DOI 10.1007/s10559-015-9768-z
Cybernetics and Systems Analysis, Vol. 51, No. 5, September, 2015

CONVERGENCE OF THE MODIFIED EXTRAGRADIENT METHOD FOR VARIATIONAL INEQUALITIES WITH NON-LIPSCHITZ OPERATORS

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UDC 517.988

Abstract. *We propose a modified extragradient method with dynamic step size adjustment to solve variational inequalities with monotone operators acting in a Hilbert space. In addition, we consider a version of the method that finds a solution of a variational inequality that is also a fixed point of a quasi-nonexpansive operator. We establish the weak convergence of the methods without any Lipschitzian continuity assumption on operators.*

Keywords: *variational inequality, monotone operator, Hilbert space, extragradient method, weak convergence.*

INTRODUCTION

Many problems in operations research and mathematical physics can be written as variational inequalities [1–4]. The solution of these inequalities is an intensively developing field of applied nonlinear analysis. By now, many methods [5–34] have been proposed, in particular, of projection type (which use the operation of metric projection on the feasible set).

As is generally known, in problems of finding a saddle point or Nash equilibrium, for the most simple projection method (an analog of the gradient projection method) to converge, strengthened monotonicity conditions should be satisfied [6, 7]. If they are not satisfied, several approaches can be applied. One of them is to regularize the original problem in order to impart the required property to it [5]. The convergence without problem modification is provided in iterative extragradient methods first proposed by Korpelevich in [21]. These methods were analyzed in many studies [22–34]. For variational inequalities and equilibrium programming problems, modifications of the Korpelevich algorithm with one metric projection onto feasible set were proposed [27, 28]. In these so-called subgradient extragradient algorithms and in the Korpelevich algorithm, the first stages of the iteration coincide, and then, to obtain the next approximation, projection onto some half-space being the support for the feasible set is carried out instead of projection onto the feasible set. In [27, 28], the weak convergence of sequences generated by the subgradient extragradient algorithm to some solution of variational inequality is proved. An obvious disadvantage of the algorithm, which impedes its wide use, is the assumption that the Lipschitz constant of the operator is known or admits a simple estimate. Moreover, in many problems, operators may not satisfy the Lipschitz condition. In the majority of studies, these are the Lipschitz operators that are considered in the algorithms of solution of variational inequalities.

In the present paper, we propose a modification of the subgradient extragradient algorithm with dynamic step size adjustment for variational inequalities with monotone non-Lipschitz operator and prove its convergence. The used step size adjustment is described in [22, 23]. We will also consider the variants of the method for variational inequalities or operator equations with a priori information about the solution, specified in the form of a set of fixed points of quasi-nonexpanding operator.

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PROBLEM STATEMENT

In what follows, H is a real Hilbert space with scalar product $(\cdot,\cdot)$ and generated norm $||\cdot||$. Let C be a nonempty subset of space H and A be an operator acting in H. Consider the variational inequality

$$\text{find } x\in C:\ (Ax, y-x)\geq 0\ \ \forall\ y\in C \tag{1}$$

and denote its set of solutions by $VI(A,C)$.

Assume that the following conditions are satisfied:

- the set $C\subseteq H$ is convex and closed;
- the operator $A:H\to H$ is monotone, uniformly continuous on bounded sets, and maps bounded sets into bounded ones;
- the set $VI(A,C)$ is nonempty.

Remark 1. If $\dim H<\infty$, then it will suffice to demand from the operator A to be monotone and continuous.

Auxiliary Information. Let P_C be an operator of metric projection onto set C, i.e., $P_C x$ be a unique element of the set C with the property $||P_C x-x||=\min_{z\in C}||z-x||$. The following characterizations of the element $P_C x$ are useful:

$$y=P_C x\Leftrightarrow y\in C \text{ and } (y-x, z-y)\geq 0\ \ \forall\ z\in C, \tag{2}$$

$$y=P_C x\Leftrightarrow y\in C \text{ and } ||y-z||^2\leq||x-z||^2-||y-x||^2\ \ \forall\ z\in C. \tag{3}$$

From inequality (2) it follows that $x\in VI(A,C)$ if and only if $x=P_C(x-\lambda Ax)$, where $\lambda>0$ [1].

If the operator $A:H\to H$ is monotone and continuous and the set $C\subseteq H$ is convex and closed, then $x\in VI(A,C)$ if and only if $x\in C$ and $(Ay, y-x)\geq 0$ for all $y\in C$ [1]. In particular, the set $VI(A,C)$ is convex and closed.

