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DISTRIBUTED CONTROL WITH THE GENERAL QUADRATIC CRITERION IN A SPECIAL NORM FOR SYSTEMS DESCRIBED BY PARABOLIC–HYPERBOLIC EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. *We obtain conditions to find the distributed optimal control for parabolic-hyperbolic equations with nonlocal boundary conditions and general quadratic criterion in the special norm. The unique solvability of systems for finding the optimal solution is established, systems' kernels are estimated, and the convergence of solutions of the problem is proved.*

Keywords: *optimal control, parabolic–hyperbolic equations, nonlocal boundary conditions, distributed control.*

INTRODUCTION

Analysis of processes described by mixed-type equations is a promising field. This is because they appear in many applications to gas and electromagnetic dynamics, in mathematical biology, theory of electron scattering, etc. For example, the paper [1] considers a boundary-value problem for parabolic–hyperbolic equations with one type of nonlocal boundary conditions, [2] analyzes problems for parabolic–hyperbolic equations in multidimentional space that arise in the analysis of problems about the motion of conductive liquid in electromagnetic field.

Control problems for models with parabolic–hyperbolic equations and nonlocal pointwise conditions have not been considered earlier. In the present paper we will derive the optimality conditions to find the distributed control with general quadratic criterion in a special norm.

PROBLEM STATEMENT

Let a controlled process $y(x,t) \in C^1(\overline{D}) \cap C^2(D_-) \cap C^{2,1}(D_+)$ satisfy in D the equation

$$L\, y(x,t) = \hat{u}(x,t), \tag{1}$$

initial

$$y(x,-\alpha) = \varphi(x) \tag{2}$$

and boundary conditions

$$y(0,t) = 0,\ \ y'(0,t) = y'(1,t), \quad -\alpha \le t \le T, \tag{3}$$

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where $D=\{(x,t): 0<x<1,\ -\alpha<t\le T,\ \alpha,\ T>0\}$, $D_-=\{(x,t):\ 0<x<1,\ -\alpha<t\le 0\}$, $D_+=\{(x,t): 0<x<1,\ 0<t\le T\}$, functions $\hat u$ and φ are assumed given and their properties in smoothness will be described below,

$$Ly=\begin{cases} y_t-y_{xx},\ t>0,\\ y_{tt}-y_{xx},\ t<0.\end{cases}$$

It is required to find control $\hat u^*(x,t)\in K$ that minimizes the functional

$$I(\hat u)=0.5\left(\hat\alpha\ \|y(\cdot,T)-\psi(\cdot)\|_D^2+\hat\beta_1\int_{-\alpha}^{0}\|y(\cdot,t)\|_D^2\,dt+\hat\beta_2\int_0^T\|y(\cdot,t)\|_D^2\,dt\right.$$

$$\left.+\hat\gamma_1\int_{-\alpha}^{0}\|v(\cdot,t)\|_D^2\,dt+\hat\gamma_2\left(\|u(\cdot,0)\|_D^2+\int_0^T\|u_t(\cdot,t)\|_D^2\,dt\right)\right)$$

$$=0.5\sum_{i=0}^{\infty}\left(\hat\alpha\,(y_i(T)-\psi_i)^2+\hat\beta_1\int_{-\alpha}^{0}y_i^2(t)\,dt+\hat\beta_2\int_0^T y_i^2(t)\,dt\right.$$

$$\left.+\hat\gamma_1\int_{-\alpha}^{0}v_i^2(t)\,dt+\hat\gamma_2\left(u_i^2(0)+\int_0^T\dot u_i^2(t)\,dt\right)\right),\tag{4}$$

where $\psi(x)$ is a fixed function, $\hat\alpha,\hat\beta_i\ge 0,\ \hat\gamma_i>0,\ i=\overline{1,2};\ \hat\alpha+\hat\beta_1+\hat\beta_2>0$, and the class of functions K is specified in [3].

Problem (1)–(3), (4) is formally equivalent to the sequence of the following finite-dimensional problems:

(i) find controls $v_0^*(t)\in C[-\alpha,0),\ u_0^*(0)\in R^1,\ \xi_0^*(t)\in L_2[0,T]$ that minimize the functional

$$I_0=0.5\left(\hat\alpha\,(y_0(T)-\psi_0)^2+\hat\beta_1\int_{-\alpha}^{0}y_0^2(t)\,dt+\hat\beta_2\int_0^T y_0^2(t)\,dt\right.$$

$$\left.+\hat\gamma_1\int_{-\alpha}^{0}v_0^2(t)\,dt+\hat\gamma_2\left(u_0^2(0)+\int_0^T\xi_0^2(t)\,dt)\right)\right),\tag{5}$$

