_1\alpha_2\ldots\alpha_l + \alpha_1\alpha_2\ldots\alpha_{l-1} + \alpha_1\alpha_2\ldots\alpha_{l-2}\alpha_l + \ldots + \alpha_2\alpha_3\ldots\alpha_l),$$

$$a_l = \alpha_1\alpha_2\ldots\alpha_l.$$

Let us expand function $y(t)$ into the Taylor series at points $l, l-2, \ldots, 0$ based on its value at point $l-1$

$$y_{n+l} = y_{n+l-1} + \frac{\tau}{1!}f_{n+l-1} + \frac{\tau^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{n+l-1} + \ldots + \frac{\tau^k}{k!}\left(\frac{\partial^{k-1} f}{\partial t^{k-1}}\right)_{n+l-1},$$

$$y_{n+l-2} = y_{n+l-1} - \frac{\tau}{1!}f_{n+l-1} + \frac{\tau^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{n+l-1} + \ldots + (-1)^k\frac{\tau^k}{k!}\left(\frac{\partial^{k-1} f}{\partial t^{k-1}}\right)_{n+l-1},$$

$$y_{n+l-3} = y_{n+l-1} - 2\frac{\tau}{1!}f_{n+l-1} + 4\frac{\tau^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{n+l-1} + \ldots + (-1)^k 2^k\frac{\tau^k}{k!}\left(\frac{\partial^{k-1} f}{\partial t^{k-1}}\right)_{n+l-1}, \tag{22}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_n = y_{n+l-1} - (l-1)\frac{\tau}{1!}f_{n+l-1} + (l-1)^2\frac{\tau^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{n+l-1} + \ldots + (-1)^k(l-1)^k\frac{\tau^k}{k!}\left(\frac{\partial^{k-1} f}{\partial t^{k-1}}\right)_{n+l-1}.$$

Multiplying sequentially the left- and right-hand sides of the first equality in (22) by the coefficient a_l defined by system (21), those of the second equality by coefficient a_{l-2}, and those of the third equality by a_{l-3}, etc., and adding termwise the obtained expressions, we obtain the equality

$$\sum_{i=0}^{l} a_i y_{n+i} = \tau\sum_{i=1}^{k-1}\frac{c_i}{(i+1)!}\left(\frac{\partial^i f}{\partial t^i}\right)_{n+l-1}, \tag{23}$$

where

$$c_1 = a_l - a_{l-2} - 2a_{l-3} - 3a_{l-4} - \ldots,$$

$$c_2 = a_l + a_{l-2} + 4a_{l-3} + 9a_{l-4} + \dots,$$

$$c_3 = a_l - a_{l-2} - 8a_{l-3} - 27a_{l-4} - \dots,$$

$$c_4 = a_l + a_{l-2} + 16a_{l-3} + 81a_{l-4} + \dots,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$c_l = a_l + (-1)^l (a_{l-2} + 2^l a_{l-3} + 3^l a_{l-4} + \dots).$$

If the right-hand side of the partial differential equation is continuous and is bounded together with the nth order derivatives, then approximation (23) will ensure high accuracy due to small coefficient in the remainder term also in case of decreasing step τ. If the right-hand side has no such derivatives, then the limiting order of accuracy of this scheme cannot be implemented. Then schemes of a smaller order of accuracy, equal to the order of existing derivatives, are usually used.

We will omit from consideration elementary two-step schemes of first and second orders (as they are investigated theoretically quite well), will construct a family of schemes of the third (in case of explicit scheme) and fourth (in case of implicit scheme) orders of accuracy and will use it as an example to analyze the main ideas of the method. As the initial expression for the characteristic polynomial $\Phi_a(q)$ we will take the quadratic trinomial

$$\Phi_a(q) = (q-1)(a_1 q - 1) = a_1 q^2 - (1 + a_1)q + 1, \tag{24}$$

whose one of the roots is the real root $q_0 = 1$. According to the corollary of the fundamental theorem of algebra, the second root (24) will also be real.

We will specify coefficient a_1 so that the characteristic polynomial (24) corresponds to the linear procedure (5) approximating the equation $\partial y / \partial t - f(t, y) = 0$. As the initial expression, we will use Eq. (23) for $l = 2$

$$\begin{aligned} &a_1 y_{n+l} - (a_1 + 1) y_{n+l-1} + y_{n+l-2} \\ &= (a_1 - 1)\frac{\tau}{1!} f_{n+l-1} + (a_1 + 1)\frac{\tau^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{n+l-1} + (a_1 - 1)\frac{\tau^3}{3!}\left(\frac{\partial^2 f}{\partial t^2}\right)_{n+l-1} \\ &+ (a_1 + 1)\frac{\tau^4}{4!}\left(\frac{\partial^3 f}{\partial t^3}\right)_{n+l-1} + \dots \end{aligned} \tag{25}$$

and will supplement it with the Taylor series for function f in the following form:

$$0 = -\tau f_{n+l-2} + \tau f_{n+l-1} - 2\frac{\tau^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{n+l-1} + 3\frac{\tau^3}{3!}\left(\frac{\partial^2 f}{\partial t^2}\right)_{n+l-1} - 4\frac{\tau^4}{4!}\left(\frac{\partial^3 f}{\partial t^3}\right)_{n+l-1} + \dots \tag{26}$$

Let us multiply (26) by $(a_1 + 1)/2$ and add the obtained expression to (25). The first two terms on the right-hand side of series (26) contain coefficients with the opposite signs and equal absolute values τ. Therefore, their algebraic sum multiplied by the same factor $(a_1 + 1)/2$ is equal to zero. Hence, for constraint (8) to hold, it is necessary to equate coefficient $a_1 - 1$ of f_{n+l-1} to one, i.e., to suppose that $a_1 = 2$. The obtained value of a_1 leads to the following formula for the explicit difference scheme:

$$y_{n+l} = y_{n+l-1} + \frac{1}{2}\left[y_{n+l-1} - y_{n+l-2} + \frac{\tau}{2}(5 f_{n+l-1} - 3 f_{n+l-2}) \right] + O(\tau^3). \tag{27}$$

For $a_1 = 2$, the roots of the characteristic equation for scheme (27) are real and different and satisfy the dependencies $q_0 = 1$ and $0 < q_1 < 1$. Hence, according to Definition 5, the homogeneous scheme (27) is strongly stable.

To construct an implicit scheme, we will supplement system (25) and (26) with the Taylor series

$$0 = -\tau f_{n+l} + \tau f_{n+l-1} + 2\frac{\tau^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{n+l-1} + 3\frac{\tau^3}{3!}\left(\frac{\partial^2 f}{\partial t^2}\right)_{n+l-1} + 4\frac{\tau^4}{4!}\left(\frac{\partial^3 f}{\partial t^3}\right)_{n+l-1} + \ldots \tag{28}$$

Let us multiply (26) by some factor A and (28) by some factor B and add the obtained expressions to (25). We will find the values of factors A and B so that the coefficients of $(\partial f/\partial t)_{n+l-1}$ and $(\partial^2 f/\partial t^2)_{n+l-1}$ vanish. From here we will obtain the system of algebraic equations

$$2A - 2B = -(a_1 + 1),$$
$$3A + 3B = -(a_1 - 1). \tag{29}$$

Since, like in (26), the first two terms on the right-hand side of (28) with equal coefficients have opposite signs, for condition (8) to be satisfied, the coefficient $a_1 - 1$ of f_{n+l-1} on the right-hand side of (25) should be equal to one. As before, the equality $a_1 = 2$ satisfies this condition. Then from the solution of system (29) we obtain $A = -11/12$, $B = 7/12$ and the following formula for the implicit difference scheme:

$$y_{n+l} = y_{n+l-1} + \frac{1}{2}\left[y_{n+l-1} - y_{n+l-2} + \frac{\tau}{12}(11 f_{n+l} + 8 f_{n+l-1} - 7 f_{n+l-2})\right] + O(\tau^4). \tag{30}$$

Let us construct another family of finite-difference schemes for $l = 3$. Let us represent the characteristic polynomial $\Phi_a(q)$ as a cubic quadronimal

$$\Phi_a(q) = (q-1)(a_1 q - 1)(a_2 q - 1)$$
$$= a_1 a_2 q^3 - (a_1 + a_2 + a_1 a_2) q^2 + (1 + a_1 + a_2) q - 1, \tag{31}$$

whose given roots are real different roots $q_0 = 1$ and $q_1 = 1/a_1 = 1/2$. According to the corollary of the fundamental theorem of algebra, the third root in (31) will also be real.

