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UNCERTAIN LINEAR SYSTEMS OF EQUATIONS: STRONG SOLVABILITY AND STRONG FEASIBILITY

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Abstract. *The authors consider strong solvability and strong feasibility of uncertain linear systems of equations in five grades (exact, quasi-exact, semi-exact, quasi-fuzzy, and fuzzy).*

Keywords: *fuzzy sets, interval matrix, fuzzy matrix interval systems of equations, uncertain systems of equations, strong solvability of systems of linear equations, strong feasibility of systems of linear equations.*

INTRODUCTION

Handling uncertainties that take place in modeling real phenomena, processes, and complex systems is an important and difficult problem. The apparatus of fuzzy sets (numbers) [1–15] is used to solve systems of equations with fuzzy data [16]. The result of the solution of such system is often required in the form of ordinary numbers. The study [17] makes the first step in the investigation of these systems by means of interval ones, namely, introduces the concepts of an uncertain linear system of equations as a set of five special interval systems of equations, of weak and strong solvability (feasibility) of an uncertain linear system of equations in five senses (fuzzy, quasi-fuzzy, semi-fuzzy, quasi-exact, and exact), and also proves the criteria of weak solvability and weak feasibility of uncertain system of equations in all the five senses. The study [18] continues these investigations for uncertain linear inequalities, where strong solvability and feasibility are analyzed.

The purpose of the present study is to analyze strong solvability and strong feasibility of uncertain linear systems of equations in the above five senses.

NECESSARY DEFINITIONS

Definition 1. By a fuzzy number A we will understand the set of pairs $A=\{a\mid\mu(a);\ a\in[a_L,a_R]\subset R^1,\ \mu(a)\in[0,1]\}$.

Definition 2. Set of numbers $a\in R^1$ in the set of pairs of fuzzy number A that form this fuzzy number is called the carrier of fuzzy number A and $\mu(a)$ is the value of the membership function of the fuzzy number A.

Definition 3. A fuzzy number $A=\{a_1\mid\mu(a_1),\dots,a_n\mid\mu(a_n)\}$ is called discrete (or a fuzzy number with discrete carrier) if $\{a_i\}_{i=1}^n$ is a discrete set, $\mu(a_i)\in[0,1]$; $\mu(a_i)>0\ \forall i=2,3,\dots,n-1$, $\mu(a_1)=\mu(a_n)=0$. Denote $a_1=a_L$ and $a_n=a_R$ for $a_1<\dots<a_n$.

Definition 4. A fuzzy number $A=\{a\mid\mu(a)\ \forall a\in[a_L,a_R]\subset R^1\}$ is called continual (or a fuzzy number with continual carrier) if the value of $\mu(a)$ is specified for all $a\in[a_L,a_R]$: $\mu(a_L)=0; 1\ge\mu(a)>0\ \forall a\in(a_L,a_R)$; $\mu(a_R)=0$.

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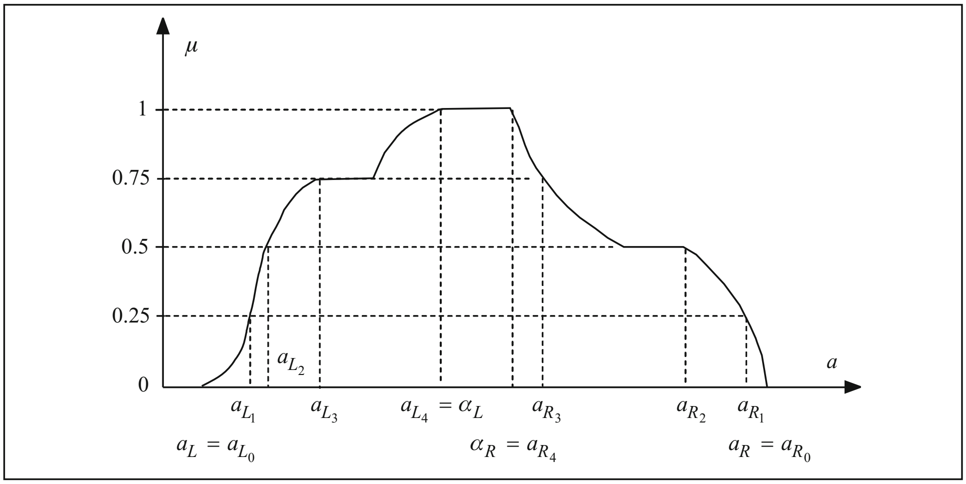

Fig. 1. Obtaining a standardized fuzzy number from a standard continual fuzzy number.