An operator $T:H\to H$ is called quasi-non-expanding if $F(T)=\{x\in H: Tx=x\}\neq\varnothing$ and $||Tx-y||\leq||x-y||$ for all $x\in H$, $y\in F(T)$ [7, 35]. The set of fixed points $F(T)$ of a quasi-non-expanding operator is closed and convex [7, 35]. An operator $S:C\to H$ is called demiclosed in $y\in H$ if for a sequence of points $x_n\in C$ from $x_n\to x$ weakly and $Sx_n\to y$ strongly it follows that $Sx=y$ [7]. For a non-expanding operator $T:C\to H$ the operator $I-T$ is demiclosed at zero [7].

To prove the weak convergence of sequences of elements of a Hilbert space, we will use the well-known Opial lemma.

LEMMA 1 [36]. Let the sequence (x_n) of elements of the Hilbert space H weakly converge to element $x\in H$. Then for all $y\in H\setminus\{x\}$ we have $\underline{\lim}_{n\to\infty}||x_n-x||<\underline{\lim}_{n\to\infty}||x_n-y||$.

Consider the function $t\mapsto||x-P_C(x-tAx)||$, $t\in\mathbb{R}$, which has the following useful property.

LEMMA 2. For $x\in H$ and $\alpha\geq\beta>0$ the inequalities hold

$$\frac{||x-P_C(x-\alpha Ax)||}{\alpha}\leq\frac{||x-P_C(x-\beta Ax)||}{\beta},$$

$$||x-P_C(x-\beta Ax)||\leq||x-P_C(x-\alpha Ax)||.$$

Proof. Suppose $x_\alpha=P_C(x-\alpha Ax)$ and $x_\beta=P_C(x-\beta Ax)$. From (2) it follows that

$$\left(\frac{x_\alpha-x+\alpha Ax}{\alpha}, x_\beta-x_\alpha\right)\geq 0,\ \left(\frac{x_\beta-x+\beta Ax}{\beta}, x_\alpha-x_\beta\right)\geq 0.$$

Adding the inequalities yields

$$0\leq\left(\frac{x-x_\alpha}{\alpha}-\frac{x-x_\beta}{\beta}, x_\alpha-x_\beta\right)=\left(\frac{x-x_\alpha}{\alpha}-\frac{x-x_\beta}{\beta}, (x-x_\beta)-(x-x_\alpha)\right),$$

whence

$$0\leq-||x-x_\alpha||^2-\frac{\alpha}{\beta}||x-x_\beta||^2+||x-x_\alpha||\,||x-x_\beta||+\frac{\alpha}{\beta}||x-x_\alpha||\,||x-x_\beta||.$$

Hence,

$$0 \geq \left(||x - x_\alpha|| - \frac{\alpha}{\beta} ||x - x_\beta|| \right) (||x - x_\alpha|| - ||x - x_\beta||).$$

Whence it follows that

$$||x - x_\alpha|| - \frac{\alpha}{\beta} ||x - x_\beta|| \leq 0,$$

as was to be shown. ■

MODIFIED EXTRAGRADIENT ALGORITHM

To solve inequality (1), we propose the following algorithm.

Algorithm 1

Initialization. Specify the numerical parameters $\sigma > 0$, $\tau \in (0,1)$, and $\theta \in (0,1)$ and element $x_0 \in H$.

Iteration Step. For $x_n \in H$ calculate $y_n = P_C(x_n - \lambda_n A x_n)$, where λ_n is obtained from the condition

$$\begin{cases} j(n) = \min\{j \geq 0 : \sigma\tau^j ||AP_C(x_n - \sigma\tau^j A x_n) - A x_n|| \leq \theta ||P_C(x_n - \sigma\tau^j A x_n) - x_n||\}, \\ \lambda_n = \sigma\tau^{j(n)}. \end{cases} \tag{4}$$

If $y_n = x_n$, then end; otherwise calculate $x_{n+1} = P_{T_n}(x_n - \lambda_n A y_n)$, where $T_n = \{z \in H : (x_n - \lambda_n A x_n - y_n, z - y_n) \leq 0\}$.

Remark 2. Algorithm 1 is a modification of the subgradient extragradient algorithms considered in [27, 28]. The dynamic step size adjustment (4) is described in [22, 23].