Let us represent equality (23) for $l = 3$ and additional Taylor series for function f in the following form:

$$a_1 a_2 y_{n+l} - (a_1 + a_2 + a_1 a_2) y_{n+l-1} + (1 + a_1 + a_2) y_{n+l-2} - y_{n+l-3}$$
$$= (a_1 a_2 - a_1 - a_2 + 1)\frac{\tau}{1!} f_{n+l-1} + (a_1 a_2 + a_1 + a_2 - 3)\frac{\tau^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{n+l-1}$$
$$+ (a_1 a_2 - a_1 - a_2 + 1)\frac{\tau^3}{3!}\left(\frac{\partial^2 f}{\partial t^2}\right)_{n+l-1} + (a_1 a_2 + a_1 + a_2 - 15)\frac{\tau^4}{4!}\left(\frac{\partial^3 f}{\partial t^3}\right)_{n+l-1}, \tag{32}$$

$$0 = -\tau f_{n+l-2} + \tau f_{n+l-1} - 2\frac{\tau^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{n+l-1} + 3\frac{\tau^3}{3!}\left(\frac{\partial^2 f}{\partial t^2}\right)_{n+l-1} - 4\frac{\tau^4}{4!}\left(\frac{\partial^3 f}{\partial t^3}\right)_{n+l-1} + \ldots, \tag{33}$$

$$0 = -\tau f_{n+l-3} + \tau f_{n+l-1} - 4\frac{\tau^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{n+l-1} + 12\frac{\tau^3}{3!}\left(\frac{\partial^2 f}{\partial t^2}\right)_{n+l-1} - 32\frac{\tau^4}{4!}\left(\frac{\partial^3 f}{\partial t^3}\right)_{n+l-1} + \ldots \tag{34}$$

Let us add (32) and Eqs. (33) and (34), multiplied beforehand by some coefficients A and B, respectively. Let us find the values of factors A and B so that the coefficients of $(\partial f/\partial t)_{n+l-1}$ and $(\partial^2 f/\partial t^2)_{n+l-1}$ vanish. As a result, we will obtain the system of equations

$$2A + 4B = -(a_1 a_2 + a_1 + a_2 - 3),$$

$$3A + 12B = -(a_1 a_2 - a_1 - a_2 + 7). \quad (35)$$

The first two terms on the right-hand sides of both (33) and (34) are equal as before and have opposite signs. Therefore, for condition (8) to be satisfied, the coefficient of f_{n+l-1} on the right-hand side of (32) should be equal to one, i.e.,

$$(a_1 a_2 - a_1 - a_2 + 1) = 1. \quad (36)$$

Equality (36) is true if and only if $a_1 = a_2 = 2$, as was to be shown. From the solution of system (35) we obtain $A = 88/12$, $B = -29/12$, and the following formula for the explicit difference scheme:

$$y_{n+l} = y_{n+l-1} + \frac{1}{4}\left[4y_{n+l-1} - 5y_{n+l-2} + y_{n+l-3} + \frac{\tau}{12}(71 f_{n+l-1} - 88 f_{n+l-2} + 29 f_{n+l-3})\right] + O(\tau^3). \quad (37)$$

The roots of the characteristic equation for scheme (37) are real and different and obey the dependences $q_0 = 1$, $0 < q_1 < 1$, and $0 < q_2 < 1$. Hence, according to Definition 5, the homogeneous scheme (37) is strongly stable.

To construct the implicit scheme, we will supplement system (32) with the Taylor series

$$0 = -\tau f_{n+l} + \tau f_{n+l-1} + 2\frac{\tau^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{n+l-1} + 3\frac{\tau^3}{3!}\left(\frac{\partial^2 f}{\partial t^2}\right)_{n+l-1} + 4\frac{\tau^4}{4!}\left(\frac{\partial^3 f}{\partial t^3}\right)_{n+l-1} + 5\frac{\tau^5}{5!}\left(\frac{\partial^4 f}{\partial t^4}\right)_{n+l-1} + \ldots \quad (38)$$

Let us multiply (38) by some factor A and add the obtained result to Eqs. (32) and (35) multiplied beforehand by B and C, respectively. We will find the values of factors A, B, and C so that the coefficients of $(\partial f/\partial t)_{n+l-1}$, $(\partial^2 f/\partial t^2)_{n+l-1}$, and $(\partial^3 f/\partial t^3)_{n+l-1}$ vanish. It will give the system of equations

$$2A - 2B - 4C = -(a_1 a_2 + a_1 + a_2 - 3),$$

$$3A + 3B + 12C = -(a_1 a_2 - a_1 - a_2 + 7), \quad (39)$$

$$4A - 4B - 32C = -(a_1 a_2 + a_1 + a_2 - 15).$$

The first two terms on the right-hand sides of series (32)–(34) have opposite signs; therefore, the algebraic sum of the coefficients with equal absolute values multiplied by the same number is equal to zero in each of these series. Hence, for condition (8) to be satisfied, the unique nonzero coefficient $(a_1 a_2 - a_1 - a_2 + 1)$ of f_{n+l-1} on the right-hand side of (31) should be equal to one, i.e., we again obtain (36) and the equality $a_1 = a_2 = 2$. The solution of system (39) yields $A = -41/24$, $B = 53/24$, and $C = -17/24$ and the following formula for the implicit difference scheme:

$$y_{n+l} = y_{n+l-1} + \frac{1}{4}[4y_{n+l-1} - 5y_{n+l-2} + y_{n+l-3} + \frac{\tau}{24}(41 f_{n+l} + 19 f_{n+l-1} - 53 f_{n+l-2} + 17 f_{n+l-3})] + O(\tau^5). \quad (40)$$

Thus, we have proved that the family of strongly stable linear l-step schemes (37) and (40) for $l = 3$ irrespective of the order of local accuracy p stipulates the characteristic polynomial $\Phi_a(q)$ with the same values a_1 and a_2 defined by equality (36). It remains to prove that these values a_1 and a_2 will be zeroes of the polynomial $\Phi_a(q)$ of strongly stable l-step scheme for another value of l.

To this end, we will construct a fourth-degree characteristic polynomial $\Phi_a(q)$

$$\Phi_a(q) = (q-1)(a_1 q - 1)(a_2 q - 1)(a_3 q - 1)$$

$$= a_1 a_2 a_3 q^4 - (a_1 a_2 + a_1 a_3 + a_2 a_3 + a_1 a_2 a_3) q^3$$

$$+ (a_1 + a_2 + a_3 + a_1 a_2 + a_1 a_3 + a_2 a_3) q^2 - (1 + a_1 + a_2 + a_3) q + 1, \tag{41}$$

whose given roots are real different roots $q_0 = 1$, $q_1 = 1/a_1 = 1/2$, and $q_2 = 1/a_2 = 1/2$. According to the corollary of the fundamental theorem of algebra, the fourth root of (41) is also real.