Definition 5. Points a of the carrier of fuzzy number A at which $\mu(a)=1$ are called peak points for the fuzzy number A.

Definition 6. A discrete fuzzy number $A=\{a_1|\mu_1,\ldots,\ a_n|\mu_n\}$, where $a_1<a_2<\ldots<a_n$, is called single-peak if there exists a unique set of successive numbers $\alpha_L=a_{i+1}, a_{i+2},\ldots, a_{i+p}=\alpha_R$ for which $\mu_{i+1}=\ldots=\mu_{i+p}=1$. Points $a_{i+1},\ldots, a_{i+p}$ are called peak of the discrete fuzzy number A. If $p=1$, then such peak of number A is called acute; otherwise, it is not acute.

Definition 7. Continual fuzzy number $A=\{a\,|\,\mu(a);\, a\in[a_L, a_R]\}$ is called single-peak if there exists a unique interval $[\alpha_L,\alpha_R]\subset(a_L,a_R)$ such that for all numbers a from this interval the membership function $\mu(a)=1$. Interval $[\alpha_L,\alpha_R]$ is called the peak of continual fuzzy number A. If $\alpha_L=\alpha_R$, then such peak of number A is called acute; otherwise, it is not acute.

Definition 8. A fuzzy number A is called normal if for any, specified in A elements of the carrier a_i^L, a_j^L from a_L to α_L, $\mu(a_i^L)\le\mu(a_j^L)$ $\forall a_i^L<a_j^L$, i.e., the membership function is non-decreasing on $[a_L,\alpha_L]$, and for any given elements of the carrier a_i^R, a_j^R from α_R to a_R, $\mu(a_i^R)\ge\mu(a_j^R)$ $\forall a_i^R>a_j^R$, i.e., the membership function is non-increasing on $[\alpha_R, a_R]$.

Definition 9. A normal single-peak number is called standard (continual or discrete).

Definition 10. A standardized fuzzy number is a discrete fuzzy number of the form $A=\{a_{L_0}|0;\, a_{L_1}|0.25;\, a_{L_2}|0.5;\, a_{L_3}|0.75;\, a_{L_4}|1;\ a_{R_4}|1;\, a_{R_3}|0.75;\ a_{R_2}|0.5;\ a_{R_1}|0.25;\ a_{R_0}|0\}$, where $a_{L_0}<a_{L_1}<a_{L_2}<a_{L_3}\le a_{L_4}<a_{R_4}<a_{R_3}<a_{R_2}<a_{R_1}<a_{R_0}$. This number can be specified by the arranged ten $A=(a_{L_0},a_{L_1},a_{L_2},a_{L_3},\ a_{L_4},a_{R_4},a_{R_3},a_{R_2},a_{R_1},a_{R_0})=(\underline{a}_0,\ \underline{a}_1,\ \underline{a}_2,\ \underline{a}_3,\ \underline{a}_4,\bar{a}_4,\ \bar{a}_3,\ \bar{a}_2,\ \bar{a}_1,\ \bar{a}_0)$. If the peak is acute, then it is obvious that $\underline{a}_4=\bar{a}_4$.

To obtain a standardized fuzzy number from a standard continual one, it is necessary to discretize the carrier according to the values $0.25i$ $(i=0,1,2,3,4)$ of the membership functions. And the ends of intervals $[\underline{a}_i,\bar{a}_i]$, $i=1,2,3,4$, are taken from the conditions $\forall a\in[\underline{a}_i,\bar{a}_i]$ $\mu(a)\in[0.25i;1]$, and for as much as small $\varepsilon>0$ $\mu(\underline{a}_i-\varepsilon)<0.25i$, $\mu(\bar{a}_i+\varepsilon)<0.25i$ (Fig. 1), $\underline{a}_0=a_L$; $\bar{a}_0=a_R$. To obtain a standardized fuzzy number from a standard discrete fuzzy number, it is possible to use the technique outlined in [19].

