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STOCHASTIC BEHAVIORAL MODELS. CLASSIFICATION*

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Abstract. *Stochastic behavioral models are specified by a difference evolutionary equation for the probabilities of binary alternatives. The classification of stochastic behavioral models is analyzed by the limit behavior of alternatives probabilities. The main property of classification is characterized by three types of equilibrium: attractive, repulsive, and dominant. The stochastic behavioral models are classified by using the stochastic approximation.*

Keywords: *evolutionary behavioral process, stochastic behavioral process, equilibrium, attractive model, repulsive model, dominant models, generator of discrete Markov process.*

INTRODUCTION

The purpose of this paper is to classify and analyze the dynamics of discrete-time behavioral processes. The discrete time instants are also called stages.

Stochastic behavioral processes are described by normalized sums of sample random variables with a finite number of values, which correspond to a set of possible decisional inferences.

The dynamics of behavioral process is determined by the regression functions of increments. The evolutionary behavioral processes are given by a solution of difference evolutionary equations.

A simple but very important case is the stochastic behavioral process, defined by normalized sums of sample random variables with two possible values, for example, ± 1. The fundamental property of the regression function of increments, in this case, means that the probability of sampling values are determined by two directing parameters, one of which promotes (increases) the probability of a certain behavioral choice, and the other inhibits (reduces) the probability of the alternatives. Such general principle of "stimulus and deterrence" [1] is present in a wide variety of natural stochastic processes that describe, for example, the interaction of molecules in chemical reactions [2] or the procedure of buying and selling in the economy [3].

The important analysis task, related to the stochastic behavioral processes, is the description of possible behavioral scenarios under unrestricted growth of the stages. In this context, an important task is to classify behavioral processes by asymptotic properties of behavioral frequencies (probabilities) for increasing number of stages.

The classification of the evolutionary behavioral process is realized by using specific properties of regression function of increments, based on the idea of Wright–Fisher [2, 4], where the regression function of increments has three

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equilibrium points, which play an essential role in models classification. The classifiers, corresponding to every equilibrium, are introduced for classification of stochastic behavioral processes by using stochastic approximation approach.

The stochastic behavioral processes considered here are similar to the stochastic learning models studied in detail in numerous papers and monographs [5, 6]). However, stochastic learning models are focused primarily on studying the asymptotic properties by unrestricted growth of the sample volume. In the meanwhile, the stochastic behavioral processes under consideration are investigated under unrestricted growth of the stages by a fixed sample volume, assuming of course that the sample size is large enough.

Stochastic behavioral models serve as a model of intelligence, driven by "stimulation–deterrence" factors. Similarly, property stochastic behavioral models "gain–loss" are used in the analysis of processes in economics. The dynamics of population genetics processes are also subject to the stochastic behavioral models properties listed above [4, 7]. In particular, in the book [8] linear stochastic behavioral models are also applied.

The paper contains three sections.

In Sec. 1, we introduce evolutionary behavioral processes [9] determined by solutions of difference evolutionary equation (Proposition 2). The increments of these processes are given by time-homogeneous regression functions.

Section 2 contains three variants of behavioral models classification [10], their justification and interpretation by the limit of evolutionary behavioral processes. The generic model classification is implemented by using classifier functions.

In Sec. 3, the classification is established for stochastic behavioral models by using a stochastic approximation approach [11] and using the classifier functions introduced in Sec. 2.

1. EVOLUTIONARY BEHAVIORAL PROCESS

The binary stochastic behavioral process is given by the averaged sum of sample values

$$S_N(k) := \frac{1}{N}\sum_{n=1}^{N} \delta_n(k), \ k \ge 0, \tag{1}$$

of independent and identically distributed by n random variables $\delta_n(k)$, $1 \le n \le N$, $k \ge 0$, which take binary values ± 1.

The binary stochastic behavioral process (1) can be represented by the difference of frequency processes:

$$S_N(k) = S_N^+(k) - S_N^-(k), \ k \ge 0. \tag{2}$$

The frequency stochastic behavioral processes $S_N^{\pm}(k)$, $k \ge 0$, are defined as follows:

$$S_N^{\pm}(k) := \frac{1}{N}\sum_{n=1}^{N} \delta_n^{\pm}(k), \ k \ge 0, \tag{3}$$

where

$$\delta_n^{\pm}(k) := I\{\delta_n(k) = \pm 1\}, \ 1 \le n \le N, \ k \ge 0.$$

The frequency behavioral processes $S_N^{\pm}(k)$, $k \ge 0$, describe the relative part of positive and negative values of the sample terms $\delta_n(k)$, $1 \le n \le N$, $k \ge 0$.

It is obvious that the identity

$$S_N^+(k) + S_N^-(k) \equiv 1 \ \ \forall k \ge 0$$

takes place.

Hence, the frequency stochastic behavioral processes (3) can be represented as follows:

$$S_N^{\pm}(k) = \frac{1}{2}[1 \pm S_N(k)], \ k \ge 0. \tag{4}$$

The properties of the binary stochastic behavioral process (1) imply the corresponding properties of the frequency stochastic behavioral processes (3), which can present a particular interest for applications. In what follows, all the three stochastic behavioral processes (1) and (3) will be analyzed in parallel.