We select coefficient a_3 so that the characteristic polynomial (41) would correspond to the linear procedure (23) approximating the equation $\partial y/\partial t - f(t, y) = 0$. To this end, we will multiply both parts of the first row of system (22) by the coefficient of q^4 in the characteristic polynomial (41), both parts of the second row by the coefficient of q^2 in the same polynomial, both parts of the third row by the coefficient of q in (41), and add these three series to the fourth row of system (22). As a result, we obtain

$$a_1 a_2 a_3 y_{n+l} - (a_1 a_2 + a_1 a_3 + a_2 a_3 + a_1 a_2 a_3) y_{n+l-1}$$

$$+ (a_1 a_2 + a_1 a_3 + a_2 a_3 + a_1 + a_2 + a_3) y_{n+l-2}$$

$$+ (a_1 + a_2 + a_3 + 1) y_{n+l-3} - y_{n+l-4}$$

$$= (a_1 a_2 a_3 - a_1 a_2 - a_1 a_3 - a_2 a_3 + a_1 + a_2 + a_3) \frac{\tau}{1!} f_{n+l-1}$$

$$+ [a_1 a_2 a_3 + a_1 a_2 + a_1 a_3 + a_2 a_3 - 3(a_1 + a_2 + a_3) + 5] \frac{\tau^2}{2!} \left(\frac{\partial f}{\partial t} \right)_{n+l-1}$$

$$+ [a_1 a_2 a_3 - a_1 a_2 - a_1 a_3 - a_2 a_3 + 7(a_1 + a_2 + a_3) - 19] \frac{\tau^3}{3!} \left(\frac{\partial^2 f}{\partial t^2} \right)_{n+l-1}$$

$$+ [a_1 a_2 a_3 + a_1 a_2 + a_1 a_3 + a_2 a_3 - 15(a_1 + a_2 + a_3) + 65] \frac{\tau^4}{4!} \left(\frac{\partial^3 f}{\partial t^3} \right)_{n+l-1} + \ldots \tag{42}$$

To satisfy condition (8), it is necessary to equate the coefficient of τf_{n+l-1} on the right-hand side of (42) to one, i.e., to suppose $a_1 a_2 a_3 - a_1 a_2 - a_1 a_3 - a_2 a_3 + a_1 + a_2 + a_3 - 1 = 1$. This equality is true if and only if

$$a_1 = a_2 = a_3 = 2, \tag{43}$$

which is the proof of the considered theorem, and the homogeneous scheme (42) is strongly stable.

Let us increase the order of scheme (42) by eliminating the derivatives $(\partial f/\partial t)_{n+l-1}$, $(\partial^2 f/\partial t^2)_{n+l-1}$, and $(\partial^3 f/\partial t^3)_{n+l-1}$, which appear in its right-hand side, by means of the series

$$0 = -\tau f_{n+l} + \tau f_{n+l-1} + 2\frac{\tau^2}{2!} \left(\frac{\partial f}{\partial t} \right)_{n+l-1} + 3\frac{\tau^3}{3!} \left(\frac{\partial^2 f}{\partial t^2} \right)_{n+l-1}$$

$$+ 4\frac{\tau^4}{4!} \left(\frac{\partial^3 f}{\partial t^3} \right)_{n+l-1} + 5\frac{\tau^5}{5!} \left(\frac{\partial^4 f}{\partial t^4} \right)_{n+l-1} + \ldots,$$

$$0=-\tau f_{n+l-2}+\tau f_{n+l-1}-2\frac{\tau^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{n+l-1}+3\frac{\tau^3}{3!}\left(\frac{\partial^2 f}{\partial t^2}\right)_{n+l-1}$$

$$-4\frac{\tau^4}{4!}\left(\frac{\partial^3 f}{\partial t^3}\right)_{n+l-1}+5\frac{\tau^4}{5!}\left(\frac{\partial^4 f}{\partial t^4}\right)_{n+l-1}+\ldots,$$

$$0=-\tau f_{n+l-3}+\tau f_{n+l-1}-4\frac{\tau^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{n+l-1}+12\frac{\tau^3}{3!}\left(\frac{\partial^2 f}{\partial t^2}\right)_{n+l-1}$$

$$-32\frac{\tau^4}{4!}\left(\frac{\partial^3 f}{\partial t^3}\right)_{n+l-1}+80\frac{\tau^5}{5!}\left(\frac{\partial^4 f}{\partial t^4}\right)_{n+l-1}+\ldots,$$

$$0=-\tau f_{n+l-4}+\tau f_{n+l-1}-6\frac{\tau^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{n+l-1}+27\frac{\tau^3}{3!}\left(\frac{\partial^2 f}{\partial t^2}\right)_{n+l-1}$$

$$-108\frac{\tau^4}{4!}\left(\frac{\partial^3 f}{\partial t^3}\right)_{n+l-1}+405\frac{\tau^5}{5!}\left(\frac{\partial^4 f}{\partial t^4}\right)_{n+l-1}+\ldots$$

As a result, we will obtain the explicit scheme in the form

$$y_{n+l}=y_{n+l-1}+\frac{1}{8}[12y_{n+l-1}-18y_{n+l-2}+7y_{n+l-3}-y_{n+l-3}$$

$$+\frac{\tau}{24}(325f_{n+l-1}-617f_{n+l-2}+415f_{n+l-3}-99f_{n+l-4})]+O(\tau^5) \qquad (44)$$

and the implicit scheme in the form

$$y_{n+l}=y_{n+l-1}+\frac{1}{8}[12y_{n+l-1}-18y_{n+l-2}+7y_{n+l-3}-y_{n+l-3}+\frac{\tau}{720}(2321f_{n+l}$$

$$+466f_{n+l-1}-4584f_{n+l-2}+3166f_{n+l-3}-649f_{n+l-4})]+O(\tau^6). \qquad (45)$$

Following the same line of reasoning, increasing the value of l in the normalized scheme (23), we can show that the characteristic polynomial with the real coefficients $\Phi_a(q)$ has the form (18), as was to be shown.

FAMILY OF STRONGLY STABLE LINEAR l-STEP FINITE-DIFFERENCE SCHEMES ON NON-UNIFORM MESH ω_τ

Without loss of generality, assume that the solution of problem (5) can be implemented numerically at the initial rated step of the difference mesh ω_τ, i.e., for $n=0$, since technical details of many proofs appear much simpler in this case because the dependence of functions in (5) on n disappears and the dependence on l becomes trivial

$$\sum_{i=0}^{l}a_i y_i-\tau_l\sum_{i=0}^{l}b_i f_i=0. \qquad (46)$$

THEOREM 5. For each strongly stable homogeneous linear multistep single-stage finite-difference scheme (5) with real coefficients a_i and b_i $(i=\overline{0,l})$ satisfying the normalization condition (6) and (8), the characteristic polynomial $\Phi_a(q)$ has only real roots satisfying the strong root criterion, is unique, and is defined as

$$\Phi_a(q) = (q - q_0)\prod_{i=1}^{l}(s_i q - 1)^i, \tag{47}$$

where the sequence of real numbers s_i $(i = \overline{1, l})$ has the following properties:

— is mutual for linear $(l = k)$- and $(l = k + 1)$-step procedures (5) on the given mesh ω_τ for $i = \overline{0, k}$;

— the values of its components on the non-uniform mesh ω_τ only depend on the relation of mesh steps and are within the interval $1 < s_i \le 2$ $(i = \overline{1, l})$.