The five intervals, $[\underline{a}_i,\bar{a}_i]$, $i=0,1,2,3,4$, in the standardized fuzzy number to describe linguistic variables are chosen according to the same principle as for the five basic meanings of the Saaty scale [20]. However, for other purposes the number of intervals can be increased by introducing the intervals corresponding to the arbitrary condition $\alpha\in[0,1]$ of the value of membership function of a fuzzy number.

In the present paper, we will only use standardized fuzzy numbers; therefore, in what follows, we will omit the word "standardized."

According to the terminology from [21], we will introduce the concepts and notation from the theory of interval matrices, necessary for the further presentation.

Consider two $m \times n$ matrices, $\underline{A}$ and $\overline{A}$, with real elements, i.e., $\underline{A}, \overline{A} \in R^{m \times n}$, if $n = 1$, then the matrix is a column vector. Let $\underline{A} \le \overline{A}$ and the signs $\le$ and $<$ for the matrices mean that the inequality $\le$ or $<$ holds point-by-point.

Definition 11. Let the set of matrices $A \in R^{m \times n}$ satisfy the condition $\underline{A} \le A \le \overline{A}$. We will call such set an interval matrix I_A, i.e., $I_A = \{A \mid \underline{A} \le A \le \overline{A}\}$.

We will call matrices $\underline{A}$ and $\overline{A}$, respectively, the lower and upper bounds of the interval matrix I_A, which we will denote by $I_A = [\underline{A}, \overline{A}]$.

Definition 12. By average matrix A_c of interval matrix I_A we will understand the matrix defined by the relation

$$A_c = \frac{1}{2}(\underline{A} + \overline{A}). \tag{1}$$

The matrix of radii Δ of the interval matrix I_A is the matrix defined by the formula

$$\Delta = \frac{1}{2}(\overline{A} - \underline{A}). \tag{2}$$

It is obvious that elements of the matrix of radii are nonnegative: $\Delta_{ij} \ge 0 \ \forall i = \overline{1, m}, \ \forall j = \overline{1, n}$. According to Definition 12, from formulas (1) and (2) we obtain $\underline{A} = A_c - \Delta$ and $\overline{A} = A_c + \Delta$. Thus, we can represent the interval matrix as $I_A = [\underline{A}, \overline{A}] = [A_c - \Delta, A_c + \Delta]$ or $I_A = \{A \mid |A - A_c| \le \Delta\}$, where the absolute value of matrix $B = (b_{ij}) \in R^{m \times n}$ is defined as the matrix $|B| = (|b_{ij}|) \in R^{m \times n}$.

Definition 13. An interval vector I_b (column) is an interval matrix with one column, i.e., $I_b \in R^{m \times 1} = R^m$.

We will also denote interval vector by $I_b = \{b \mid \underline{b} \le b \le \overline{b}\}$, where $\underline{b}, \overline{b} \in R^m$ are its lower and upper bounds, respectively.

By the average vector b_c of interval vector we will understand $b_c = \frac{1}{2}(\underline{b} + \overline{b})$, and the vector of radii for I_b is vector $\delta = \frac{1}{2}(\overline{b} - \underline{b})$. Thus, we represent the interval vector as $I_b = [\underline{b}, \overline{b}] = [b_c - \delta; b_c + \delta]$.

Being based on the concept of interval matrix, we will introduce the concept of a fuzzy matrix.

Definition 14. A fuzzy matrix A^f is a five-layer table (a matrix, an array) consisting on each layer t of interval matrices I_A^t, $t \in \{0, 1, 2, 3, 4\}$, where t is layer number. Matrix I_A^t for $t = \text{const}$ is an interval matrix of the form $I_A^t = [\underline{A}^t, \overline{A}^t]$. Here, $\underline{A}^t = (\underline{a}_{ijt}) \in R^{m \times n}$ and $\overline{A}^t = (\overline{a}_{ijt}) \in R^{m \times n}$, $t \in \{0, 1, 2, 3, 4\}$.