The dynamics, by $k \geq 0$, of evolutionary behavioral processes (1) will be studied in terms of the following conditional probabilities:

$$C(k+1) := E[\delta_n(k+1) \mid S_N(k) = C(k)],\ 1 \leq n \leq N,\ \ k \geq 0, \tag{5}$$

and

$$P_\pm(k+1) := E[\delta_n^\pm(k+1) \mid S_N^\pm(k) = P_\pm(k)],\ 1 \leq n \leq N,\ \ k \geq 0. \tag{6}$$

It is easy to see that the conditional probabilities (5) and (6) have the following representations:

$$P_\pm(k+1) := E[S_N^\pm(k+1) \mid S_N^\pm(k) = P_\pm(k)],\ \ k \geq 0, \tag{7}$$

$$C(k+1) := E[S_N(k+1) \mid S_N(k) = C(k)],\ \ k \geq 0. \tag{8}$$

It is obvious that the conditional expectations (7) and (8) satisfy the following relation (see (2)):

$$C(k) = P_+(k) - P_-(k),\ \ k \geq 0.$$

Relations (7) and (8) imply that the conditional expectations (5) and (6) do not depend on the sample size N.

The dynamics of the evolutionary behavioral processes will be studied in terms of increments:

$$\Delta P_\pm(k+1) := P_\pm(k+1) - P_\pm(k),\ \ \Delta C(k+1) := C(k+1) - C(k),\ \ k \geq 0. \tag{9}$$

Proposition 1. The probabilities increments (9) are given by the difference evolutionary equations

$$\Delta P_\pm(k+1) = \mp V(P_+(k), P_-(k)),\ \ k \geq 0, \tag{10}$$

and their difference is given by the following difference evolutionary equation:

$$\Delta C(k+1) = -V(C(k)),\ \ k \geq 0, \tag{11}$$

where the regression functions of increments are

$$V(p_+, p_-) := p_+ p_-(V_+ p_+ - V_- p_-), \tag{12}$$

and also

$$V(c) := 2V(p_+, p_-). \tag{13}$$

Here, by definition (see (4))

$$p_\pm = \frac{1}{2}[1 \pm c].$$

The linear component of the regression function of increments (11) is defined by the real-valued directing parameters $V_\pm$, which satisfy the condition $-1 < V_\pm < +1$.

Remark 1. The regression function of increments (11) contains the linear term $(V_+ p_+ - V_- p_-)$, which reflects the main principle of interactions in behavior processes, "stimulation-deterrence." The additional multiplier $p_+ p_-$ corresponds to the property of absorption at the extreme points (0,1). Similar multipliers are present in the Wright–Fisher genetic model [4, Ch.10; 7, Ch. 12].

The linear component in (12) has equilibrium points

$$V(\rho_+^0, \rho_-^0) = 0$$

defined as follows:

$$\rho_\pm^0 = 0,\ \ \rho_\mp^0 = 1,$$

and the equilibrium values (ρ_+, ρ_-) determined by the equations

$$V_+\rho_+ - V_-\rho_- = 0,\ \ \rho_+ + \rho_- = 1.$$

Hence, the equilibrium values (ρ_+, ρ_-) are expressed as follows:

$$\rho_\pm = V_\mp / V, \ V := V_+ + V_-.$$

The difference

$$\rho := \rho_+ - \rho_- \tag{14}$$

is the equilibrium of the regression function of increments (13): $V(\rho) = 0$.

The equilibrium points $(\rho_\pm, \rho)$ allow expressing the linear component of the regression function of increments (11) as follows:

$$V(p_+, p_-) = \pm V p_+ p_- (p_\pm - \rho_\pm). \tag{15}$$

Similarly, the regression function of increments in (12), using the relations (11) and (14), acquires the following form:

$$V(c) = \frac{1}{4} V(1 - c^2)(c - \rho). \tag{16}$$

The regression function of increments representations (15), (16) allow formulating the difference evolutionary equations (10) and (11) in the following form.

Proposition 2. The evolutional behavioral processes $P_\pm(k)$ and $C(k)$, $k \ge 0$, are determined by solutions of the following difference evolutionary equations:

$$\Delta P_\pm(k+1) = \mp V P_+(k) P_-(k)(P_\pm(k) - \rho_\pm), \tag{17}$$

$$\Delta C(k+1) = -\frac{1}{4} V(1 - C^2(k))(C(k) - \rho). \tag{18}$$

Remark 2. The evolutional behavioral processes defined by the evolutionary equations (10) are characterized by two parameters $V_\pm$, whereas the evolutionary behavioral processes defined by the evolutionary equations (17) and (18) are characterized by two parameters V and $\rho_\pm$ in (17) and V and ρ in (18), respectively.

Now, it is possible to classify the evolutionary behavioral processes $C(k)$ and $P_\pm(k)$, $k \ge 0$, by varying the parameters V, $V_\pm$, ρ or $\rho_\pm$.

2. CLASSIFICATION OF THE MODELS

2.1. Classification of Evolutionary Behavioral Models. Evolutional behavioral processes given by the difference evolutionary equations (17) or (18) are characterized by the properties of the regression functions of increments (15) or (16).

Consider, for definiteness, the frequency evolutional behavioral processes $P_+(k)$, $k \ge 0$. The models classification can be implemented in different ways.

Firstly, it is natural to characterize different models by the directing parameters $V_\pm$ and $V = V_+ + V_-$. It is possible to use Eqs. (10), (11).