It is clear that if $y_n = x_n$, then x_n belongs to the set C and is a solution of the variational inequality. Indeed, the equality $x_n = P_C(x_n - \lambda_n A x_n)$ is equivalent to the inequality

$$(x_n - x_n + \lambda_n A x_n, y - x_n) \geq 0 \ \forall\, y \in C.$$

Let us show that procedure (4) is always executed in a finite number of steps.

LEMMA 3. The rule (4) of choice of parameter λ_n is correct, i.e., $j(n) < +\infty$.

Proof. Let $x_n \in VI(A,C)$. Then $x_n = P_C(x_n - \sigma A x_n)$ and $j(n) = 0$. Consider the situation $x_n \notin VI(A,C)$ and suppose that for all $j \in \mathbb{N}$ the inequality

$$\sigma\tau^j ||AP_C(x_n - \sigma\tau^j A x_n) - A x_n|| > \theta ||P_C(x_n - \sigma\tau^j A x_n) - x_n||$$

holds, whence

$$\lim_{j\to\infty} ||P_C(x_n - \sigma\tau^j A x_n) - x_n|| = 0.$$

From the uniform continuity of the operator A on bounded sets it follows that $\lim_{j\to\infty} ||AP_C(x_n - \sigma\tau^j A x_n) - A x_n|| = 0$.

Thus,

$$\lim_{j\to\infty} \frac{||P_C(x_n - \sigma\tau^j A x_n) - x_n||}{\sigma\tau^j} = 0. \tag{5}$$

Assume $y_n^j = P_C(x_n - \sigma\tau^j A x_n)$. We have

$$\left(\frac{y_n^j - x_n}{\sigma\tau^j}, x - y_n^j \right) + (A x_n, x - y_n^j) \geq 0 \ \forall\, x \in C. \tag{6}$$

Passing to the limit in (6) and taking into account (5), we obtain that $(A x_n, x - x_n) \geq 0 \ \forall\, x \in C$, i.e., $x_n \in VI(A,C)$. We have arrived at a contradiction. ■

Remark 3. In the proof of Lemma 3, we did not use the monotonicity of the operator A.

Let us pass to the proof of the weak convergence of the algorithm.

WEAK CONVERGENCE OF ALGORITHM 1

First, let us prove an important inequality, which relates the distances from the points generated by the algorithm to the set $VI(A,C)$.

LEMMA 4. For the sequences (x_n) and (y_n) generated by the algorithm, the inequality

$$||x_{n+1}-z||^2 \le ||x_n - z||^2 - (1-\theta^2)||x_n - y_n||^2 \quad (7)$$

holds, where $z \in VI(A,C)$.

Proof. Following the same line of reasoning as in [28, proof of Lemma 3], we obtain the inequality

$$||x_{n+1}-z||^2 \le ||x_n - z||^2 - ||x_n - y_n||^2 - ||y_n - x_{n+1}||^2$$
$$+ 2\lambda_n(Ax_n - Ay_n, x_{n+1} - y_n). \quad (8)$$

We estimate the term $2\lambda_n(Ax_n - Ay_n, x_{n+1} - y_n)$ in (8) as follows:

$$2\lambda_n(Ax_n - Ay_n, x_{n+1} - y_n) \le 2\lambda_n ||Ax_n - Ay_n|| \, ||x_{n+1} - y_n||$$
$$\le 2\theta ||x_n - y_n|| \, ||x_{n+1} - y_n|| \le \theta^2 ||x_n - y_n||^2 + ||x_{n+1} - y_n||^2. \quad (9)$$

Considering estimate (9) in (8), we obtain the desired inequality (7). ■

Remark 4. Estimating the term $2\lambda_n(Ax_n - Ay_n, x_{n+1} - y_n)$ in (8) in other way, we obtain the useful inequality

$$||x_{n+1}-z||^2 \le ||x_n - z||^2 - (1-\theta)||x_n - y_n||^2 - (1-\theta)||x_{n+1} - y_n||^2, \quad (10)$$

where $z \in VI(A,C)$.

Let us now formulate one of the main results of the study.

THEOREM 1. The sequences (x_n) and (y_n), generated by Algorithm 1, weakly converge to some point $z \in VI(A,C)$.

Proof. From inequality (7) it follows that the sequence (x_n) is Fejer one with respect to the set $VI(A,C)$, i.e.,

$$||x_{n+1}-z|| \le ||x_n - z|| \ \forall n \in \mathbb{N}, \ \forall z \in VI(A,C).$$

In particular, the sequence (x_n) is bounded.