We will denote by $a_{ijt} \in [\underline{a}_{ijt}, \overline{a}_{ijt}]$ an element of matrix A^f, where $\underline{a}_{ijt}$ and $\overline{a}_{ijt}$ are parameters of the fuzzy standardized number $a_{ij} = (\underline{a}_{ij0}, \underline{a}_{ij1}, \underline{a}_{ij2}, \underline{a}_{ij3}, \underline{a}_{ij4}, \overline{a}_{ij4}, \overline{a}_{ij3}, \overline{a}_{ij2}, \overline{a}_{ij1}, \overline{a}_{ij0})$, i and j, respectively, are the numbers of the row and column, and t is the layer number of matrix A^f. Matrix I_A^t is called layer t of matrix A^f, and matrix A^f is five-layer one.

In the case where $n = 1$, matrix A^f is called a fuzzy vector (column) with m fuzzy coordinates and is denoted by b^f. Vector $I_b^t \in R^m$ is called layer t of vector b^f, $t = 0, 1, 2, 3, 4$, and vector b^f is five-layer one.

Example 1. Let a be a standardized fuzzy number: $a = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$, i.e., $a = (0|0; 1|0.25; 2|0.5; 3|0.75; 4|1; 5|1; 6|0.75; 7|0.5; 8|0.25; 9|0)$. Consider a single-element fuzzy vector (number) $A^f = (I_b^t)$, $t = 0, 1, 2, 3, 4$ for $m = 1$ and $n = 1$, where $I_b^0 = [0; 9]$, $I_b^1 = [1; 8]$, $I_b^2 = [2; 7]$, $I_b^3 = [3; 6]$, and $I_b^4 = [4; 5]$. In this case, A^f is a five-layer matrix with one column and one row, i.e., fuzzy number $a = A^f$.

Let us present the definition of interval linear systems of equations from [21].

Definition 15. An interval linear system of equations

$$I_A x = I_b \tag{3}$$

is the set of all systems of linear equations

$$Ax = b, \tag{4}$$

where

$$A \in I_A; \ b \in I_b. \tag{5}$$

Using Definition 15, we will introduce the concept of an uncertain linear system of equations according to [17].

Definition 16. An uncertain linear system of equations

$$A^f x = b^f \tag{6}$$

is the set of five interval linear systems of equations

$$\left[\begin{array}{l} I_A^4 x = I_b^4, \\ I_A^3 x = I_b^3, \\ I_A^2 x = I_b^2, \\ I_A^1 x = I_b^1, \\ I_A^0 x = I_b^0, \end{array}\right. \tag{7}$$

where A^f and I_A^t, b^f and I_b^t $(t = 0, 1, \ldots, 4)$ are interrelated according to Definition 14.

Definition 17. The system of linear equation (4) is called solvable if it has some solution, and is called feasible if it has a nonnegative solution.

Definition 18 [21]. The interval linear system of equations (3) is called strongly solvable (feasible) if each system (4) with data (5) is solvable (is feasible).

Let us associate the interval linear system of equations $I_A^t x = I_b^t$, $t \in \{0, 1, 2, 3, 4\}$, with the set with the number t of systems of linear equation of the form (4) with data of the form (5):

$$A^t x = b^t, \tag{8}$$

$$A^t \in I_A^t; \ b^t \in I_b^t. \tag{9}$$

According to [17], let us define strong solvability (feasibility) of system (6).

Definition 19. An uncertain linear system of equations (6) is called strongly solvable (feasible) in exact sense if for $t = 4$ each of systems (8) with data (9)

$$A^4 x = b^4 \tag{10}$$

is solvable (is feasible). We will call this type of strong solvability (feasibility) the fourth type.

Definition 20. Uncertain linear system of equations (6) is called strongly solvable (feasible) in quasi-exact sense if for $t = 3$ each of systems (8)

$$A^3 x = b^3 \tag{11}$$

with data (9) is solvable (feasible).

Definition 21. Uncertain linear system of equations (6) is called strongly solvable (feasible) in semi-exact (or semi-fuzzy) sense if for $t = 2$ each of systems (8)

$$A^2 x = b^2 \tag{12}$$

with data from (9) is solvable (feasible). This type of strong solvability (feasibility) is called the second type.

Definition 22. Uncertain linear system of equations (6) is called strongly solvable (feasible) in quasi-fuzzy sense if for $t = 1$ each system from systems (8)

$$A^1 x = b^1 \tag{13}$$

with data from (9) is solvable (feasible). We will call this type of strong solvability (feasibility) the first type.