Behavioral model classification-I

The directing parameters V, $V_\pm$, $V = V_+ + V_-$ characterize the following behavioral models:

$\mathbb{M}A$:	$V > 0,\ V_\pm \ge 0$	attractive,
$\mathbb{M}R$:	$V < 0,\ V_\pm \le 0$	repulsive,
$\mathbb{M}A+$:	$V > 0,\ V_+ \le 0 \le V_-$	attractive, dominant $+$,
$\mathbb{M}R+$:	$V < 0,\ V_+ \le 0 \le V_-$	repulsive, dominant $+$,
$\mathbb{M}A-$:	$V > 0,\ V_- \le 0 \le V_+$	attractive, dominant $-$,
$\mathbb{M}R-$:	$V < 0,\ V_- \le 0 \le V_+$	repulsive, dominant $-$.

Behavioral model classification-II

The directing parameters V, $\rho_\pm$, $\rho_+ + \rho_- = 1$ characterize the following behavioral models:

$\mathbb{M}A$:	$V > 0,\ 0 < \rho_\pm < 1$	attractive,
$\mathbb{M}R$:	$V < 0,\ 0 < \rho_\pm < 1$	repulsive,

$\mathbb{M}A+$: $V>0,\ \rho_- \le 0 \le \rho_+$ attractive, dominant $+$,
$\mathbb{M}R+$: $V<0,\ \rho_+ \le 0 \le \rho_-$ repulsive, dominant $+$,
$\mathbb{M}A-$: $V>0,\ \rho_+ \le 0 \le \rho_-$ attractive, dominant $-$,
$\mathbb{M}R-$: $V<0,\ \rho_- \le 0 \le \rho_+$ repulsive, dominant $-$.

Behavioral model classification-III

The directing parameters V, ρ characterize the following behavioral models:

$\mathbb{M}A$: $V>0,\ |\rho|<1$ attractive,
$\mathbb{M}R$: $V<0,\ |\rho|<1$ repulsive,
$\mathbb{M}A+$: $V>0,\ \rho\ge 1$ attractive, dominant $+$,
$\mathbb{M}R+$: $V<0,\ \rho\le -1$ repulsive, dominant $+$,
$\mathbb{M}A-$: $V>0,\ \rho\le -1$ attractive, dominant $-$,
$\mathbb{M}R-$: $V<0,\ \rho\ge 1$ repulsive, dominant $-$.

The classification of evolutionary behavioral models is based on the analysis of asymptotic behavior of evolutionary behavioral processes (7) and (8) given by the solutions of difference evolutionary equations (17) and (18).

2.2. Justification of Classification-I and Classification-II. Without loss of generality, only frequency evolutionary behavioral process $P_+(k)$, $k\ge 0$, is considered for definiteness.

The dynamics of evolutionary behavioral processes given by the solution of evolutionary equation (17) is determined by the regression functions of increments in the following form:

$$\Delta P_+(k+1) = -VP_+(k)P_-(k)(P_+(k)-\rho_+). \tag{19}$$

The corresponding regression function of increments

$$V_+(p) := -Vp(1-p)(p-\rho_+)$$

has zeros at the points 0, 1, ρ_+, in which its first derivative assumes the following values:

$$V_+'(0) = V\rho_+ = V_-,\ V_+'(\rho_+) = -V\rho_+\rho_-,\ V_+'(1) = V\rho_- = V_+.$$

To justify Classification-II, we start considering.

■ The model $\mathbb{M}A$, where the equilibrium value ρ_+ of the frequency behavioral process satisfies the inequality

$$0<\rho_+<1. \tag{20}$$

Suppose for definiteness that the initial value of the frequency behavioral probability $P_+(0)>\rho_+$, with the consequence that $P_+(0)-\rho_+>0$. Then, by (19) it is clear (by induction) that $\Delta P_+(k)<0$, $k\ge 0$. Hence, $P_+(k)$, $k\ge 0$, is a decreasing and lower bounded random sequence (Fig. 1).

This implies that

$$\exists \lim_{k\to\infty} P_+(k) = P_+^* \ge \rho_+. \tag{21}$$

So that, by the convergence (21) and by (19), the following limit equation takes place:

$$0 = -Vp_+^* p_-^*(p_+^* - \rho_+). \tag{22}$$

Therefore,

$$p_+^* = \rho_+. \tag{23}$$

In a similar way, supposing $P_+(0)<\rho_+$ yields

$$\lim_{k\to\infty} P_+(k) = p_+^* \le \rho_+$$

and Eq. (23) remains valid.

So, the model $\mathbb{M}A$ has an attractive equilibrium point $0<\rho_+<1$ and $\lim_{k\to\infty} P_+(k) = \rho_+$.

■ The model $\mathbb{M}R$, where the frequency behavioral probabilities with equilibrium ρ_+ satisfy inequality (20).

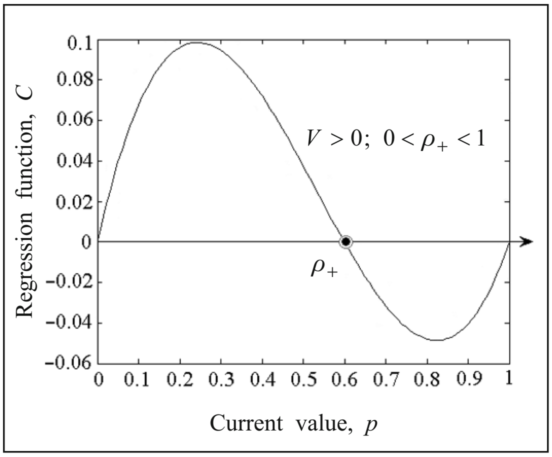

Fig. 1. Function $V_+(p)$ in the model $\mathbb{M}A$.