Let us fix the number $N \in \mathbb{N}$ and consider inequalities (7) for all numbers $1, 2, \ldots, N$. Adding them, we obtain

$$||x_{n+1}-z||^2 \le ||x_1 - z||^2 - (1-\theta^2)\sum_{n=1}^{N} ||x_n - y_n||^2. \quad (11)$$

Inequality (11) yields the convergence of the number series $\sum_n ||x_n - y_n||^2$. Thus,

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \quad (12)$$

Consider the subsequence (x_{n_k}) weakly converging to some point $z \in H$. Then $y_{n_k} \to z$ weakly and $z \in C$. Let us show that $z \in VI(A,C)$.

Two variants are possible: (i) the sequence (λ_{n_k}) does not tend to zero; (ii) the sequence (λ_{n_k}) tends to zero.

Consider variant (i). We may assume that $\lambda_{n_k} \ge \lambda$ for all sufficiently large k and some $\lambda > 0$. We have $(y_{n_k} - x_{n_k} + \lambda_{n_k} Ax_{n_k}, x - y_{n_k}) \ge 0 \ \forall x \in C$. Whence, using the monotonicity of operator A, we deduce the estimate

$$0 \le \frac{(y_{n_k} - x_{n_k} + \lambda_{n_k} Ax_{n_k}, x - y_{n_k})}{\lambda_{n_k}} = \frac{(y_{n_k} - x_{n_k}, x - y_{n_k})}{\lambda_{n_k}}$$
$$+ (Ax_{n_k}, x_{n_k} - y_{n_k}) + (Ax_{n_k}, x - x_{n_k})$$
$$\le \frac{(y_{n_k} - x_{n_k}, x - y_{n_k})}{\lambda_{n_k}} + (Ax_{n_k}, x_{n_k} - y_{n_k}) + (Ax, x - x_{n_k}).$$

Passing to the limit and taking into account (12), we obtain $(Ax, x-z) \geq 0 \ \forall x \in C$. Hence, $z \in VI(A, C)$.

Consider variant (ii). Let $\lim_{k\to\infty} \lambda_{n_k} = 0$. Assume $z_{n_k} = P_C(x_{n_k} - \mu_{n_k} A x_{n_k})$, where $\mu_{n_k} = \lambda_{n_k} \tau^{-1} = \sigma \tau^{j(n_k)-1} > \lambda_{n_k} > 0$. Applying Lemma 2 yields $\|x_{n_k} - z_{n_k}\| \leq \frac{1}{\tau} \|x_{n_k} - y_{n_k}\| \to 0$.

In particular, the sequence (z_{n_k}) is bounded and $z_{n_k} \to z$ weakly. From the uniform continuity of the operator A on bounded sets it follows that $\|Ax_{n_k} - Az_{n_k}\| \to 0$, and the inequality $\mu_{n_k} \|Az_{n_k} - Ax_{n_k}\| > \theta \|z_{n_k} - x_{n_k}\|$ yields the asymptotics

$$\frac{\|x_{n_k} - z_{n_k}\|}{\mu_{n_k}} \to 0. \tag{13}$$

Further, we have $(z_{n_k} - x_{n_k} + \mu_{n_k} A x_{n_k}, x - z_{n_k}) \geq 0 \ \forall x \in C$, whence we deduce the estimate

$$0 \leq \frac{(z_{n_k} - x_{n_k}, x - z_{n_k})}{\mu_{n_k}} + (Ax_{n_k}, x_{n_k} - z_{n_k}) + (Ax, x - x_{n_k}).$$

Making passage to the limit and taking into account (13), we obtain $(Ax, x-z) \geq 0 \ \forall x \in C$, whence $z \in VI(A, C)$.

Let us show that $x_n \to z$ weakly. Then from (12) it follows that $y_n \to z$ weakly. Let us reason by contradiction. Let there exist a subsequence (x_{m_k}) such that $x_{m_k} \to z'$ weakly and $z \neq z'$. It is clear that $z' \in VI(A, C)$. Let us apply Lemma 1 twice. We get

$$\lim_{n\to\infty} \|x_n - z\| = \lim_{k\to\infty} \|x_{n_k} - z\| < \lim_{k\to\infty} \|x_{n_k} - z'\| = \lim_{n\to\infty} \|x_n - z'\|$$

$$= \lim_{k\to\infty} \|x_{m_k} - z'\| < \lim_{k\to\infty} \|x_{m_k} - z\| = \lim_{n\to\infty} \|x_n - z\|,$$

which is impossible. Thus, $z = z'$. ■

Remark 5. The weak limit $z \in VI(A, C)$ of the Fejer sequence (x_n) generated by Algorithm 1 has the property $P_{VI(A,C)} x_n \to z$ strongly [7]. If the set $VI(A, C)$ is an affine manifold, then $x_n \to P_{VI(A,C)} x_0$ strongly [7].