Proof. Let us replace Taylor series (22) of the expansion of function $y(t)$ on the uniform mesh $\overline{\omega}_\tau$ at points $l, l-2, \dots, 0$ based on its value at point $l-1$ with the Taylor series on the non-uniform mesh ω_τ

$$y_l = y_{l-1} + \frac{\tau_l}{1!} f_{l-1} + \frac{\tau_l^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{l-1} + \dots + \frac{\tau_l^p}{p!}\left(\frac{\partial^{p-1} f}{\partial t^{p-1}}\right)_{l-1},$$

$$y_{l-2} = y_{l-1} - \frac{\tau_{l-1}}{1!} f_{l-1} + \frac{\tau_{l-1}^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{l-1} + \dots + (-1)^p \frac{\tau_{l-1}^p}{p!}\left(\frac{\partial^{p-1} f}{\partial t^{p-1}}\right)_{l-1}, \tag{48}$$

$$y_{l-3} = y_{l-1} - 2\frac{\tau_{l-1} + \tau_{l-2}}{1!} f_{l-1} + \frac{(\tau_{l-1} + \tau_{l-2})^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{l-1}$$

$$+ \dots + (-1)^p \frac{(\tau_{l-1} + \tau_{l-2})^p}{p!}\left(\frac{\partial^{p-1} f}{\partial t^{p-1}}\right)_{l-1},$$

....

$$y_0 = y_{l-1} - \frac{\tau_{l-1} + \tau_{l-2} + \dots + \tau_0}{1!} f_{l-1} + \frac{(\tau_{l-1} + \tau_{l-2} + \dots + \tau_0)^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{l-1}$$

$$+ \dots + (-1)^p \frac{(\tau_{l-1} + \tau_{l-2} + \dots + \tau_0)^p}{p!}\left(\frac{\partial^{p-1} f}{\partial t^{p-1}}\right)_{l-1}.$$

Multiplying sequentially the left- and right-hand sides of the first equality in (48) by the coefficient a_l, defined by system (21), those of the second equality by the coefficient a_{l-2}, of the third equality by a_{l-3}, etc., and then adding the obtained expressions termwise, we arrive at the equality

$$\sum_{i=0}^{l} a_i y_i = \tau_l \sum_{i=0}^{p} \frac{c_i}{(i+1)!}\left(\frac{\partial^i f}{\partial t^i}\right)_{l-1}, \tag{49}$$

where

$$c_0 = a_l - \frac{\tau_{l-1}}{\tau_l} a_{l-2} + \frac{\tau_{l-1} + \tau_{l-2}}{\tau_l} a_{l-3} - \frac{\tau_{l-1} + \tau_{l-2} + \tau_{l-3}}{\tau_l} a_{l-4} + \dots,$$

$$c_1 = \tau_l\left[a_l + \left(\frac{\tau_{l-1}}{\tau_l}\right)^2 a_{l-2} + \left(\frac{\tau_{l-1} + \tau_{l-2}}{\tau_l}\right)^2 a_{l-3} + \left(\frac{\tau_{l-1} + \tau_{l-2} + \tau_{l-3}}{\tau_l}\right)^2 a_{l-4} + \dots\right],$$

$$c_2 = \tau_l^2\left[a_l - \left(\frac{\tau_{l-1}}{\tau_l}\right)^3 a_{l-2} - \left(\frac{\tau_{l-1} + \tau_{l-2}}{\tau_l}\right)^3 a_{l-3} + \left(\frac{\tau_{l-1} + \tau_{l-2} + \tau_{l-3}}{\tau_l}\right)^3 a_{l-4} + \dots\right],$$

$$c_3 = \tau_l^3 \left[a_l + \left(\frac{\tau_{l-1}}{\tau_l} \right)^4 a_{l-2} + \left(\frac{\tau_{l-1} + \tau_{l-2}}{\tau_l} \right)^4 a_{l-3} + \left(\frac{\tau_{l-1} + \tau_{l-2} + \tau_{l-3}}{\tau_l} \right)^4 a_{l-4} + \ldots \right],$$

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$$c_p = \tau_l^p \left[a_l + (-1)^{p+1} \left(\frac{\tau_{l-1}}{\tau_l} \right)^{p+1} a_{l-2} + (-1)^{p+1} \left(\frac{\tau_{l-1} + \tau_{l-2}}{\tau_l} \right)^{p+1} a_{l-3} \right.$$

$$\left. + (-1)^{p+1} \left(\frac{\tau_{l-1} + \tau_{l-2} + \tau_{l-3}}{\tau_l} \right)^{p+1} a_{l-4} + \ldots \right]. \tag{50}$$

Neglecting on the right-hand side of (49) all the terms containing derivatives of function $f(t)$, we obtain the scheme of order $p = 1$

$$\sum_{i=0}^{l} a_i y_i = \tau_l c_0 f_{l-1} + O(\bar{\tau}^2), \tag{51}$$

where $\bar{\tau} = \max_{i=\overline{0,l}} \tau_i$ is the maximum step of mesh ω_τ.

To satisfy condition (8), it is necessary to equate coefficient c_0 of f_{l-1} to one, i.e., to suppose

$$a_l - \frac{\tau_{l-1}}{\tau_l} a_{l-2} + \frac{\tau_{l-1} + \tau_{l-2}}{\tau_l} a_{l-3} - \frac{\tau_{l-1} + \tau_{l-2} + \tau_{l-3}}{\tau_l} a_{l-4} + \ldots + a_0 \left(\sum_{i=0}^{l} \tau_{l-i} \right) \Big/ \tau_l = 1. \tag{52}$$

It is possible to increase the order of local accuracy of the explicit l-step scheme (51) by supplementing (49) with the Taylor series with respect to the function f_{l-1}

$$0 = -f_l + f_{l-1} + \frac{\tau_l}{1!} \left(\frac{\partial f}{\partial t} \right)_{l-1} + \frac{\tau_l^2}{2!} \left(\frac{\partial^2 f}{\partial t^2} \right)_{l-1} + \ldots + \frac{\tau_l^p}{p!} \left(\frac{\partial^p f}{\partial t^p} \right)_{l-1},$$

$$0 = -f_{l-2} + f_{l-1} - \frac{\tau_{l-1}}{1!} \left(\frac{\partial f}{\partial t} \right)_{l-1} + \frac{\tau_{l-1}^2}{2!} \left(\frac{\partial^2 f}{\partial t^2} \right)_{l-1} + \ldots + (-1)^p \frac{\tau_{l-1}^p}{p!} \left(\frac{\partial^p f}{\partial t^p} \right)_{l-1},$$

$$0 = -f_{l-3} + f_{l-1} - \frac{\tau_{l-1} + \tau_{l-2}}{1!} \left(\frac{\partial f}{\partial t} \right)_{l-1}$$

$$+ \frac{(\tau_{l-1} + \tau_{l-2})^2}{2!} \left(\frac{\partial^2 f}{\partial t^2} \right)_{l-1} + \ldots + (-1)^p \frac{(\tau_{l-1} + \tau_{l-2})^p}{p!} \left(\frac{\partial^p f}{\partial t^p} \right)_{l-1}, \tag{53}$$

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$$0 = -f_0 + f_{l-1} - \frac{\sum_{i=1}^{l-1} \tau_{l-i}}{1!} \left(\frac{\partial f}{\partial t} \right)_{l-1} + \frac{\left(\sum_{i=1}^{l-1} \tau_{l-i} \right)^2}{2!} \left(\frac{\partial f}{\partial t} \right)_{l-1} + \ldots + (-1)^p \frac{\left(\sum_{i=1}^{l-1} \tau_{l-i} \right)^p}{p!} \left(\frac{\partial^p f}{\partial t^p} \right)_{l-1}$$

and eliminating from the system of equations (49) and (53) derivatives of function f of order $k = (\overline{1, p})$.

The first two terms on the right-hand sides of each series (53) have coefficients with opposite signs and absolute values equal to one. Therefore, during the elimination of derivatives of function f of kth order from the system of equations (49), (53), the algebraic sum of coefficients of function f in (53) multiplied by an arbitrary and the same factor will be equal to zero; hence, equality (52) will ensure condition (8).