on the solutions of the boundary-value problem

$$\frac{d^2y_0(t)}{dt^2}=v_0(t),\ t\in(-\alpha,0),\ y_0(-\alpha)=\varphi_0;$$

$$\frac{dy_0(t)}{dt}=u_0(0)+\int_0^t\xi_0(\tau)\,d\tau,\ t\in(0,T];$$

$$y_0(0-)=y_0(0+),\ \ \dot y_0(0-)=\dot y_0(0+)=u_0(0);\tag{6}$$

(ii) find controls $v_i^*(t)\in C[-\alpha,0),\ u_i^*(0)\in R^1,\ \xi_i^*(t)\in L_2[0,T],\ i=\overline{2k-1,2k}$ that minimize the functional

$$I_k=0.5\sum_{i=2k-1}^{2k}\left(\hat\alpha(y_i(T)-\psi_i)^2+\hat\beta_1\int_{-\alpha}^{0}y_i^2(t)\,dt+\hat\beta_2\int_0^T y_i^2(t)\,dt\right.$$

$$\left.+\hat\gamma_1\int_{-\alpha}^{0}v_i^2(t)\,dt+\hat\gamma_2\left(u_i^2(0)+\int_0^T\xi_i^2(t)\,dt\right)\right),\tag{7}$$

on the solutions of the boundary-value problem

$$\frac{dy_{2k-1}(t)}{dt}=-\lambda_k^2\ y_{2k-1}(t)+u_{2k-1}(0)+\int_0^t \xi_{2k-1}(\tau)d\tau,\ t>0,$$

$$\frac{d^2 y_{2k-1}(t)}{dt^2}=-\lambda_k^2 y_{2k-1}(t)+v_{2k-1}(t),\ t<0,$$

$$y_{2k-1}(-\alpha)=\varphi_{2k-1}, \tag{8}$$

$$\frac{dy_{2k}(t)}{dt}=-\lambda_k^2\ y_{2k}(t)-2\lambda_k\ y_{2k-1}(t)+u_{2k}(0)+\int_0^t \xi_{2k}(\tau)d\tau,\ t>0,$$

$$\frac{d^2 y_{2k}(t)}{dt^2}=-\lambda_k^2 y_{2k}(t)-2\lambda_k y_{2k-1}(t)+v_{2k}(t),\ t<0,$$

$$y_{2k}(-\alpha)=\varphi_{2k}; \tag{9}$$

$$y_i(0-)=y_i(0+),\ i=\overline{2k-1,2k},$$

$$\dot y_{2k-1}(0-)=\dot y_{2k-1}(0+)=-\lambda_k^2 y_{2k-1}(0+)+u_{2k-1}(0),$$

$$\dot y_{2k}(0-)=\dot y_{2k}(0+)=-\lambda_k^2 y_{2k}(0+)-2\lambda_k y_{2k-1}(0+)+u_{2k}(0). \tag{10}$$

OPTIMALITY CONDITIONS

Optimality Conditions for Problem (i). Since functional (5), (6) is strictly convex in controls, it has a unique point of minimum in $C[-\alpha,0)\times R^1\times L_2(0,T)$, which is characterized by the optimality conditions [4]

$$\hat\gamma_1\ v_0(t)+\int_{-\alpha}^0 K_{0,1}^{(1)}(t,\tau)v_0(\tau)\,d\tau+K_{0,2}^{(1)}(t)\,u_0(0)+\int_0^T K_{0,3}^{(1)}(t,\tau)\xi_0(\tau)\,d\tau$$

$$=M_{0,1}^{(1)}(t)\varphi_0+M_{0,2}^{(1)}(t)\psi_0,\ t\in[-\alpha,0),$$

$$\hat\gamma_2 u_0(0)+\int_{-\alpha}^0 K_{0,1}^{(2)}(\tau)v_0(\tau)\,d\tau+K_{0,2}^{(2)}\ u_0(0)+\int_0^T K_{0,3}^{(2)}(\tau)\xi_0(\tau)\,d\tau=M_{0,1}^{(2)}\ \varphi_0+M_{0,2}^{(2)}\psi_0,$$

$$\hat\gamma_2\xi_0(t)+\int_{-\alpha}^0 K_{0,1}^{(3)}(t,\tau)v_0(\tau)\,d\tau+K_{0,2}^{(3)}(t)\,u_0(0)+\int_0^T K_{0,3}^{(3)}(t,\tau)\xi_0(\tau)\,d\tau$$

$$=M_{0,1}^{(3)}(t)\varphi_0+M_{0,2}^{(3)}(t)\psi_0,\ t\in(0,T], \tag{11}$$

where

$$\Phi_{0,+}^0(t)=1,\ V_{0,+}^0(t,\tau)=-(\alpha+\tau),\ U_{0,+}^0(t)=\alpha,\ U_{0,+}^0(t,\tau)=1,$$

$$\Phi_{0,-}^0(t)=1,\ V_{0,-}^0(t,\tau)=-(\alpha+t),\ U_{0,-}^0(t)=\alpha+t,\ V_{0,-}^0(t,\tau)=t-\tau;$$