Remark 6. The asymptotics (12) can be specified up to the following:

$$\lim_{n\to\infty} \sqrt{n} \|x_n - y_n\| = 0. \tag{14}$$

Indeed, if (14) does not hold, then $\|x_n - y_n\| \geq \mu n^{-1/2}$ for some $\mu > 0$ and all sufficiently large n. Hence, the series $\sum_n \|x_n - y_n\|^2$ diverges. We have obtained an inconsistency.

VARIANT OF ALGORITHM 1 FOR THE VARIATIONAL INEQUALITY WITH A PRIORI INFORMATION

Consider a variant of the method for solving the variational inequality (1), which is a fixed point of the given operator.

Let $S : H \to H$ be a quasi-non-expanding operator such that $I - S$ is an operator demiclosed at zero and having the set of fixed points $F(S) = \{x \in H : Sx = x\}$. Assume that $VI(A, C) \cap F(S) \neq \varnothing$.

Remark 7. Let $g : H \to \mathbb{R}$ be a convex differentiable function. If the set $D = \{x \in H : g(x) \leq 0\} \neq \varnothing$, then it can be treated as the set of fixed points of the quasi-non-expanding operator

$$Sx = \begin{cases} x - \dfrac{g(x)}{\|g'(x)\|^2} g'(x) & \text{if } x \notin D, \\ x & \text{if } x \in D, \end{cases}$$

where $g'(x) \in H$ is the derivative of g at point $x \in H$ [35]. For the operator $I - S$ to be demiclosed at zero, it will suffice that g is bounded on any bounded set [35].

To find elements of the set $VI(A,C)\cap F(S)$, consider the following algorithm.

Algorithm 2

Initialization. Specify the numerical parameters $\sigma > 0$, $\tau \in (0,1)$, $\theta \in (0,1)$, element $x_0 \in H$, and sequence $(\delta_n) \subseteq [a,b] \subseteq (0,1)$.

Iteration Step. For $x_n \in H$ calculate $y_n = P_C(x_n - \lambda_n A x_n)$, where λ_n is obtained from the condition

$$\begin{cases} j(n) = \min\{j \ge 0 : \sigma\tau^j \| AP_C(x_n - \sigma\tau^j Ax_n) - Ax_n \| \le \theta \| P_C(x_n - \sigma\tau^j Ax_n) - x_n \|\}, \\ \lambda_n = \sigma\tau^{j(n)}. \end{cases}$$

Calculate $x_{n+1} = \delta_n x_n + (1-\delta_n) SP_{T_n}(x_n - \lambda_n Ay_n)$, where $T_n = \{z \in H : (x_n - \lambda_n Ax_n - y_n, z - y_n) \le 0\}$.

Remark 8. The study [27] proposes a method to find elements of the set $VI(A,C)\cap F(S)$ for non-expanding operator S and the Lipschitz monotone operator A with constant step $\lambda \in (0, 1/L)$, where $L > 0$ is the Lipschitz constant of the operator A. Algorithm 2 is a modification of this method with dynamic step size adjustment.

The following theorem takes place.

THEOREM 2. The sequences (x_n) and (y_n) generated by Algorithm 2 weakly converge to some point $z \in VI(A,C)\cap F(S)$.

Proof. Suppose $z_n = P_{T_n}(x_n - \lambda_n Ay_n)$. Since the operator S is quasi-non-expanding, for all $z \in VI(A,C)\cap F(S)$ we obtain

$$\|x_{n+1} - z\|^2 \le \|\delta_n(x_n - z) + (1-\delta_n)(Sz_n - z)\|^2$$

$$= \delta_n \|x_n - z\|^2 + (1-\delta_n)\|Sz_n - z\|^2 - \delta_n(1-\delta_n)\|x_n - Sz_n\|^2$$

$$\le \delta_n \|x_n - z\|^2 + (1-\delta_n)\|z_n - z\|^2 - \delta_n(1-\delta_n)\|x_n - Sz_n\|^2. \quad (15)$$