Definition 23. Uncertain linear system of equations (6) is called strongly solvable (feasible) in fuzzy sense if for $t=0$ each of the systems (8)

$$A^0 x = b^0 \tag{14}$$

with data from (9) is solvable (feasible). We will call this type of strong solvability (feasibility) zero type.

GENERAL PROBLEM STATEMENT

Criteria of weak solvability and weak feasibility in all five senses for the uncertain linear system of equations (6) are obtained in [17]. In [18], this approach is developed for the analysis of linear uncertain systems of inequalities, which are defined similarly to Definition 16, i.e., uncertain linear system of inequalities $A^f x \le b^f$ is a set of five interval linear systems of inequalities $I_A^t x \le I_b^t$, where $t = 0, 1, \ldots, 4$, and A^f, I_A^t, b^f, and I_b^t are interrelated according to Definition 14. In [18], criteria of strong solvability and strong feasibility of uncertain linear systems of inequalities in all five senses are obtained.

In the present paper, we formulate the problem of obtaining the criteria of strong solvability and feasibility for uncertain linear systems of equations. The results of the analysis of systems of fuzzy equations and inequalities of the form $A_1^f x \le b_1^f$ and $A_2^f x = b_2^f$ are important and necessary to solve optimization problems, where these uncertain systems are the condition that specifies the feasible domain.

CRITERIA OF STRONG SOLVABILITY AND STRONG FEASIBILITY OF UNCERTAIN LINEAR SYSTEMS OF EQUATIONS

Denote unit vector by $e = (1, 1, \ldots, 1)^{\mathrm{T}} \in R^m$, where T denotes transposition. If the dimension of vector e is not specified, it can be easily determined from the context. Let $I_A^t = [A_c^t - \Delta^t; A_c^t + \Delta^t]$ be given interval $(m \times n)$-matrix defined by (7), and vector $y \in Y_m \subset R^m$, where $Y_m = \{y \in R^m \,\big|\, |y| = e\}$. Denote by D_z for vector $z \in R^m$ a square matrix from $R^{m \times m}$, in which its elements are arranged on the principal diagonal and the other elements are zero, i.e.,

$$D_z = \operatorname{diag}(z_1, \ldots, z_m) = \begin{pmatrix} z_1 & 0 & \ldots & 0 & 0 \\ 0 & z_2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & z_{m-1} & 0 \\ 0 & 0 & \ldots & 0 & z_m \end{pmatrix}. \tag{15}$$

Let us introduce matrices

$$A_{yz}^t = A_c^t - D_y \Delta^t D_z, \tag{16}$$

where D_y and D_z are matrices of the form (15) and $y \in Y_m$ and $z \in Y_n$. If y or z is a unit vector e, then D_y and D_z are, respectively, unit matrices. Therefore, $A_{ye}^t = A_c^t - D_y \Delta^t$ and $A_{ez}^t = A_c^t - \Delta^t D_z$.

Similarly, for the interval m-dimensional vector $I_b^t = [b_c^t - \delta^t; b_c^t + \delta^t]$ defined in (7), we will introduce vector $b_y^t = b_c^t + D_y \delta^t$. According to the introduced definitions, $A_{yz}^t \in I_A^t$ and $b_y^t \in I_b^t$ (16), and also for elements of matrix A_{yz}^t and vector b_y^t we have $(A_{yz}^t)_{ij} = \begin{cases} \overline{a}_{ij}^t, & y_i z_j = -1, \\ \underline{a}_{ij}^t, & y_i z_j = 1; \end{cases}$ $(b_y^t)_i = \begin{cases} \overline{b}_i, & y_i = 1, \\ \underline{b}_i, & y_i = -1. \end{cases}$

Denote the convex hull of set $M \subset R^m$ by conv M. As is generally known, it is the intersection of all convex subsets of R^m containing M.

Let $\bar{x}, \bar{x}^t, \bar{\bar{x}}, \bar{\bar{x}}^t \in R^n$.