$$K_{0,1}^{(1)}(t,\tau)=\hat{\alpha}V_{0,+}^{0}(T,t)V_{0,+}^{0}(T,\tau)+\hat{\beta}_1\left(\int_{-\alpha}^{0}V_{0,-}^{0}(\xi,t)V_{0,-}^{0}(\xi,\tau)d\xi\right.$$

$$+\int_{\tau}^{0}V_{0,-}^{0}(\xi,t)V_{0,-}^{0}(\xi,\tau)d\xi+\int_{t}^{0}V_{0,-}^{0}(\xi,t)V_{0,-}^{0}(\xi,\tau)d\xi$$

$$\left.+\begin{cases}\int_{t}^{0}V_{0,-}^{0}(\xi,t)V_{0,-}^{0}(\xi,\tau)d\xi, & \tau\le t,\\ \int_{\tau}^{0}V_{0,-}^{0}(\xi,t)V_{0,-}^{0}(\xi,\tau)d\xi, & \tau>t\end{cases}\right)+\hat{\beta}_2\int_{0}^{T}V_{0,+}^{0}(\xi,t)V_{0,+}^{0}(\xi,\tau)d\xi,$$

$$K_{0,2}^{(1)}(t)=\hat{\alpha}\left(U_{0,+}^{0}(T)+\int_{0}^{T}U_{0,+}^{0}(T,\tau)\,d\tau\right)V_{0,+}^{0}(T,t)+\hat{\beta}_1\int_{-\alpha}^{0}U_{0,-}^{0}(\tau)V_{0,-}^{0}(\tau,t)\,d\tau$$

$$+\hat{\beta}_2\int_{0}^{T}\left(U_{0,+}^{0}(\tau)+\int_{0}^{\tau}U_{0,+}^{0}(\tau,\xi)\,d\xi\right)V_{0,+}^{0}(\tau,t)\,d\tau,$$

$$K_{0,3}^{(1)}(t,\tau)=\hat{\alpha}V_{0,+}^{0}(T,t)\int_{\tau}^{T}U_{0,+}^{0}(T,\mu)\,d\mu+\hat{\beta}_2\int_{\tau}^{T}\int_{\tau}^{\xi}U_{0,+}^{0}(\xi,\mu)d\mu\,V_{0,+}^{0}(\xi,t)d\xi,$$

$$M_{0,1}^{(1)}(t)=-\hat{\alpha}\,\Phi_{0,+}^{0}(T)V_{0,+}^{0}(T,t)-\hat{\beta}_1\int_{-\alpha}^{0}\Phi_{0,-}^{0}(\xi)V_{0,-}^{0}(\xi,t)d\xi$$

$$-\hat{\beta}_2\int_{0}^{T}\Phi_{0,+}^{0}(\xi)V_{0,+}^{0}(\xi,t)d\xi,\ M_{0,2}^{(1)}(t)=\hat{\alpha}V_{0,+}^{0}(T,t);\ K_{0,1}^{(2)}(t)=K_{0,2}^{(1)}(t),$$

$$K_{0,2}^{(2)}=\hat{\alpha}\left(U_{0,+}^{0}(T)+\int_{0}^{T}U_{0,+}^{0}(T,\tau)\,d\tau\right)^2+\hat{\beta}_1\int_{-\alpha}^{0}(U_{0,-}^{0}(\xi))^2\,d\xi+\hat{\beta}_2\int_{0}^{T}\left(U_{0,+}^{0}(\xi)+\int_{0}^{\xi}U_{0,+}^{0}(\xi,\tau)\,d\tau\right)^2\,d\xi,$$

$$K_{0,3}^{(2)}(t)=\hat{\alpha}\left(U_{0,+}^{0}(T)+\int_{0}^{T}U_{0,+}^{0}(T,\tau)d\tau\right)\int_{t}^{T}U_{0,+}^{0}(T,\tau)d\tau$$

$$+\hat{\beta}_2\int_{0}^{T}\left(U_{0,+}^{0}(\xi)+\int_{0}^{\xi}U_{0,+}^{0}(\xi,\tau)d\tau\right)\int_{t}^{\xi}U_{0,+}^{0}(\xi,\mu)d\mu\,d\xi,$$

$$M_{0,1}^{(2)}=-\hat{\alpha}\left(U_{0,+}^{0}(T)+\int_{0}^{T}U_{0,+}^{0}(T,\tau)\,d\tau\right)\Phi_{0,+}^{0}(T)-\hat{\beta}_1\int_{-\alpha}^{0}U_{0,-}^{0}(\xi)\,\Phi_{0,-}^{0}(\xi)d\xi$$

$$-\hat{\beta}_2\int_{0}^{T}\left(U_{0,+}^{0}(\xi)+\int_{0}^{\xi}U_{0,+}^{0}(\xi,\tau)\,d\tau\right)\Phi_{0,+}^{0}(\xi)d\xi,$$