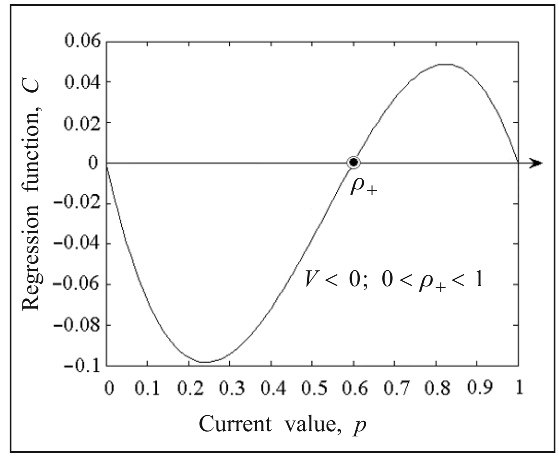

Fig. 2. Function $V_+(p)$ in the model $\mathbb{M}R$.

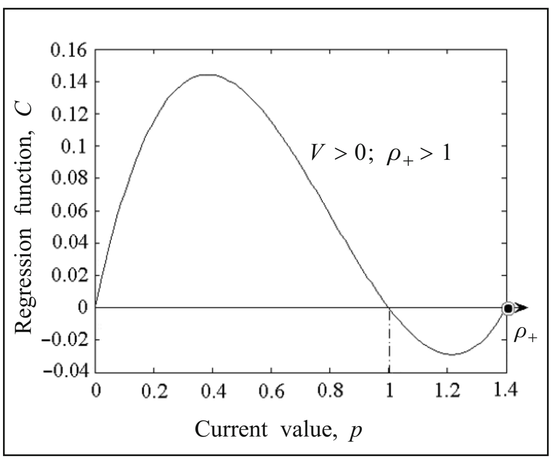

Fig. 3. Function $V_+(p)$ in the model $\mathbb{M}A+$.

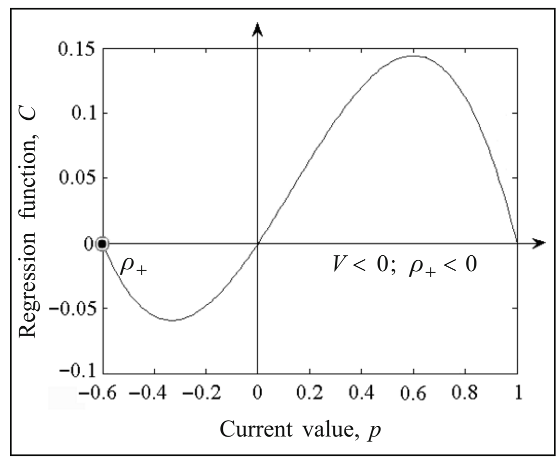

Fig. 4. Function $V_+(p)$ in the model $\mathbb{M}R+$.

If $P_+(0)>\rho_+$, the increments satisfy the relations $\Delta P_+(k)>0$, $k\ge 0$. Hence, $P_+(k)$, $k\ge 0$, is an increasing and upper bounded random sequence. This implies that $\lim_{k\to\infty} P_+(k)=p_+^*$. Now, the limit equation (22) ensures that $p_+^*=1$.

Similarly, if $P_+(0)<\rho_+$, the convergence to zero of the random sequence $P_+(k)$, $k\ge 0$, is obtainable. Obviously (Fig. 2), there are the following relations:

$$P_+(k)\le 0 \ \text{ for } \ 0\le P_+(0)\le \rho_+,$$

$$P_+(k)\ge 0 \ \text{ for } \ \rho_+\le P_+(0)\le 1.$$

Thus, the frequency behavioral probability $P_+(k)$ increases in the interval $(\rho_+,1)$ and decreases in the interval $(0,\rho_+)$.

Hence, the equilibrium point ρ_+ is repulsive, i.e., there takes place the convergence

$$\lim_{k\to\infty} P_+(k)=p_+^*=\begin{cases}0 & \text{if } \ 0<P_+(0)<\rho_+;\\ 1 & \text{if } \ \rho_+<P_+(0)<1.\end{cases}$$

■ The model $\mathbb{M}A+$: $V>0$, $V_+\le 0\le V_-$. In this model, the equilibrium point satisfies the inequalities (Fig. 3)

$$\rho_-\le 0,\ \ \rho_+\ge 1.$$

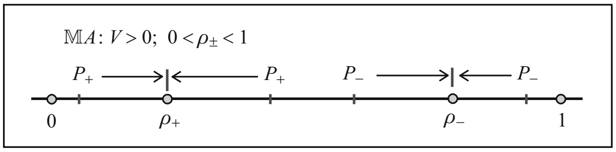

Fig. 5. Behavior of $P_{\pm}(k)$ in the model $\mathbb{M}A$ with increasing k.

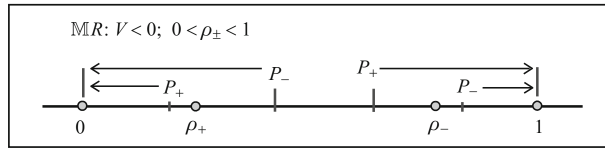

Fig. 6. Behavior of $P_{\pm}(k)$ in the model $\mathbb{M}R$ with increasing k.