Equality (52) contains $l+1$ unknowns a_i $(i=\overline{0,l})$. At first sight, the problem has no solution since there is only one equation to find $l+1$ unknowns. However, it is not so. We can prove the existence of a unique solution by the induction method. Let us begin the proof with the l-step scheme for $l=2$. As the initial expression for the characteristic polynomial $\Phi_a(q)$ (7), we will take the quadratic trinomial

$$\Phi_a(q)=(q-1)(a_1q-1)=a_1q^2-(1+a_1)q+1, \tag{54}$$

one of whose roots is the real root $q_0=1$. According to the corollary of the fundamental theorem of algebra, the second root of (54) is also real.

Let us specify coefficient a_1 so that the characteristic polynomial (54) would correspond to the linear procedure (46) approximating the equation $\partial y/\partial t-f(t,y)=0$. To this end, let us multiply both parts of the first row in (48) by a_1 and add to the second row in (48). As a result, we obtain

$$a_1y_l-(a_1+1)y_{l-1}+y_{l-2}=\left(a_1-\frac{\tau_{l-1}}{\tau_l}\right)\frac{\tau_l}{1!}f_{l-1}+\left[a_1+\left(\frac{\tau_{l-1}}{\tau_l}\right)^2\right]\frac{\tau_l^2}{2!}\left(\frac{\partial f}{\partial t}\right)_{l-1}$$

$$+\left[a_1-\left(\frac{\tau_{l-1}}{\tau_l}\right)^3\right]\frac{\tau_l^3}{3!}\left(\frac{\partial^2 f}{\partial t^2}\right)_{l-1}+\left[a_1+\left(\frac{\tau_{l-1}}{\tau_l}\right)^4\right]\frac{\tau_l^4}{4!}\left(\frac{\partial^3 f}{\partial t^3}\right)_{l-1}+\ldots \tag{55}$$

Neglecting on the right-hand side of (55) all the terms that contain derivatives of function $f(t)$, we obtain scheme of order $p=1$

$$a_1y_l-(1+a_1)y_{l-1}+y_{l-2}=\left(a_1-\frac{\tau_{l-1}}{\tau_l}\right)\tau_lf_{l-1}+O(\overline{\tau}^2). \tag{56}$$

To satisfy condition (8), it is necessary to equate the coefficient of τ_lf_{l-1} to one, i.e., to suppose $a_1-\tau_{l-1}/\tau_l=1$. Whence $a_1=1+\tau_{l-1}/\tau_l=1+\gamma$.

For $\tau_{l-1}>0$, $\tau_l>0$, and all γ we obtain $1<a_1\le 2$; hence, the characteristic equation of the family of three-point schemes (55) has two different real roots: $q_0=1$ and $0<q_1<1$. According to Definition 5, the family of homogeneous schemes (55) is strongly stable.

It is possible to increase the order of local accuracy of explicit l-step scheme for $l=2$ by supplementing Eq. (55) with the second Taylor series from (53) and eliminating the first derivative $(\partial f/\partial t)_{l-1}$ from these equations. Multiplying the left- and right-hand sides of the second Taylor series from (53) by

$$\frac{1}{2}\frac{\tau_l}{\tau_{l-1}}\left[a_1+\left(\frac{\tau_{l-1}}{\tau_l}\right)^2\right]\tau_l$$

and adding the obtained expression to (55) yield

$$a_1y_l-(1+a_1)y_{l-1}+y_{l-2}=\frac{\tau_l}{2}(f_{l-1}-f_{l-2})\left[a_1+\left(\frac{\tau_{l-1}}{\tau_l}\right)^2\right]$$

$$+\left(a_1-\frac{\tau_{l-1}}{\tau_l}\right)\tau_lf_{l-1}+O(\overline{\tau}^3). \tag{57}$$

Since in the first term of the right-hand side of the obtained equation functions f_{l-1} and f_{l-2} in round brackets have opposite signs, the algebraic sum of their factors

$$+\frac{\tau_l}{2}\left[a_1+\left(\frac{\tau_{l-1}}{\tau_l}\right)^2\right] \text{ and } -\frac{\tau_l}{2}\left[a_1+\left(\frac{\tau_{l-1}}{\tau_l}\right)^2\right]$$

is equal to zero. Hence, for condition (18) to hold, the coefficient $a_1-\tau_{l-1}/\tau_l$ of f_{l-1} in the second term on the right-hand side of (57) should be identically equal to one. Equality (55) satisfies this condition, which is stated in the theorem. Substituting the value of a_1 (55) into (57), we obtain the final formula for the explicit difference scheme of order $p=2$.

To construct the l-step implicit scheme for $l=2$ of the third order, we will supplement Eq. (55) with the first and second Taylor series for the function f from (53). Let us multiply the first series by the factor A, indefinite yet, the second series by indefinite factor B, and add the obtained expressions to (55). Let us find the values of factors A and B so that the sums of the coefficients for $(\partial f/\partial t)_{l-1}$ and $(\partial^2 f/\partial t^2)_{l-1}$ would vanish. From here we obtain the system of the algebraic equations

$$\tau_l A-\tau_{l-1}B+\frac{\tau_l^2}{2!}\left[a_1+\left(\frac{\tau_{l-1}}{\tau_l}\right)^2\right]=0,$$

$$\frac{\tau_l^2}{2}A+\frac{\tau_{l-1}^2}{2}B+\frac{\tau_l^3}{3!}\left[a_1-\left(\frac{\tau_{l-1}}{\tau_l}\right)^3\right]=0,$$

whose solution has the form

$$A=-\frac{1}{6}\frac{\tau_l^2}{\tau_l+\tau_{l-1}}\left[a_1\left(2+3\frac{\tau_{l-1}}{\tau_l}\right)+\left(\frac{\tau_{l-1}}{\tau_l}\right)^3\right],$$

$$B=\frac{1}{6}\frac{\tau_l^2}{\tau_l+\tau_{l-1}}\left\{a_1\left[3-2\left(\frac{\tau_{l-1}}{\tau_l}\right)^2\right]+5\left(\frac{\tau_{l-1}}{\tau_l}\right)^2\right\}. \tag{58}$$

Before showing the result of the addition of Eq. (55) and two first series in (53) multiplied beforehand by the obtained values of A and B, respectively, let us mention that the first two terms on the right-hand sides of these series with equal coefficients have opposite signs; therefore, the algebraic sum of the coefficients with equal absolute values multiplied by the same number is equal to zero in each of these series. Hence, to satisfy condition (8), the unique nonzero coefficient $a_1-\tau_{l-1}/\tau_l$ of f_{l-1} on the right-hand side of (55) should be identically equal to one, which agrees well with (55) and is stated in the theorem.

Combining the system of equations that consists of Eq. (55) and two first series in (53) with regard for the found values of unknowns A and B, we obtain the following formula for strongly stable implicit difference scheme of order $p=3$:

$$a_1 y_l-(1+a_1)y_{l-1}+y_{l-2}$$

$$=-Af_l+\left[\left(a_1-\frac{\tau_{l-1}}{\tau_l}\right)\tau_l+A+B\right]f_{l-1}-Bf_{l-2}+O(\bar{\tau}^4). \tag{59}$$

Thus, we have proved that the family of strongly stable linear l-step schemes (55) for $l=2$, irrespectively of the order of local accuracy p, ensures the characteristic polynomial (54) with the same value a_1 defined by Eq. (55). It remains to prove that the roots of the polynomial $\Phi_a(q)$ (47) of the strongly stable homogeneous l-step scheme remain real for another value of l as well.

To this end, let us consider the family of finite-difference schemes for $l=3$. We will construct the characteristic polynomial (47) with preset values of real and different roots $q_0=1$ and $q_1=1/a_1=1+\tau_{l-1}/\tau_l$

$$\Phi_a(q)=(q-1)(a_1q-1)(a_2q-1)$$

$$=a_1a_2q^3-(a_1+a_2+a_1a_2)q^2+(1+a_1+a_2)q-1=\beta_1q^3-\beta_2q^2+\beta_3q-1, \tag{60}$$

where the notation is introduced: $\beta_1=a_1a_2$, $\beta_2=a_1+a_2+a_1a_2$, and $\beta_3=1+a_1+a_2$.