$$M_{0,2}^{(2)}=\hat{\alpha}\left(U_{0,+}^{0}(T)+\int_{0}^{T}U_{0,+}^{0}(T,\tau)\,d\tau\right);$$

$$K_{0,1}^{(3)}(t,\tau)=K_{0,3}^{(1)}(\tau,t), K_{0,2}^{(3)}(t)=K_{0,3}^{(2)}(t),$$

$$K_{0,3}^{(3)}(t,\tau)=\hat{\alpha}\int_t^T U_{0,+}^0(T,\mu)d\mu\int_\tau^T U_{0,+}^0(T,\xi)d\xi+\hat{\beta}_2\begin{cases}\int_t^T\int_\tau^\xi U_{0,+}^0(\xi,\mu)d\mu\int_t^\xi U_{0,+}^0(\xi,\nu)d\nu d\xi, \ \tau\le t,\\ \int_\tau^T\int_\tau^\xi U_{0,+}^0(\xi,\mu)d\mu\int_t^\xi U_{0,+}^0(\xi,\nu)d\nu d\xi, \ \tau> t,\end{cases}$$

$$M_{0,1}^{(3)}(t)=-\hat{\alpha}\,\Phi_{0,+}^0(T)\int_t^T U_{0,+}^0(T,\mu)\,d\mu-\hat{\beta}_2\int_t^T\Phi_{0,+}^0(\tau)\int_t^\tau U_{0,+}^0(\tau,\mu)d\mu d\tau,$$

$$M_{0,2}^{(3)}(t)=\hat{\alpha}\int_t^T U_{0,+}^0(T,\mu)\,d\mu.$$

Let us establish the unique solvability of system (11) by defining the operator $A_0\ \theta_0(\cdot)=\Gamma_{3\times3}\theta_0(t)+A_0\ \theta_0(\cdot)$, where $(\theta_0(t))'=(v_0(t),u_0(0),\xi_0(t))\in L_2(-\alpha,0)\times R^1\times L_2(0,T)$, $\Gamma_{3\times3}=\mathrm{diag}\{\hat{\gamma}_1,\hat{\gamma}_2,\hat{\gamma}_2\}$, operator A_0 is defined by the remaining terms in the left-hand sides of the system of equations (11).

It is obvious that operator A_0 acts from $L_2(-\alpha,0)\times R^1\times L_2(0,T)$ into $L_2(-\alpha,0)\times R^1\times L_2(0,T)$, is linear and continuous. The following lemma is true.

LEMMA 1. System (11) has a unique solution in the space $C(-\alpha,0)\times R^1\times L_2(0,T)$.

Proof. The space $L_2(-\alpha,0)\times R^1\times L_2(0,T)$ is Hilbert with the natural scalar product

$$\langle\theta_0,\widetilde{\theta}_0\rangle_3=\int_{-\alpha}^0 v_0(t)\widetilde{v}_0(t)\,dt+u_0(0)\,\widetilde{u}_0(0)+\int_0^T\xi_0(t)\,\widetilde{\xi}_0(t)\,dt,$$

where $(\theta_0(t))'=(v_0(t),u_0(0),\xi_0(t))$, $(\widetilde{\theta}_0(t))'=(\widetilde{v}_0(t),\widetilde{u}_0(0),\widetilde{\xi}_0(t))$.

In functional (5), let us select the part, quadratic in controls $v_0(t)$, $t\in[-\alpha,0)$ and $u_0(0),\xi_0(t)\in[0,T]$, and subtract

$$0.5\left(\hat{\gamma}_1\int_{-\alpha}^0 v_0^2(t)\,dt+\hat{\gamma}_2\left(u_0^2(0)+\int_0^T\xi_0^2(t)\,dt\right)\right)$$

from it, i.e., consider the functional

$$\widetilde{I}_0=0.5\Bigg[\hat{\alpha}\left(\int_{-\alpha}^0 V_{0,+}^0(T,\tau)\,v_0(\tau)\,d\tau+\left(U_{0,+}^0(T)+\int_0^T U_{0,+}^0(T,\mu)\,d\mu\right)u_0(0)\right.$$

$$\left.+\int_0^T\int_\tau^T U_{0,+}^0(T,\mu)d\mu\,\xi_0(\tau)\,d\tau\right)^2+\hat{\beta}_1\int_{-\alpha}^0\left(\int_{-\alpha}^0 V_{0,-}^0(t,\tau)\,v_0(\tau)\,d\tau+U_{0,-}^0(t)\,u_0(0)\right.$$

$$\left.+\int_{-\alpha}^t V_{0,-}^0(t,\tau)v_0(\tau)\,d\tau\right)^2dt+\hat{\beta}_2\int_0^T\left(\int_{-\alpha}^0 V_{0,+}^0(t,\tau)\,v_0(\tau)\,d\tau\right.$$

$$\left.+\left(U_{0,+}^0(t)+\int_0^t U_{0,+}^0(t,\mu)\,d\mu\right)u_0(0)+\int_0^t\int_\tau^t U_{0,+}^0(t,\mu)d\mu\,\xi_0(\tau)\,d\tau\right)^2dt\Bigg].$$