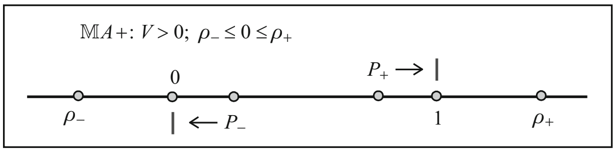

Fig. 7. Behavior of $P_{\pm}(k)$ in the model $\mathbb{M}A+$ with increasing k.

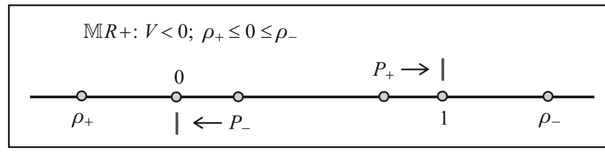

Fig. 8. Behavior of $P_{\pm}(k)$ in the model $\mathbb{M}R+$ with increasing k.

Accordingly, the regression function of increments satisfies $V_+(p) \ge 0$ for all $p \in [0,1]$. Consequently, the probability of a positive alternative increases and we have the following limit (see (22)):

$$\lim_{k\to\infty} P_+(k) = 1. \tag{24}$$

■ The model $\mathbb{M}R+$: $V < 0$, $V_+ \le 0 \le V_-$. In this model, the equilibrium point satisfies the inequalities $\rho_+ \le 0$, $\rho_- \ge 1$.

The regression function of increments satisfies the inequality (Fig. 4) $V_+(p) \ge 0$ for all $p \in [0,1]$.

Hence, the behavior of a positive alternative probability is described by the limit relation (24).

■ The model $\mathbb{M}A-$: $V > 0$, $V_- \le 0 \le V_+$. As above, there is a limit:

$$\lim_{k\to\infty} P_+(k) = 0. \tag{25}$$

■ The model $\mathbb{M}R-$: $V < 0$, $V_- \le 0 \le V_+$. Again, in a similar way as above, there is the limit (25).

2.3. Evolutionary Behavioral Models Interpretation. The classification of the behavioral models presented in Sec. 2.2 refers to the limit behavior analysis (as the number of stages $k \to \infty$) of the increments $\Delta P_{\pm}(k)$, which are solutions of the difference equations (17).

■ In the model $\mathbb{M}A$ (Fig. 5), the probabilities of the alternatives $P_{\pm}(k)$, $k \ge 0$, tend to equilibrium values $\rho_{\pm}$ for any initial value $P_{\pm}(0)$. The equilibriums $\rho_{\pm}$ are attractive.

Namely, we have

$$\lim_{k\to\infty} P_{\pm}(k) = \rho_{\pm}.$$

■ In the model $\mathbb{M}R$ (Fig. 6), the behavior of the alternative probabilities $P_{\pm}(k)$ as $k \to \infty$ depends on the initial conditions. The equilibriums $\rho_{\pm}$ are repulsive.

Specifically, we have

$$\lim_{k\to\infty} P_{\pm}(k) = 0, \text{ if } \; P_{\pm}(0) < \rho_{\pm}; \; \lim_{k\to\infty} P_{\pm}(k) = 1, \text{ if } \; P_{\pm}(0) > \rho_{\pm}.$$

■ In the models $\mathbb{M}A+$ and $\mathbb{M}R+$ (dominant+), for any initial value $P_{\pm}(0)$, the advantage of the alternative + is verified: $P_+(k) \to 1$ as $k \to \infty$, while $P_-(k) \to 0$ as $k \to \infty$. In this case, there are two different situations.

◊ If $V > 0$ (Fig. 7), the attraction of the positive alternative probability to equilibrium $\rho_+ \ge 1$ occurs.

More specifically,

$$\lim_{k\to\infty} P_+(k) = 1, \; \lim_{k\to\infty} P_-(k) = 0. \tag{26}$$

◊ If $V < 0$ (Fig. 8), the repulsion of the positive alternative probability from the equilibrium $\rho_+ \le 0$ occurs.

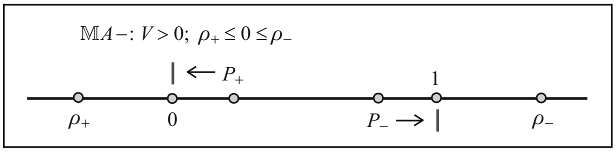

Fig. 9. Behavior of $P_\pm(k)$ in the model $\mathbb{M}A-$ with increasing k.

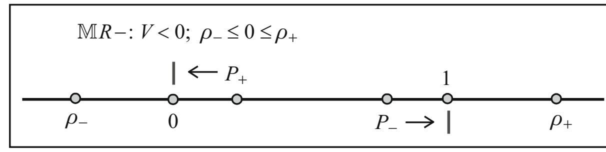

Fig. 10. Behavior of $P_\pm(k)$ in the model $\mathbb{M}R-$ with increasing k.

Again, the limits in (26) hold true.

■ In the models $\mathbb{M}A-$ and $\mathbb{M}R-$ (dominant $-$), for any initial value $P_\pm(0)$, the advantage of the alternative is verified: $P_-(k) \to 1$ as $k \to \infty$, while $P_+(k) \to 0$ as $k \to \infty$. Also, in this case, there are two different situations.

◊ If $V>0$ (Fig. 9), the attraction of the positive alternative probability to the equilibrium $\rho_+ \le 0$ occurs. Accordingly,

$$\lim_{k\to\infty} P_+(k) = 0, \quad \lim_{k\to\infty} P_-(k) = 1. \tag{27}$$

◊ If $V<0$ (Fig. 10), the repulsion of the positive alternative probability from the equilibrium $\rho_+ \ge 1$ occurs.