THEOREM 1. Uncertain linear system of equations $A^f x = b^f$ of the form (6) is strongly solvable of type t, $t = 0, 1, 2, 3, 4$, if and only if for each vector $y \in Y_m$ the linear system of equations

$$A^t_{ye}\bar{x}^t - A^t_{-ye}\bar{\bar{x}}^t = b^t_y, \tag{17}$$

$$\bar{x}^t \geq 0,\ \bar{\bar{x}}^t \geq 0, \tag{18}$$

has a solution $\bar{x}^t_y$, $\bar{\bar{x}}^t_y$ ($\bar{x}^t_y \in R^n$, $\bar{\bar{x}}^t_y \in R^n$). If it is true, i.e., the uncertain linear system of equations $A^f x = b^f$ is strongly solvable of type t, then for any $A^t \in I^t_A$ and $b^t \in I^t_b$ system $A^t x = b^t$ has a solution in the set conv $\{\bar{x}^t_y - \bar{\bar{x}}^t_y \mid y \in Y_m\}$.

Proof. According to Definitions 19–23 (see (10)–(14)), uncertain system (6) is called strongly solvable of type $t \in \{0, 1, 2, 3, 4\}$ if each of systems (8) with data (9) is solvable. According to Definition 18, it means strong solvability of the interval system $I^t_A = I^t_b$. For strong solvability of interval systems, the Rohn criterion [22] is known (see also Theorem 2.14 from [21]). Immediate application of the Rohn criterion proves the theorem.

If we use the notation and definitions in terms of which Theorem 1 and relation (17) are formulated, we can also give the following interpretation of system (17). For $y_i = 1$ the ith rows of matrices A_{ye} and A_{-ye} are equal to the ith rows $\underline{A}_i$ and $\overline{A}_i$ of matrices $\underline{A}$ and $\overline{A}$, respectively, and $(b_y)_i = \overline{b}_i$, i.e., for $y_i = 1$ the ith equation in system (17) is as follows:

$$(\underline{A}\bar{x} - \overline{A}\bar{\bar{x}})_i = \overline{b}_i. \tag{19}$$

Similarly, for $y_i = -1$ the ith equation in system (17) is

$$(\overline{A}\bar{x} - \underline{A}\bar{\bar{x}})_i = \underline{b}_i. \tag{20}$$

Thus, the set of systems (17) for all $y \in Y_m$ is the set of all 2^m systems whose ith equations have the form (19) or (20) depending on the ith coordinate of vector y ($y_i = \pm 1$). It proves (see also [21]) the following statement.

THEOREM 2. Analyzing uncertain linear system of equations (6) for strong solvability of type $t \in \{0, 1, 2, 3, 4\}$ is NP-hard.

As is known (see [21]), vector $x \in R^n$ is called strong solution of the interval linear system of equations $I^t_A x = b^t$ of the form (3) if it satisfies the system $Ax = b$ of the form (4) for any $A \in I_A$ and $b \in I_b$.

Definition 24. We will call vector $x^t \in R^n$ a strong solution of type t ($t \in \{0, 1, 2, 3, 4\}$) of the uncertain linear system of equations $A^f x = b^f$ of the form (6) if it is a strong solution of the interval linear system of equations $I^t_A x = b^t$ of the form (3).

The following characterization of strong solutions of type t of an uncertain linear system of equations takes place.

THEOREM 3. Vector $x^t \in R^n$ is a strong solution of type t of the uncertain linear system of equations $A^f x = b^f$ of the form (6) if and only if it satisfies the system

$$A^t_c x^t = b^t_c, \tag{21}$$

$$\Delta^t |x^t| = 0, \tag{22}$$

$$\delta^t = 0, \tag{23}$$

where $A^t_c = \frac{1}{2}(\underline{A}^t + \overline{A}^t)$, $b^t_c = \frac{1}{2}(\overline{b}^t + \underline{b}^t)$, $\Delta^t = \frac{1}{2}(\overline{A}^t - \underline{A}^t)$, and $\delta^t = \frac{1}{2}(\overline{b}^t - \underline{b}^t)$.

Proof. According to Definition 24, vector x^t is called a strong solution of type t of the uncertain linear system of equations (6) if it is a strong solution of the interval linear system of equations $I_A^t x = b^t$ of the form (3). For strong solutions of interval linear systems, the criterion is known (see Theorem 2.16 in [21]). Simple application of this criterion to system $I_A^t x = b^t$ proves the theorem.