It is clear that $\widetilde{I}_0 \geq 0$. Now, let us scalarly multiply the value of the operator $A_0 \theta_0(\cdot)$ by $\theta_0(t)$, i.e., consider the quadratic form

$$\Pi_0 = \langle A_0\ \theta_0(\cdot), \theta_0(\cdot) \rangle\!\rangle_3 = \int_{-\alpha}^{0} \int_{-\alpha}^{0} K_{0,1}^{(1)}(t,\tau)\, v_0(\tau)\, d\tau v_0(t)\, dt + \int_{-\alpha}^{0} K_{0,2}^{(1)}(t)\, v_0(t)\, dt$$

$$\times u_0(0) + \int_{-\alpha}^{0} \int_{0}^{T} K_{0,3}^{(1)}(t,\tau)\, \xi_0(\tau)\, d\tau\, v_0(t)\, dt + \int_{-\alpha}^{0} K_{0,1}^{(2)}(\tau)\, v_0(\tau)\, d\tau u_0(0)$$

$$+ K_{0,2}^{(2)}\, u_0^2(0) + \int_{0}^{T} K_{0,3}^{(2)}(\tau)\, \xi_0(\tau)\, d\tau\, u_0(0) + \int_{0}^{T} \int_{-\alpha}^{0} K_{0,1}^{(3)}(t,\tau)\, v_0(\tau)\, d\tau\, \xi_0(t)\, dt$$

$$+ \int_{0}^{T} K_{0,2}^{(3)}(t)\, \xi_0(t)\, dt\, u_0(0) + \int_{0}^{T} \int_{0}^{T} K_{0,3}^{(3)}(t,\tau)\, \xi_0(\tau)\, d\tau\, \xi_0(t)\, dt.$$

Substituting the explicit form of kernels $K_{0,i}^{(j)},\ i, j = \overline{1,3}$, into the quadratic form Π_0, we obtain the equality $\Pi_0 = 2\widetilde{I}_0$. This yields the positive definiteness of the operator A_0 and the unique solvability of system (11) in the space $L_2(-\alpha, 0) \times R^1 \times L_2(0,T)$. From the first equation of this system we find $|v_0(t_1) - v_0(t_2)| < C\,|t_1 - t_2|$ for any $t_1, t_2 \in [-\alpha, 0]$.

Optimality Conditions for Problem (ii). Due to the strict convexity of functional (7)–(10), this problem has a unique solution from the space $(C(-\alpha,0) \times R^1 \times L_2(0,T))^2$, which is characterized by the necessary and sufficient optimality conditions [4]

$$\hat{\gamma}_1\, v_i(t) + \sum_{j=2k-1}^{2k} \left(\int_{-\alpha}^{0} K_{j,1}^{(1,i)}(t,\tau) v_j(\tau)\, d\tau + K_{j,2}^{(1,i)}(t)\, u_j(0) \right.$$

$$\left. + \int_{0}^{T} K_{j,3}^{(1,i)}(t,\tau) \xi_j(\tau)\, d\tau \right) = \sum_{j=2k-1}^{2k} (M_{j,1}^{(1,i)}(t)\, \varphi_j + M_{j,2}^{(1,i)}(t) \psi_j),\ \ t \in [-\alpha, 0),$$

$$\hat{\gamma}_2\, u_i(0) + \sum_{j=2k-1}^{2k} \left(\int_{-\alpha}^{0} K_{j,1}^{(2,i)}(\tau) v_j(\tau)\, d\tau + K_{j,2}^{(2,i)}\, u_j(0) + \int_{0}^{T} K_{j,3}^{(2,i)}(\tau)\, \xi_j(\tau)\, d\tau \right)$$

$$= \sum_{j=2k-1}^{2k} (M_{j,1}^{(2,i)} \varphi_j + M_{j,2}^{(2,i)}\, \psi_j),$$

$$\hat{\gamma}_2\, \xi_i(t) + \sum_{j=2k-1}^{2k} \left(\int_{-\alpha}^{0} K_{j,1}^{(3,i)}(t,\tau) v_j(\tau)\, d\tau + K_{j,2}^{(3,i)}(t)\, u_j(0) \right.$$

$$\left. + \int_{0}^{T} K_{j,3}^{(3,i)}(t,\tau) \xi_j(\tau)\, d\tau \right) = \sum_{j=2k-1}^{2k} (M_{j,1}^{(3,i)}(t)\, \varphi_j + M_{j,2}^{(3,i)}(t) \psi_j),\ \ t \in (0,T],\ i = \overline{2k-1, 2k}, \tag{12}$$

whose kernels are obtained similarly to problem (i) from solutions of the boundary-value problems (8)–(10) from [3].