Again, the relations (27) hold true.

2.4. Discussion of the Behavioral Models Classification. The foregoing classification of behavioral models can be used to explain phenomena of collective behavior and to predict trends in collective decisions.

1. For parameters of the model $\mathbb{M}A$, the preferred choice of alternatives varies at each stage, and there is an attracting equilibrium point $\rho_\pm$. The behavioral probabilities $P_\pm(k)$ converge, as $k \to \infty$, to the equilibrium values, $\rho_\pm$.

This means that, in the behavioral process, in the long run, the stationary frequencies $\rho_\pm$ of choice of alternatives are observable: $N\rho_+$ subjects select the alternative $+$, and $N\rho_-$ subjects select the alternative $-$.

2. For parameters of the model $\mathbb{M}R$, there is a repulsive equilibrium point, which distinguishes the initial values. For initial values $P_\pm(0) < \rho_\pm$, the alternatives probabilities $\pm$ converge to 0 as $k \to \infty$; for $P_\pm(0) > \rho_\pm$, the alternatives probabilities converge to 1. Accordingly, there are two absorbing equilibrium points: 0 or 1.

This law seems to deserve the greatest attention of specialists. The following interpretation of this law can be given, for example, in the education system.

Each team class has a certain average intelligence ρ, $0<\rho<1$, and the result of a long behavioral process strongly depends on the initial conditions. If the proportion of scholars taking the right decision exceeds ρ, then eventually the whole class will do the same. If that proportion is less than ρ, then in the behavioral process the class average intelligence will reduce to zero.

This law describes the effects of the intelligence of individual subjects on the collective intelligence.

3. For parameters of the model $\mathbb{M}A\pm$ and $\mathbb{M}R\pm$, the advantage of one of the alternatives is preserved at any stage: for all initial conditions there exists a unique equilibrium absorbing point as $k \to \infty$: $P_+(k) \to 1$, $P_-(k) \to 0$ or $P_+(k) \to 0$, $P_-(k) \to 1$.

In this case, one of the alternatives dominates the selection.

The regularities of the behavioral process formulated above require experimental verification.

2.5. Classifiers. The evolutionary behavioral process is given by a solution of difference evolutionary equation (18) with regression function of increments (16).

Now, we introduce classifier functions, which discriminate the behavioral models of the Classification-III.

The classifier functions are constructed using truncated regression function of increments by the elimination of the corresponding equilibrium multipliers. The classifier functions are defined by the following expressions:

$$V_\rho(c) := -V(1-c^2), \; |c|<1, \tag{28}$$

for equilibrium point ρ, and

$$V_{\pm 1}(c) := \pm V(1 \pm c)(c-\rho), \; |c|<1, \tag{29}$$

for extremal points ± 1.

The classifier functions (28) and (29) have an essential property: they assume negative values for all $c \in (-1, +1)$ in the corresponding Classification-III models.

The attractive model $\mathbb{M}A: V > 0,\ |\rho| < 1$ is discriminated, by the classifier (28), which takes negative value: $V_\rho(c) = -\frac{1}{4}V(1-c^2) < 0$ for all $c \in (-1,\ +1)$.

The regression function of increments (16) in the difference evolutionary equation (18) has the following representation:

$$V(c) = \frac{1}{4}V_\rho(c)(c-\rho).$$

Hence, the evolutionary behavioral process determined by solution of the difference equation (18) is increasing for $c \in (-1, \rho)$ and decreasing for $c \in (\rho, +1)$.

The repulsive model $\mathbb{M}R: V < 0,\ |\rho| < 1$ is discriminated, by the classifiers (29), as follows:

$$V_{+1}(c) := V(1+c)(c-\rho) < 0 \text{ for } c \in (\rho, +1), \tag{30}$$

and

$$V_{-1}(c) := -V(1-c)(c-\rho) < 0 \text{ for } c \in (-1, \rho). \tag{31}$$

The regression function of increments (16) in the difference evolutionary equation (18) has the following representation:

$$V(c) = \begin{cases} \frac{1}{4}V_{+1}(c)(c-1) & \text{for } c \in (\rho, +1); \\ \frac{1}{4}V_{-1}(c)(c+1) & \text{for } c \in (-1, \rho). \end{cases}$$

The inequalities (30) and (31) mean that the evolutionary behavioral processes determined by the difference evolutionary equation (18) is increasing for $c \in (\rho, +1)$ up to $c_1 = 1$ and decreasing for $c \in (-1, \rho)$ up to $c_{-1} = -1$. Such a comportment of the evolutionary behavioral process is classified as repulsive property of the extremal points $\rho \in \{-1, +1\}$.

The model $\mathbb{M}A+: V > 0,\ \rho \ge +1$ (attractive dominant $+$) and the model $\mathbb{M}R+:\ V < 0,\ \rho \le -1$ (repulsive dominant $+$) are discriminated by the classifier (29) as follows:

$$V_{+1}(c) := V(1+c)(c-\rho) < 0 \text{ for all } c \in (-1, +1).$$

For these models, the regression function of increments (16) in the difference evolutionary equation (18) is used in the following representation:

$$V(c) = \frac{1}{4}V_{+1}(c)(c-1) \text{ for all } c \in (-1,\ +1).$$

Hence, the evolutionary behavioral processes are increasing for all $c \in (-1, +1)$, up to the value $+1$ for both models $\mathbb{M}A+$ and $\mathbb{M}R+$.