From (22) it follows that variable $x_j^t = 0$ for all j, for which the jth column Δ_j^t of matrix Δ^t is nonzero ($\Delta_j^t \neq 0$). Denote $J_n = \{1, 2, \ldots, n\}$, $J = \{j \mid \Delta_j^t \neq 0\}$; $\bar{J} = J_n \setminus J$ is the complement of J in the universal set J_n. Then we can write conditions (21) and (22) as

$$\sum_{j \in \bar{J}} (A_c^t)_j x_j^t = b_c^t, \tag{24}$$

$$x_j^t = 0, \ j \in J, \tag{25}$$

where $(A_c^t)_j$ is the jth column of matrix A_c. Solving system (23)–(25) yields the strong solution x^t of the type t of uncertain system (6) if it exists, which happens rather rarely.

According to Definitions 18–23, description of strong feasibility of type t for $t = 0, 1, 2, 3, 4$ can be obtained from strong solvability.

THEOREM 4. An uncertain linear system of equations $A^f x = b^f$ of the form (6) is strongly feasible of type t for $t = 0, 1, 2, 3, 4$ if and only if $\forall y \in Y_m$ system

$$A_{ye}^t x^t = b_y^t \tag{26}$$

has a nonnegative solution x_y^t. Moreover, if such solutions exist, then for any $A^t \in I_A^t$ and for any $b^t \in I_b^t$ system $A^t x = b^t$ has solutions in the set

$$\text{conv}\,\{x_y^t \mid \ y \in Y_m\}. \tag{27}$$

Proof. If the system $A^f x = b^f$ is strongly feasible of type t ($t = 0, 1, 2, 3, 4$), then each system (26) has a nonnegative solution x_y^t since $\forall y \in Y_m$ $A_{ye}^t \in I_A^t$ and $b_y^t \in I_b^t$. Let us show the opposite. Let $\forall y \in Y_m$ system (26) have a nonnegative solution x_y^t. Denote $\bar{x}_y^t = x_y^t$ and $\bar{\bar{x}}_y^t = 0$ $\forall y \in Y_m$. Substituting $\bar{x}_y^t$ and $\bar{\bar{x}}_y^t$ into (17) and (18), we can easily prove the feasibility of this system. But according to Theorem 1 it means that each system $A^f x = b^f$ for any $A^t \in I_A^t$ and $b^t \in I_b^t$ has a solution in the set conv $\{\bar{x}_y^t - \bar{\bar{x}}_y^t \mid y \in Y_m\}$, which for $\bar{x}_y^t = x_y^t$ and $\bar{\bar{x}}_y^t = 0$ coincides with set (27). Hence, uncertain system $A^f x = b^f$ is strongly feasible of type t for $t = 0, 1, 2, 3, 4$, as was to be shown.

On the basis of Theorem 2.18 from [21], we may state that analyzing an uncertain linear system of equations for strong feasibility is NP-hard.

By virtue of Theorem 1 from [17] and Definitions 19–27 of strong feasibility and solvability of a system of linear fuzzy equations, the following statements are true.

THEOREM 5. If vector x^t is a strong solution of type t for $t = 0, 1, 2, 3$ of an uncertain linear system of equations $A^f x = b^f$, then it is a strong solution of type $t+1$ of this system.

This theorem and its proof are similar to Theorem 3 from [17].

THEOREM 6. If an uncertain linear system of equations $A^f x = b^f$ is:
— strongly solvable in fuzzy sense, then it is strongly solvable in quasi-fuzzy sense;
— strongly solvable in quasi-fuzzy sense, then it is strongly solvable in semi-fuzzy sense;
— strongly solvable in semi-fuzzy sense, then it is strongly solvable in quasi-exact sense;
— strongly solvable in quasi-exact sense, then it is strongly solvable in exact sense.

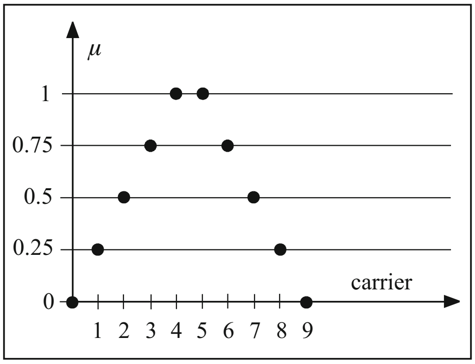

Fig. 2. Fuzzy number a.