Let us establish the unique solvability of system (12) by defining the operator

$$A_k\ \theta_k(\cdot) = \Gamma_{6\times 6} \theta_k(t) + A_k\ \theta_k(\cdot),$$

where $(\theta_k(t))' = (v_{2k-1}(t), v_{2k}(t), u_{2k-1}(0), u_{2k}(0), \xi_{2k-1}(t), \xi_{2k}(t)) \in (L_2(-\alpha,0) \times R^1 \times L_2(0,T))^2$, $\Gamma_{6\times 6} = \mathrm{diag}\,\{\hat{\gamma}_1, \hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_2, \hat{\gamma}_2, \hat{\gamma}_2\}$, operator A_k is defined by the remaining terms in the left-hand sides of the system of equations (12).

It is obvious that operator A_k acts from the space $(L_2(-\alpha,0)\times R^1\times L_2(0,T))^2$ into the space $(L_2(-\alpha,0)\times R^1\times L_2(0,T))^2$ and is linear and continuous. The following lemma is true.

LEMMA 2. System (12) has a unique solution in the space $(C(-\alpha,0)\times R^1\times L_2(0,T))^2$.

The proof coincides with the proof of Lemma 1 where system (11) is replaced with system (12) and the space $L_2(-\alpha,0)\times R^1\times L_2(0,T)$ with the space $(L_2(-\alpha,0)\times R^1\times L_2(0,T))^2$ with the scalar product

$$\langle\theta_k,\widetilde{\theta}_k\rangle_6=\sum_{i=2k-1}^{2k}\left(\int_{-\alpha}^{0}v_i(t)\widetilde{v}_i(t)\,dt+u_i(0)\,\widetilde{u}_i(0)+\int_0^T\xi_i(t)\widetilde{\xi}_i(t)\,dt\right),$$

where

$$(\theta_k(t))'=(v_{2k-1}(t),v_{2k}(t),u_{2k-1}(0),u_{2k}(0),\xi_{2k-1}(t),\xi_{2k}(t)),(\widetilde{\theta}_k(t))'$$

$$=(\widetilde{v}_{2k-1}(t),\widetilde{v}_{2k}(t),\widetilde{u}_{2k-1}(0),\widetilde{u}_{2k}(0),\widetilde{\xi}_{2k-1}(t),\widetilde{\xi}_{2k}(t)).$$

SUBSTANTIATION OF THE RESULTS

To substantiate the optimality conditions obtained above, the controls found from them should satisfy conditions from [3]. To test them, we will establish the following estimates.

First, let us estimate the components of the system of equations (12):

$$\|K_{2k-1,1}^{(1,2k-1)}\|_{C(-\alpha,0)\times C(-\alpha,0)}\le\frac{C_1\,\hat{\alpha}}{\lambda_k^2\exp(2\lambda_k^2T)}+\frac{C_2\hat{\beta}_1}{\lambda_k^2}+\frac{C_3\hat{\beta}_2}{\lambda_k^4},$$

$$\|K_{2k,1}^{(1,2k-1)}\|_{C(-\alpha,0)\times C(-\alpha,0)}=\|K_{2k-1,1}^{(1,2k)}\|_{C(-\alpha,0)\times C(-\alpha,0)}\le\frac{C_1\,\hat{\alpha}}{\lambda_k^3\exp(2\lambda_k^2T)}+\frac{C_2\hat{\beta}_1}{\lambda_k^2}+\frac{C_3\hat{\beta}_2}{\lambda_k^5},$$

$$\|K_{2k-1,2}^{(1,2k-1)}\|_{C(-\alpha,0)}=\|K_{2k-1,1}^{(2,2k-1)}\|_{C(-\alpha,0)}\le\frac{C_1\hat{\alpha}}{\lambda_k^2\exp(\lambda_k^2T)}+\frac{C_2\,\hat{\beta}_1}{\lambda_k^3}+\frac{C_3\hat{\beta}_2}{\lambda_k^4},$$

$$\|K_{2k,2}^{(1,2k-1)}\|_{C(-\alpha,0)}=\|K_{2k-1,1}^{(2,2k)}\|_{C(-\alpha,0)}\le\frac{C_1\hat{\alpha}}{\lambda_k^3\exp(\lambda_k^2T)}+\frac{C_2\,\hat{\beta}_1}{\lambda_k^3}+\frac{C_3\,\hat{\beta}_2}{\lambda_k^5},$$

$$\|K_{2k-1,3}^{(1,2k-1)}\|_{C(-\alpha,0)\times C(0,T)}=\|K_{2k-1,1}^{(3,2k-1)}\|_{C(-\alpha,0)\times C(0,T)}\le\frac{C_1\,\hat{\alpha}}{\lambda_k^2\exp(\lambda_k^2T)}+\frac{C_3\,\hat{\beta}_2}{\lambda_k^4},$$