According to the corollary of the fundamental theorem of algebra, the third root of (60) will also be real. We will select the coefficient a_2 so that the characteristic polynomial (60) would corresponded to the linear procedure (15) approximating the equation $\partial y/\partial t-f(t,y)=0$. To this end, we multiply both parts of the first series of system (48) by the coefficient of q^3 in the characteristic polynomial (60), both parts of the second series by the coefficient of q in the same polynomial, and add these two series to the third series of system (48). As a result, we obtain

$$\beta_1y_l-\beta_2y_{l-1}+\beta_3y_{l-2}-y_{l-3}=(\beta_1-\beta_3s_1+s_{1,2})\tau_lf_{l-1}$$
$$+\frac{1}{2!}\tau_l(\beta_1+\beta_3s_1^2-s_{1,2}^2)\tau_l\left(\frac{\partial f}{\partial t}\right)_{l-1}$$
$$+\frac{1}{3!}\tau_l^2(\beta_1-\beta_3s_1^3+s_{1,2}^3)\tau_l\left(\frac{\partial^2 f}{\partial t^2}\right)_{l-1}+\frac{1}{4!}\tau_l^3(\beta_1+\beta_3s_1^4-s_{1,2}^4)\tau_l\left(\frac{\partial^3 f}{\partial t^3}\right)_{l-1}$$
$$+\frac{1}{5!}\tau_l^4(\beta_1-\beta_3s_1^5+s_{1,2}^5)\tau_l\left(\frac{\partial^4 f}{\partial t^4}\right)_{l-1}+O(\bar{\tau}^6), \tag{61}$$

where

$$s_1=\frac{\tau_{l-1}}{\tau_l},\ s_{1,2}=\frac{\tau_{l-1}+\tau_{l-2}}{\tau_l}.$$

From this set of schemes we will first consider the explicit schemes. Neglecting on the right-hand side of (61) all the terms that contain derivatives of function $f(t)$, we obtain the scheme of order $p=1$

$$\beta_1y_l-\beta_2y_{l-1}+\beta_3y_{l-2}-y_{l-3}=(\beta_1-\beta_3s_1+s_{1,2})\tau_lf_{l-1}+O(\bar{\tau}^2). \tag{62}$$

To satisfy condition (8), it is necessary to equate the coefficient of τ_lf_{l-1} to one, i.e., to put

$$a_1a_2-(a_1+a_2+1)\frac{\tau_{l-1}}{\tau_l}+\frac{\tau_{l-2}+\tau_{l-1}}{\tau_l}=1. \tag{63}$$

For $a_1=1+\tau_{l-1}/\tau_l$, from (63) it follows that

$$a_2=1+a_1\frac{\tau_{l-1}}{\tau_l}-\frac{\tau_{l-2}}{\tau_l}=1+\frac{\tau_{l-1}}{\tau_l}\left(1+\frac{\tau_{l-1}}{\tau_l}\right)-\frac{\tau_{l-2}}{\tau_l}. \tag{64}$$

Using the notation for steps ratio, we obtain herefrom

$$a_2=1+\frac{\tau_{l-1}}{\tau_l}\left(1+\frac{\tau_{l-1}}{\tau_l}\right)-\frac{\tau_{l-2}\tau_{l-1}}{\tau_{l-1}\tau_l}=1+\gamma(1+\gamma)-\gamma^2=1+\gamma,$$

i.e., for all γ we obtain $1<a_2\le 2$. Thus, the roots of the characteristic equation for the family of schemes (61) are real and different and obey the dependences $q_0=1$, $0<q_1<1$, and $0<q_2<1$. Hence, according to Definition 5, the family of four-point homogeneous schemes (61) is strongly stable.

It is possible to increase the order of local accuracy of the explicit l-step scheme for $l=3$ by supplementing (61) by the second Taylor series for the function f from (53). Eliminating the first derivative $(\partial f/\partial t)_{l-1}$ in these equations, multiplying the left- and right-hand sides of the second series in (53) by the factor

$$\frac{1}{2}\frac{\tau_l}{\tau_{l-1}}\left[a_{l-2}a_{l-1}+(a_{l-2}+a_{l-1}+1)\left(\frac{\tau_{l-1}}{\tau_l}\right)^2-\left(\frac{\tau_{l-2}+\tau_{l-1}}{\tau_l}\right)^2\right],$$

and adding the obtained expression to (61) yields the following formula for the explicit difference scheme of order $p=2$:

$$\beta_1 y_l-\beta_2 y_{l-1}+\beta_3 y_{l-2}-y_{l-3}=(\beta_1-\beta_3 s_1+s_{1,2})\tau_l f_{l-1}$$

$$+\frac{\tau_l}{2}\left\{\left[\left(2+\frac{\tau_l}{\tau_{l-1}}\right)\beta_1-\beta_3 s_1+\left(1-\frac{\tau_{l-2}}{\tau_{l-1}}\right)s_{1,2}\right]f_{l-1}-\frac{\tau_l}{\tau_{l-1}}(\beta_1+\beta_3 s_1^2-s_{1,2}^2)f_{l-2}\right\}+O(\bar{\tau}^3). \quad (65)$$

Let us construct explicit strongly stable homogeneous *l*-step scheme for $l=3$ of the third local order of accuracy by supplementing Eq. (61) with the second and third Taylor series for function f from (53). Multiply the second series from (53) by the yet indefinite factor A, the third series by the indefinite factor B, and add the obtained expressions to Eq. (61). Determine the values of factors A and B so that the sum of the coefficients of $(\partial f/\partial t)_{l-1}$ and $(\partial^2 f/\partial t^2)_{l-1}$ would become zero. From here we obtain the system of algebraic equations

$$-s_1 A-s_{1,2}B+\frac{1}{2}(\beta_1+s_1^2\beta_3-s_{1,2}^2)=0,$$

$$s_1^2 A+s_{1,2}^2 B+\frac{1}{3}(\beta_1-s_1^3\beta_3+s_{1,2}^3)=0,$$

whose solution has the form

$$A=\frac{1}{6}\frac{\tau_l^2}{\tau_{l-1}\tau_{l-2}}\left[(2+3s_{1,2})\beta_1+s_1^2\left(s_{1,2}+2\frac{\tau_{l-2}}{\tau_l}\right)\beta_3-s_{1,2}^3\right],$$

$$B=-\frac{1}{6}\frac{\tau_l^3}{\tau_{l-2}(\tau_{l-1}+\tau_{l-2})}\left[(2+3s_{1,2})\beta_1+s_1^3\beta_3+\frac{2\tau_{l-2}-\tau_{l-1}}{\tau_l}s_{1,2}^2\right]. \quad (66)$$

The system of equations that consists of Eq. (61), the second and third series from (53) with regard for the found values of the unknowns A and B, and Eqs. (60) and (47) allows us to obtain the following formula for the homogeneous strongly stable explicit difference scheme of order $p=3$:

$$\beta_1 y_l-\beta_2 y_{l-1}+\beta_3 y_{l-2}-y_{l-3}$$

$$=[A+B+\tau_l(\beta_1-\beta_3 s_1+s_{1,2})]f_{l-1}-Af_{l-2}-Bf_{l-3}+O(\bar{\tau}^4). \quad (67)$$