Using (10) in (15) for the estimate of the term $(1-\delta_n)\|z_n - z\|^2$, we obtain the inequality

$$\|x_{n+1} - z\|^2 \le \|x_n - z\|^2 - (1-\delta_n)(1-\theta)\|x_n - y_n\|^2$$

$$-(1-\delta_n)(1-\theta)\|z_n - y_n\|^2 - \delta_n(1-\delta_n)\|x_n - Sz_n\|^2. \quad (16)$$

From inequality (16) it follows that the sequence (x_n) is Fejer with respect to the set $VI(A,C)\cap F(S)$, i.e.,

$$\|x_{n+1} - z\| \le \|x_n - z\| \ \forall n \in \mathbb{N}, \ \forall z \in VI(A,C)\cap F(S).$$

In particular, the sequence (x_n) is bounded. Moreover, the inequalities hold

$$\|x_n - y_n\|^2 + \|z_n - y_n\|^2 \le \frac{\|x_n - z\|^2 - \|x_{n+1} - z\|^2}{(1-\delta_n)(1-\theta)},$$

$$\|x_n - Sz_n\|^2 \le \frac{\|x_n - z\|^2 - \|x_{n+1} - z\|^2}{\delta_n(1-\delta_n)},$$

whence it follows

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|z_n - y_n\| = \lim_{n\to\infty} \|x_n - Sz_n\| = 0. \quad (17)$$

Consider the subsequence (x_{n_k}) weakly converging to some point $z \in H$. Then (y_{n_k}) and (z_{n_k}) weakly converge to z and $z \in C$. Following the same line of reasoning as in the proof of Theorem 1, we obtain that $z \in VI(A,C)$. It remains to show that $z \in F(S)$. Since

$$\|z_n - Sz_n\| \le \|z_n - y_n\| + \|y_n - x_n\| + \|x_n - Sz_n\|,$$

from (17) it follows that $\lim_{n\to\infty} \|z_n - Sz_n\| = 0$.

The operator $I-S$ is demiclosed at zero. Hence, from $z_{n_k} \to z$ weakly and $\lim_{k\to\infty} ||z_{n_k} - Sz_{n_k}|| = 0$ we obtain that $z \in F(S)$.

Similarly to the proof of Theorem 1, we show that $x_n \to z$ weakly. Then from (17) it follows that $y_n \to z$ weakly. ■

Let us consider the operator equation with a priori information given as the set of fixed points of the quasi-non-expanding operator $T: H \to H$:

$$Ax = f, \; x \in F(T), \tag{18}$$

where $f \in H$. Similar problems are considered in [35].

Algorithm 2 for problem (18) takes the following form.

Algorithm 3

Initialization. Specify the numerical parameters $\sigma > 0$, $\tau \in (0,1)$, $\theta \in (0,1)$, element $x_0 \in H$, and sequence $(\delta_n) \subseteq [a,b] \subseteq (0,1)$.

Iteration Step. For $x_n \in H$ calculate $y_n = x_n - \lambda_n (Ax_n - f)$, where λ_n is obtained from the condition

$$\begin{cases} j(n) = \min \{ j \geq 0: ||A(x_n - \sigma\tau^j (Ax_n - f)) - Ax_n|| \leq \theta ||Ax_n|| \}, \\ \lambda_n = \sigma\tau^{j(n)}. \end{cases}$$

Calculate

$$z_n = x_n - \lambda_n (Ay_n - f),$$

$$x_{n+1} = \delta_n x_n + (1-\delta_n) Tz_n.$$

The result below is a special case of Theorem 2.

THEOREM 3. Let operator $A: H \to H$ be monotone, uniformly continuous on bounded sets, and map bounded sets into bounded sets. Let operator $T: H \to H$ be quasi-non-expanding and operator $I-T$ be demiclosed at zero. Suppose that $VI(A,C) \cap F(T) \neq \varnothing$ for $f \in H$. Then the sequences (x_n), (y_n), and (z_n) generated by Algorithm 3 weakly converge to some point $z \in A^{-1} f \cap F(T)$.

CONCLUSIONS

In the paper, we have proposed a modified extragradient method with dynamic step size adjustment to solve variational inequalities with monotone operators acting in a Hilbert space. The operators are not supposed to be Lipschitz. We have also considered a version of the method to solve a variational inequality with a priori information described as an inclusion of a quasi-non-expanding operator in the set of fixed points. The main theoretical result is theorems about the weak convergence of the methods. Strongly converging versions of the proposed methods can be obtained by using the iterated regularization method [5, 16, 29] or the hybrid method from [37].

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