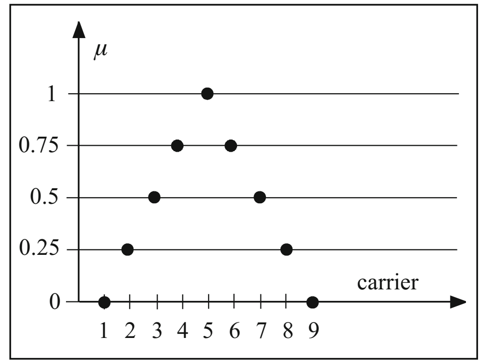

Fig. 3. Fuzzy number b.

TABLE 1. Weak and Strong Solvability and Feasibility (in the Sense t)

t	**Weak Solvability**			**Weak Feasibility**		**Strong Solvability**			**Strong Feasibility**	
	Example	Solution	Result	Solution	Result	Example	Solution	Result	Solution	Result
0	$9x = 9$	$x = 1$	+	$x = 1$	+	$0 \cdot x = 1$	$x \in \varnothing$	–	$\varnothing$	–
1	$8x = 8$	$x = 1$	+	$x = 1$	+	$\alpha \cdot x = \beta$	$x = \beta / \alpha$	+	$\beta / \alpha > 0$	+
2	$7x = 7$	$x = 1$	+	$x = 1$	+	$\alpha \cdot x = \beta$	$x = \beta / \alpha$	+	$\beta / \alpha > 0$	+
3	$6x = 6$	$x = 1$	+	$x = 1$	+	$\alpha \cdot x = \beta$	$x = \beta / \alpha$	+	$\beta / \alpha > 0$	+
4	$5x = 5$	$x = 1$	+	$x = 1$	+	$\alpha \cdot x = 5$	$x = 5 / \alpha$	+	$5 / \alpha > 0$	+

The **proof** follows from the definitions of strong solvability, Theorem 1 from [17], and Theorem 5.

If there is no strong solution x^t of type t of system $A^f x = b^f$, then there is no strong solution x^{t-1} of type $t-1$ of this system for $t = 1, 2, 3, 4$.

Example 2. Let system $A^f x = b^f$ be one equation $a^f x = b^f$, where a^f and b^f are sets of intervals, the respective fuzzy numbers a and b are shown in Figs. 2 and 3, i.e., $a = \{0|0; 1|0.25; 2|0.5; 3|0.75; 4|1; 5|1; 6|0.75; 7|0.5; 8|0.25; 9|0\}$ and $b = \{1|0; 2|0.25; 3|0.5; 4|0.75; 5|1; 6|0.75; 7|0.5; 8|0.25; 9|0\}$. Then $A^f = ([0, 9] = I_A^0, [1, 8] = I_A^1, [2, 7] = I_A^2, [3, 6] = I_A^3, [4, 5] = I_A^4)$, where layers of matrix A^f are intervals $I_A^0 = [0, 9], \ldots, I_A^4 = [4, 5]$, and $b^f = ([1, 9] = I_b^0, [2, 8] = I_b^1, [3, 7] = I_b^2, [4, 6] = I_b^3, [5, 5] = I_b^4)$, where layers of vector b^f are intervals $I_b^0 = [1, 9], \ldots, I_b^4 = [5, 5]$.

Let us write the set of interval systems:

— in the fuzzy case $t = 0$, $[0, 9]x = [1, 9]$;
— in the quasi-fuzzy case $t = 1$, $[1, 8]x = [2, 8]$;
— in the semi-fuzzy case $t = 2$, $[2, 7]x = [3, 7]$;
— in the quasi-exact case $t = 3$, $[3, 6]x = [4, 6]$;
— in the exact case $t = 4$, $[4, 5]x = [5, 5]$.

The solution is presented in Table 1, from which we can see that the equation $a^f x = b^f$ is weakly solvable and weakly feasible [17] in any of the five types, but has strong feasibility and strong solvability of four types (exact, quasi-exact, semi-exact, and quasi-fuzzy) and has no strong feasibility and solvability in fuzzy sense.

CONCLUSIONS

In the paper, we have obtained the criteria of strong feasibility and strong solvability for an uncertain linear system of equations in five senses (exact, quasi-exact, semi-exact, quasi-fuzzy, and fuzzy). In what follows, it is expedient to analyze the possibility of using the obtained results to solve optimization problems with fuzzy data.

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