$$\|K_{2k,3}^{(1,2k-1)}\|_{C(-\alpha,0)\times C(0,T)}=\|K_{2k-1,1}^{(3,2k)}\|_{C(-\alpha,0)\times C(0,T)}\le\frac{C_1\,\hat{\alpha}}{\lambda_k^3\exp(\lambda_k^2T)}+\frac{C_3\,\hat{\beta}_2}{\lambda_k^5},$$

$$\|M_{2k-1,1}^{(1,2k-1)}\|_{C(-\alpha,0)}\le\frac{C_1\,\hat{\alpha}}{\lambda_k\exp(2\lambda_k^2T)}+\frac{C_2\,\hat{\beta}_1}{\lambda_k}+\frac{C_3\hat{\beta}_2}{\lambda_k^3},$$

$$\|M_{2k,1}^{(1,2k-1)}\|_{C(-\alpha,0)}\le\frac{C_1\hat{\alpha}}{\lambda_k^2\exp(2\lambda_k^2T)}+\frac{C_2\,\hat{\beta}_1}{\lambda_k}+\frac{C_3\hat{\beta}_2}{\lambda_k^4},$$

$$\|M_{2k-1,2}^{(1,2k-1)}\|_{C(-\alpha,0)},\ \|M_{2k,2}^{(1,2k-1)}\|_{C(-\alpha,0)}\le\frac{C_1\hat{\alpha}}{\lambda_k\exp(\lambda_k^2T)};$$

$$\| K_{2k,1}^{(1,2k)} \|_{C(-\alpha,0)\times C(-\alpha,0)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^4 \exp(2\lambda_k^2 T)} + \frac{C_2 \hat{\beta}_1}{\lambda_k^2} + \frac{C_3 \hat{\beta}_2}{\lambda_k^6},$$

$$\| K_{2k-1,2}^{(1,2k)} \|_{C(-\alpha,0)} = \| K_{2k,1}^{(2,2k-1)} \|_{C(-\alpha,0)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^3 \exp(\lambda_k^2 T)} + \frac{C_2 \hat{\beta}_1}{\lambda_k^3} + \frac{C_3 \hat{\beta}_2}{\lambda_k^5},$$

$$\| K_{2k,2}^{(1,2k)} \|_{C(-\alpha,0)} = \| K_{2k,1}^{(2,2k)} \|_{C(-\alpha,0)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^4 \exp(\lambda_k^2 T)} + \frac{C_2 \hat{\beta}_1}{\lambda_k^3} + \frac{C_3 \hat{\beta}_2}{\lambda_k^6},$$

$$\| K_{2k-1,3}^{(1,2k)} \|_{C(-\alpha,0)\times C(0,T)} = \| K_{2k,1}^{(3,2k-1)} \|_{C(-\alpha,0)\times C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^3 \exp(\lambda_k^2 T)} + \frac{C_3 \hat{\beta}_2}{\lambda_k^5},$$

$$\| K_{2k,3}^{(1,2k)} \|_{C(-\alpha,0)\times C(0,T)} = \| K_{2k,1}^{(3,2k)} \|_{C(-\alpha,0)\times C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^4 \exp(\lambda_k^2 T)} + \frac{C_3 \hat{\beta}_2}{\lambda_k^6},$$

$$\| M_{2k-1,1}^{(1,2k)} \|_{C(-\alpha,0)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^2 \exp(2\lambda_k^2 T)} + \frac{C_2 \hat{\beta}_1}{\lambda_k} + \frac{C_3 \hat{\beta}_2}{\lambda_k^4},$$

$$\| M_{2k,1}^{(1,2k)} \|_{C(-\alpha,0)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^3 \exp(2\lambda_k^2 T)} + \frac{C_2 \hat{\beta}_1}{\lambda_k} + \frac{C_3 \hat{\beta}_2}{\lambda_k^5}, \quad \| M_{2k,2}^{(1,2k)} \|_{C(-\alpha,0)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^2 \exp(\lambda_k^2 T)};$$

$$| K_{2k-1,2}^{(2,2k-1)} | \le \frac{C_1 \hat{\alpha}}{\lambda_k^2} + \frac{C_2 \hat{\beta}_1}{\lambda_k^4} + \frac{C_3 \hat{\beta}_2}{\lambda_k^2}, \quad | K_{2k,2}^{(2,2k-1)} | = | K_{2k-1,2}^{(2,2k)} | \le \frac{C_1 \hat{\alpha}}{\lambda_k^3} + \frac{C_2 \hat{\beta}_1}{\lambda_k^4} + \frac{C_3 \hat{\beta}_2}{\lambda_k^5},$$

$$\| K_{2k-1,3}^{(2,2k-1)} \|_{C(0,T)} = \| K_{2k-1,2}^{(3,2k-1)} \|_{C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^2} + \frac{C_3 \hat{\beta}_2}{\lambda_k^2},$$

$$\| K_{2k,3}^{(2,2k-1)} \|_{C(0,T)} = \| K_{2k-1,1}^{(3,2k)} \|_{C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^3} + \frac{C_3 \hat{\beta}_2}{\lambda_k^3},$$