Similarly, the limit behavior of the models $\mathbb{M}A-$ and $\mathbb{M}R-$ are obtainable.

3. STOCHASTIC BEHAVIORAL MODEL

3.1. Basic Definitions. Let us focus our attention on the stochastic behavioral models described by averaged sums

$$S_N(k) := \frac{1}{N}\sum_{n=1}^{N}\delta_n(k),\ k \ge 0, \tag{32}$$

of sample values $\delta_n(k),\ 1 \le n \le N,\ k \ge 0$, which take two values ± 1 with the conditional expectation of increments (5) that satisfy the difference evolutionary equation (18).

The stochastic behavioral processes based on the averaged sums (32) are determined by two components:
— evolutionary component $V(S_N(k))$ described by the evolutionary equation (18);
— stochastic component described by the martingale-difference

$$\Delta\mu_N(k+1) = \Delta S_N(k+1) - E[\Delta S_N(k+1) \mid S_N(k)], \ k \ge 0. \tag{33}$$

So, the stochastic behavioral process is determined by a solution of the stochastic difference equation

$$\Delta S_N(k+1) = -V(S_N(k)) + \Delta\mu_N(k+1), \ k \ge 0. \tag{34}$$

The stochastic component (33) is characterized by its two first moments

$$E\Delta\mu_N(k) \equiv 0, \ k \ge 0,$$

$$E[(\Delta\mu_N(k))^2 \mid S_N(k) = c] = \sigma^2(c)/N, \ k \ge 0, \ \sigma^2(c) := \frac{1}{4}(1 - c^2).$$

3.2. Stochastic Behavioral Models in a Stochastic Approximation Scheme. The convergence problem of stochastic behavioral processes described by the difference stochastic equation (34) are more complicated than the analogous problems of convergence of evolutionary behavioral processes described by the difference evolutionary equation (18).

In order to get an effective result, the stochastic approximation approach can be used.

At the beginning, consider a stochastic behavioral process normalized by the sequence $a(t_k) := a/t_k$, $t_k = k/N$, satisfying the stochastic approximation procedure [11]:

$$\sum_{k=1}^{\infty} a(t_k) = \infty, \ \sum_{k=1}^{\infty} a^2(t_k) < \infty, \ k \ge 0. \tag{35}$$

The normalized stochastic behavioral process in $t_k = k/N$, $k \ge 1$, is considered in discrete-continuous time scale with the step $\Delta := 1/N$:

$$\Delta\alpha_N(t) = a(t)\Delta S_N(k+1), \ t_k \le t < t_{k+1}, \ k \ge 0.$$

The next step is the normalization of the directing parameter V and of the stochastic component increments, as described hereunder.

Definition 1. The stochastic behavioral process, under the stochastic approximation conditions (35), is determined by a solution of the difference stochastic equation

$$\Delta\alpha_N(t) = a(t)[-V(\alpha_N(t))\Delta + \sigma(\alpha_N(t))\Delta\mu_N(t+1)], \tag{36}$$

where $V(c)$ is defined in (16) and the stochastic component satisfies the following conditions:

$$E\Delta\mu_N(t+\Delta) \equiv 0, \ E[(\Delta\mu_N(t+\Delta))^2 \mid \alpha_N(t)] = \Delta = 1/N.$$

Remark 3. The stochastic behavioral process $\alpha_N(t_k)$, $t_k = k\Delta$, $k \ge 1$, is a discrete Markov process, characterized by its two conditional first moments:

$$E[\Delta\alpha_N(t) \mid \alpha_N(t)] = -a(t)V(\alpha_N(t))\Delta, \tag{37}$$

$$E[(\Delta\alpha_N(t) + a(t)V(c)\Delta)^2 \mid \alpha_N(t) = c] = a^2(t)\sigma^2(c)\Delta. \tag{38}$$

The second moment of the stochastic behavioral process increments, defined by the difference stochastic equation (36), is calculated using (37) and (38):

$$E[(\Delta\alpha_N^2(t) \mid \alpha_N(t) = c] = a^2(t)B_N(c)\Delta, \ B_N(c) := \sigma^2(c) + V^2(c)\Delta. \tag{39}$$

3.3. Classification-III for Stochastic Behavioral Processes. The classification of stochastic behavioral processes, according to the behavioral model Classification-III, described above in Sec. 2.3, is based on the limit theorems for a normalized stochastic behavioral process in the stochastic approximation scheme.

The behavioral model classification is characterized by an almost surely convergence of the normalized behavioral process $\alpha_N(t)$.