To construct the *l*-step implicit scheme for $l=3$ of fourth order, let us supplement Eq. (61) with the first, second, and third Taylor series for the function f from (53). Let us multiply the first series from (53) by the yet indefinite factor A, the second series by the indefinite factor B, the third series by the indefinite factor C and add the obtained expressions to Eq. (61). Let us determine the values of factors A, B, and C so that the sums of the coefficients of $(\partial f/\partial t)_{l-1}$, $(\partial^2 f/\partial t^2)_{l-1}$, and $(\partial^3 f/\partial t^3)_{l-1}$ would become zero. From here we obtain the system of algebraic equations

$$A-s_1 B-s_{1,2}C+\frac{\tau_l}{2}(\beta_1+s_1^2\beta_3-s_{1,2}^2)=0,$$

$$A+s_1^2 B+s_{1,2}^2 C+\frac{\tau_l}{3}(\beta_1-s_1^3\beta_3+s_{1,2}^3)=0,$$

$$A-s_1^3 B-s_{1,2}^3 C+\frac{\tau_l}{4}(\beta_1+s_1^3\beta_3-s_{1,2}^3)=0,$$

whose solution has the form

$$A = -\frac{\tau_l^3}{12(\tau_l + \tau_{l-1})(\tau_l + \tau_{l-1} + \tau_{l-2})}\left\{ [1 + s_1 + (2 + 3s_1)(1 + 2s_{1,2})]\beta_1 \right.$$

$$\left. + s_1^3\left(s_{1,2} + \frac{\tau_{l-2}}{\tau_l} \right)\beta_1 - s_{1,2}^3 \frac{\tau_{l-1} + \tau_{l-2}}{\tau_l} \right\},$$

$$B = \frac{\tau_l^3}{12\tau_{l-1}\tau_{l-2}(\tau_l + \tau_{l-1})}\left\{ \left(1 + 2\frac{\tau_{l-1} - \tau_{l-2}}{\tau_l} \right)\beta_1 \right.$$

$$\left. + s_1^3\left[2 + s_1 + 2\frac{\tau_{l-2}}{\tau_{l-1}}(3 + 2s_1) \right]\beta_3 - s_{1,2}^3(2 + s_{1,2}) \right\}, \tag{68}$$

$$C = -\frac{\tau_l^4}{12\tau_{l-2}(\tau_{l-1} + \tau_{l-2})(\tau_l + \tau_{l-1} + \tau_{l-2})}\left[(1 + 2s_1)\beta_1 + s_1^3\left(2 + \frac{\tau_{l-1}}{\tau_l} \right)\beta_3 \right.$$

$$\left. + s_{1,2}^3\left(\frac{3\tau_{l-2} - \tau_{l-1}}{\tau_l} + 2\frac{2\tau_{l-2} - \tau_{l-1}}{\tau_{l-1} + \tau_{l-2}} \right) \right].$$

The system of equations that consists of Eq. (61), the first, second, and third series from (53) with regard for the found values of the unknowns *A*, *B*, and *C*, and Eqs. (60) and (47), allows us to obtain the following formula for the homogeneous strongly stable implicit difference scheme of order $p = 4$:

$$\beta_1 y_l - \beta_2 y_{l-1} + \beta_3 y_{l-2} - y_{l-3}$$

$$= -Af_l + (A + B + C + \beta_1 - \beta_3 s_1 + s_{1,2})\tau_l f_{l-1} - Bf_{l-2} - Cf_{l-3} + O(\bar{\tau}^5). \tag{69}$$

This process of the construction of strongly stable linear *l*-step schemes within the framework of the proof of Theorem 5 has shown that inside each family of homogeneous schemes, i.e., for a specific value of *l*, coefficients of the characteristic polynomial Φ_a are only the functions of steps of the difference mesh and remain real irrespective of the local order of the scheme.

To complete the proof of Theorem 5, it is necessary to show that the roots of the characteristic polynomial Φ_a remain real for the multistep scheme for $l = 4$. To this end, let us construct characteristic polynomial Φ_a with the preset values of real roots $q_0 = 1$, $q_1 = 1/a_1$, and $q_2 = 1/a_2$:

$$\Phi_a(q) = (q-1)(a_1 q - 1)(a_2 q - 1)(a_3 q - 1)$$

$$= a_1 a_2 a_3 q^4 - (a_1 a_2 + a_1 a_3 + a_2 a_3 + a_1 a_2 a_3)q^3 + (a_1 + a_2 + a_3 + a_1 a_2 + a_1 a_3 + a_2 a_3)q^2$$

$$+ (a_1 + a_2 + a_3 + 1)q + 1 = \beta_1 q^4 - \beta_2 q^3 + \beta_3 q^2 - \beta_4 q + 1, \tag{70}$$

where $a_1 = 1 + \tau_{l-1}/\tau_l$, $a_2 = 1 + a_1\tau_{l-1}/\tau_l - \tau_{l-2}/\tau_l$ and the notation is introduced $\beta_1 = a_1 a_2 a_3$, $\beta_2 = a_1 a_2 + a_1 a_3$ $+ a_2 a_3 + a_1 a_2 a_3$, $\beta_3 = a_1 + a_2 + a_3 + a_1 a_2 + a_1 a_3 + a_2 a_3$, and $\beta_4 = 1 + a_1 + a_2 + a_3$.

We will select the coefficient a_3 so that the characteristic polynomial (70) would correspond to the linear procedure (15) approximating the equation $\partial y/\partial t - f(t, y) = 0$. To this end, let us multiply both parts of the first series of system (48) by the coefficient of q^4 in the characteristic polynomial (70), both parts of the second series by the coefficient of q^2 in the same polynomial, both parts of the third series by the coefficient of q in (70), and add these three series to the fourth series of system (48). As a result we obtain

$$\beta_1 y_l - \beta_2 y_{l-1} + \beta_3 y_{l-2} - \beta_4 y_{l-3} + y_{l-4} = (\beta_1 - s_1\beta_3 + s_{1,2}\beta_4 - s_{1,3})\tau_l f_{l-1}$$

$$+\frac{1}{2!}\tau_l(\beta_1 + s_1^2\beta_3 - s_{1,2}^2\beta_4 + s_{1,2}^2)\tau_l\left(\frac{\partial f}{\partial t}\right)_{l-1} + \frac{1}{3!}\tau_l^2(\beta_1 - s_1^3\beta_3 + s_{1,2}^3\beta_4 - s_{1,2}^3)\tau_l\left(\frac{\partial^2 f}{\partial t^2}\right)_{l-1}$$

$$+\frac{1}{4!}\tau_l^3(\beta_1 + s_1^4\beta_3 - s_{1,2}^4\beta_4 + s_{1,2}^4)\tau_l\left(\frac{\partial^3 f}{\partial t^3}\right)_{l-1}$$

$$+\frac{1}{5!}\tau_l^4(\beta_1 - s_1^5\beta_3 + s_{1,2}^5\beta_4 - s_{1,2}^5)\tau_l\left(\frac{\partial^4 f}{\partial t^4}\right)_{l-1} + O(\bar{\tau}^6), \tag{71}$$

where

$$s_{1,3} = \frac{\tau_{l-1} + \tau_{l-2} + \tau_{l-3}}{\tau_l}.$$

To satisfy condition (8), it is necessary to equate the coefficient of $\tau_l f_{l-1}$ to one, i.e., to put

$$\beta_1 - s_1\beta_3 + s_{1,2}\beta_4 - s_{1,3} = a_1a_2a_3 - (a_1 + a_2 + a_3 + a_1a_2 + a_1a_3 + a_2a_3)\frac{\tau_{l-1}}{\tau_l}$$

$$+ (a_1 + a_2 + a_3 + 1)\frac{\tau_{l-2} + \tau_{l-1}}{\tau_l} - \frac{\tau_{l-3} + \tau_{l-2} + \tau_{l-1}}{\tau_l} = 1. \tag{72}$$

From (72) it follows that

$$a_3 = \frac{1 + \dfrac{\tau_{l-3}}{\tau_l} + \left[a_1a_2 - (a_1 + a_2)\dfrac{\tau_{l-2}}{\tau_{l-1}}\right]\dfrac{\tau_{l-1}}{\tau_l}}{a_1a_2 - (a_1 + a_2)\dfrac{\tau_{l-2}}{\tau_{l-1}} + \dfrac{\tau_{l-2}}{\tau_l}}. \tag{73}$$

From here, using the notation for steps ratio

$$a_1 = 1 + \frac{\tau_{l-1}}{\tau_l} = 1 + \gamma,$$

$$a_2 = 1 + \frac{\tau_{l-1}}{\tau_l}\left(1 + \frac{\tau_{l-1}}{\tau_l}\right) - \frac{\tau_{l-2}}{\tau_l} = 1 + \gamma,$$

we obtain

$$a_3 = \frac{1 + \dfrac{\tau_{l-3}}{\tau_{l-2}}\dfrac{\tau_{l-2}}{\tau_{l-1}}\dfrac{\tau_{l-1}}{\tau_l} + \left[a_1a_2 - (a_1 + a_2)\dfrac{\tau_{l-2}}{\tau_{l-1}}\right]\dfrac{\tau_{l-1}}{\tau_l}}{a_1a_2 - (a_1 + a_2)\dfrac{\tau_{l-2}}{\tau_{l-1}} + \dfrac{\tau_{l-2}}{\tau_{l-1}}\dfrac{\tau_{l-1}}{\tau_l}}$$

$$= \frac{1 + \gamma^3 + [(1+\gamma)^2 - 2(1+\gamma)\gamma]\gamma}{(1+\gamma)^2 - 2(1+\gamma)\gamma - \gamma^2} = 1 + \gamma,$$

i.e., for all γ we obtain $1 < a_3 \le 2$. Thus, the roots of the characteristic equation for the family of schemes (71) are real and different and obey the dependences $q_0 = 1$, $0 < q_1 < 1$, $0 < q_2 < 1$, and $0 < q_3 < 1$. Hence, according to Definition 5, the family of homogeneous four-point schemes (61) is strongly stable.

In [3], in the problem of modeling the circulation of atmosphere to obtain the difference representation Λ of derivatives of the first and second orders appearing in the differential operator D of hydrodynamic and heat and mass transfer equations, approximation was carried out by difference operators of fourth order. Therefore, obviously, it is necessary to construct, for ordinary differential equation $\partial y / \partial t - f(t, y) = 0$, a strongly stable explicit difference scheme of order $p = 4$, corresponding to the order of approximation of the differential operator D in the equations of the problem of circulation of atmosphere. Of course, it will be useful to write the scheme for an arbitrary nth point of the difference mesh.

Let us supplement Eq. (71) with the second, third, and fourth Taylor series for function f from (53). Multiply the second series from (53) by the yet indefinite factor A, the third series by the indefinite factor B, the fourth row by the indefinite factor C, and add the obtained expressions to Eq. (71). Let us find the values of factors A, B, and C so that the sum of the coefficients for $(\partial f / \partial t)_{n+l-1}$, $(\partial^2 f / \partial t^2)_{n+l-1}$, and $(\partial^3 f / \partial t^3)_{n+l-1}$ would become zero. From here we obtain the system of algebraic equations

$$s_1 A + s_{1,2} B + s_{1,3} C = \frac{\tau_{n+l}}{2} \vartheta_1,$$

$$s_1^2 A + s_{1,2}^2 B + s_{1,3}^2 C = -\frac{\tau_{n+l}}{3} \vartheta_2, \tag{74}$$

$$s_1^3 A + s_{1,2}^3 B + s_{1,3}^3 C = \frac{\tau_{n+l}}{4} \vartheta_3,$$

where the following notation is introduced:

$$s_1 = \frac{\tau_{n+l-1}}{\tau_{n+l}}, \; s_{1,2} = \frac{\tau_{n+l-1} + \tau_{n+l-2}}{\tau_{n+l}}, \; s_{1,3} = \frac{\tau_{n+l-1} + \tau_{n+l-2} + \tau_{n+l-3}}{\tau_{n+l}},$$

$$\vartheta_1 = \beta_1 + s_1^2 \beta_3 - s_{1,2}^2 \beta_4 + s_{1,3}^2,$$

$$\vartheta_2 = \beta_1 - s_1^3 \beta_3 + s_{1,2}^3 \beta_4 - s_{1,3}^3,$$

$$\vartheta_3 = \beta_1 + s_1^4 \beta_3 - s_{1,2}^4 \beta_4 + s_{1,3}^4.$$

If in addition we introduce the notation

$$s_2 = \frac{\tau_{n+l-2}}{\tau_{n+l}}, \; s_3 = \frac{\tau_{n+l-3}}{\tau_{n+l}}, \; s_{2,3} = \frac{\tau_{n+l-2} + \tau_{n+l-3}}{\tau_{n+l}},$$

then the solution of the system of equations (74) becomes

$$A = \frac{\tau_{n+l}}{12 s_1 s_2 s_{2,3}} [6 s_{1,2} s_{1,3} \vartheta_1 + 4(s_{1,2} + s_{1,3}) \vartheta_2 + 3\vartheta_3],$$

$$B = -\frac{\tau_{n+l}}{12 s_2 s_3 s_{1,2}} [6 s_1 s_{1,3} \vartheta_1 + 4(s_1 + s_{1,3}) \vartheta_2 + 3\vartheta_3],$$

$$C = \frac{\tau_{n+l}}{12 s_3 s_{2,3} s_{1,3}} [6 s_1 s_{1,2} \vartheta_1 + 4(s_1 + s_{1,2}) \vartheta_2 + 3\vartheta_3].$$

The system of equations consisting of Eq. (71), the second, third, and fourth series from (53) with regard for the found values of unknowns A, B, and C, and equalities (70) and (73), allows us to obtain the following formula for strongly stable explicit difference scheme of order $p = 4$:

$$\beta_1 y_{n+l} - \beta_2 y_{n+l-1} + \beta_3 y_{n+l-2} - \beta_4 y_{n+l-3} + y_{n+l-4}$$

$$= [A + B + C + \tau_{n+l}(\beta_1 - s_1 \beta_3 + s_{1,2} \beta_4 - s_{1,3})] f_{n+l-1}$$

$$- A f_{n+l-2} - B f_{n+l-3} - C f_{n+l-4} + O(\bar{\tau}^5).$$

Thus, the proved Theorem 5 allows constructing the family of single-stage l-step explicit and implicit numerical methods for the solution of problem (1), (2). However, the question arises: should we specify the value of l arbitrarily large if the right-hand side of the differential equation (1), (2) is continuous and bounded together with the derivatives of lth order? Naturally, an increase in l increases the cost of the solution, especially as applied to the numerical solution of systems of partial differential equations. Hence, the choice of one value of l or another depends on the specific problem being solved. It is hardly expedient to use large values of l when methods with bounded values of l are applicable and substantial improvement of the results is not expected.

CONCLUSIONS

The implicit schemes (30), (40), and (45) we have constructed, including additional approximate value of $f(t_{n+l}, y_{n+l})$, do not provide explicit expressions to find y_{n+l}, but are only the equations with respect to this unknown. In this case, one can use the two-stage algorithm: first, find y_{n+l} by the explicit scheme whose order of accuracy is $O(\tau^{l-1})$, then, having calculated $f(t_{n+l}, y_{n+l})$, find y_{n+l} by the implicit scheme of order $O(\tau^{l})$. The value of y_{n+l} can also be found by the implicit scheme, using any method of successive approximations. Iterations substantially complicate the process of evaluation of y_{n+l} and are an essential shortcoming of implicit methods. It is especially true for the cases where $f(t_{n+l}, y_{n+l})$ is the value of difference expressions for convective and diffusion terms of nonlinear differential equations of hydrodynamics [3] since the nonlinearity of the equations does not allow ensuring a priori the convergence of the respective iterative process.

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