THEOREM 1. The behavioral model Classification-III give the following limit results:

$\mathbb{M}A: V>0,\ |\rho|<1$

$$P1 \lim_{t\to\infty} \alpha_N(t) = \rho \quad \text{as} \quad t\to\infty; \tag{40}$$

$\mathbb{M}R: V<0,\ |\rho|<1$

$$P1 \lim_{t\to\infty} \alpha_N(t) = \begin{cases} -1 & \text{if} \quad \alpha_N(0) < \rho, \\ +1 & \text{if} \quad \alpha_N(0) > \rho, \end{cases} \quad \text{as} \quad t\to\infty; \tag{41}$$

$\mathbb{M}A+: V>0,\ \rho\ge+1;\ \mathbb{M}R+: V<0,\ \rho\le-1$

$$P1 \lim_{t\to\infty} \alpha_N(t) = +1 \ \text{ as } \ t\to\infty \ \ \forall \alpha_N(0): |\alpha_N(0)|<1; \tag{42}$$

$\mathbb{M}A-: V>0,\ \rho\le-1;\ \mathbb{M}R-: V<0,\ \rho\ge+1$

$$P1 \lim_{t\to\infty} \alpha_N(t) = -1 \ \text{ as } \ t\to\infty \ \ \forall \alpha_N(0): |\alpha_N(0)|<1. \tag{43}$$

Proof. According to the approach proposed in the monograph by Nevelson–Hasminskii [11, Ch. 2, Sec.7], the generator of the discrete Markov process (37)–(39) is used as follows:

$$L_N\Phi(c) = NE[\Phi(c+\Delta\alpha_N(t)) - \Phi(c) \mid \alpha_N(t)=c] \tag{44}$$

on the test functions

$$\Phi(c) := (c-c_0)^2,\ |c|\le 1,\ c_0\in\{-1,+1,\rho\}. \tag{45}$$

LEMMA 1. On the test functions (45), generator (44) has the following representations:

$$L_N(c-c_0)^2 = -2a(t)V_{c_0}(c)(c-c_0)^2 + a^2(t)B_N(c), \tag{46}$$

where

$$B_N(c) := \sigma^2(c) + V^2(c)/N, \tag{47}$$

and by definition, the classifiers are as follows:

$$V_{c_0}(c) := \begin{cases} V_\rho(c) & \text{for} \quad c_0=\rho, \\ V_{\pm1}(c) & \text{for} \quad c_0=\pm1. \end{cases} \tag{48}$$

Theorems 2.7.1 and 2.7.2 from the Nevelson–Hasminskii monograph [11], adapted to the stochastic behavioral processes (36), are represented below.

THEOREM 2 [11, Ch. 2, Sec. 7]. Let there exist a non-negative function $V(c)$, $|c|\le 1$, with zero point c_0: $V(c_0)=0$, satisfying the inequality

$$\sup_{|c-c_0|\ge h} [V(c)(c-c_0)] < 0 \ \text{ for a fixed } \ h>0 \tag{49}$$

and the additional positive component defined by (47) be bounded:

$$|B_N(c)| = |\sigma^2(c) + V^2(c)/N| \le K.$$

Then the stochastic behavioral process $\alpha_N(t)$, $t \ge 0$, converges with probability 1 to the equilibrium point c_0:

$$\alpha_N(t) \xrightarrow{P1} c_0, \text{ as } t \to \infty.$$

The main condition (49) of Theorem 2, using Lemma 1, is transformed as follows:

$$V(c)(c-c_0) = \frac{1}{4} V_{c_0}(c)(c-c_0)^2, \; c \in \{-1, +1, \rho\}.$$

Hence, by Theorem 2.7.2 [11], the stochastic behavioral process $\alpha_N(t)$, $t \ge 0$, characterized by the generator (46)–(48), converges with probability 1, as $t \to \infty$, to the equilibrium point $c_0 \in \{-1, +1, \rho\}$. The limits (40)–(43) of Theorem 1 are proved.

CONCLUSIONS

The statistical data analysis in the stochastic behavioral process consists primarily of the original parameters $V_\pm$ evaluation. In a model of stochastic behavioral process with a number of alternatives bigger than two, the behavior regularities become naturally more complex. However, in essence, the main features are always the same: after a sufficiently large number of stages, the behavioral process of a large population of subjects is characterized by three types of equilibria: attracting, repulsive, and absorbing. In processes with many alternatives, the behavior equilibrium state (in the parameter space) can be a point, a line, a plane, etc., up to an R-dimensional subspace.

REFERENCES

1. N. M. Amosov, Algorithms of the Intellect [in Russian], Naukova Dumka, Kyiv (1979).
2. D. Koroliouk, V. S. Koroliuk, and N. Rosato, "Equilibrium process in biomedical data analysis: The Wright–Fisher model," Cybern. Syst. Analysis, **50**, No. 6, 890–897 (2014).
3. A. N. Shiryaev, Essentials of Stochastic Finance: Facts, Models, Theory, World Scientific, London (1999).
4. S. N. Ethier and T. G. Kurtz, Markov Processes: Characterization and Convergence, Wiley, New York (1986).
5. V. N. Vapnik, The Nature of Statistical Learning Theory, Springer, New York (2000).
6. O. Bousquet, S. Boucheron, and G. Lugosi, "Introduction to statistical learning theory," in: Advanced Lectures on Machine Learning, Springer, Berlin–Heidelberg (2004), pp. 169–207.
7. A. V. Skorokhod, F. C. Hoppensteadt, and H. Salehi, Random Perturbation Methods with Applications in Science and Engineering, Springer-Verlag, New York (2002).
8. R. R Bush and F. Mosteller, Stochastic Models for Learning, Wiley, New York (1955).
9. D. Koroliouk, "Binary statistical experiments with persistent non-linear regression," Theor. Probability and Math. Statist., No. 91, 71–80 (2015).
10. D. Koroliouk, "Classification of binary deterministic statistical experiments with persistent regression," Cybern. Syst. Analysis, **51**, No. 4, 644–649 (2015).
11. M. B. Nevelson and R. Z. Hasminskii, "Stochastic approximation and recursive estimation," in: Translations of Mathematical Monographs, Amer. Math. Soc., Issue 47, New York (1973).