$$| M_{2k-1,1}^{(2,2k-1)} | \le \frac{C_1 \hat{\alpha}}{\lambda_k \exp(\lambda_k^2 T)} + \frac{C_2 \hat{\beta}_1}{\lambda_k^2} + \frac{C_3 \hat{\beta}_2}{\lambda_k^3}, \quad | M_{2k,1}^{(2,2k-1)} | \le \frac{C_1 \hat{\alpha}}{\lambda_k^2 \exp(\lambda_k^2 T)} + \frac{C_2 \hat{\beta}_1}{\lambda_k^2} + \frac{C_3 \hat{\beta}_2}{\lambda_k^4},$$

$$| M_{2k-1,2}^{(2,2k-1)} | \le \frac{C_1 \hat{\alpha}}{\lambda_k^2}, \quad | M_{2k,2}^{(2,2k-1)} | \le \frac{C_1 \hat{\alpha}}{\lambda_k};$$

$$| K_{2k,2}^{(2,2k)} | \le \frac{C_1 \hat{\alpha}}{\lambda_k^4} + \frac{C_2 \hat{\beta}_1}{\lambda_k^4} + \frac{C_3 \hat{\beta}_2}{\lambda_k^4}, \quad \| K_{2k-1,3}^{(2,2k)} \|_{C(0,T)} = \| K_{2k,2}^{(3,2k-1)} \|_{C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^3} + \frac{C_3 \hat{\beta}_2}{\lambda_k^3},$$

$$\| K_{2k,3}^{(2,2k)} \|_{C(0,T)} = \| K_{2k,2}^{(3,2k)} \|_{C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^4} + \frac{C_3 \hat{\beta}_2}{\lambda_k^4}, \quad | M_{2k-1,1}^{(2,2k)} | \le \frac{C_1 \hat{\alpha}}{\lambda_k^2 \exp(\lambda_k^2 T)} + \frac{C_2 \hat{\beta}_1}{\lambda_k^2} + \frac{C_3 \hat{\beta}_2}{\lambda_k^4},$$

$$| M_{2k,1}^{(2,2k)} | \le \frac{C_1 \hat{\alpha}}{\lambda_k^3 \exp(\lambda_k^2 T)} + \frac{C_2 \hat{\beta}_1}{\lambda_k^2} + \frac{C_3 \hat{\beta}_2}{\lambda_k^5}, \quad | M_{2k,2}^{(2,2k)} | \le \frac{C_1 \hat{\alpha}}{\lambda_k^2};$$

$$\| K_{2k-1,3}^{(3,2k-1)} \|_{C(0,T)\times C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^2} + \frac{C_3 \hat{\beta}_2}{\lambda_k^2}, \quad \| K_{2k,3}^{(3,2k-1)} \|_{C(0,T)\times C(0,T)} = \| K_{2k-1,3}^{(3,2k)} \|_{C(0,T)\times C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^3} + \frac{C_3 \hat{\beta}_2}{\lambda_k^3},$$

$$\| M^{(3,2k-1)}_{2k-1,1} \|_{C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k \exp(\lambda_k^2 T)} + \frac{C_3 \hat{\beta}_2}{\lambda_k^3}, \ \| M^{(3,2k-1)}_{2k,1} \|_{C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^2 \exp(\lambda_k^2 T)} + \frac{C_3 \hat{\beta}_2}{\lambda_k^4},$$

$$\| M^{(3,2k-1)}_{2k-1,2} \|_{C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^2}, \ \| M^{(3,2k-1)}_{2k,2} \|_{C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k};$$

$$\| K^{(3,2k)}_{2k,3} \|_{C(0,T)\times C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^4} + \frac{C_3 \hat{\beta}_2}{\lambda_k^4}, \ \| M^{(3,2k)}_{2k-1,1} \|_{C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^2 \exp(\lambda_k^2 T)} + \frac{C_3 \hat{\beta}_2}{\lambda_k^4},$$

$$\| M^{(3,2k)}_{2k,1} \|_{C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^3 \exp(\lambda_k^2 T)} + \frac{C_3 \hat{\beta}_2}{\lambda_k^5}, \ \| M^{(3,2k)}_{2k,2} \|_{C(0,T)} \le \frac{C_1 \hat{\alpha}}{\lambda_k^2}. \tag{13}$$

From the positive definiteness of the operator A_k and estimates (13), we obtain

$$\|\theta_k\|_6 \le C \sum_{j=2k-1}^{2k} \Bigg[\sum_{i=2k-1}^{2k} (\| M^{(1,i)}_{j,1} \|_{C(-\alpha,0)} + |M^{(2,i)}_{j,1}| + \| M^{(3,i)}_{j,1} \|_{C(0,T)}) |\varphi_j|$$

$$+ \sum_{i=2k-1}^{2k} (\| M^{(1,i)}_{j,2} \|_{C(-\alpha,0)} + |M^{(2,i)}_{j,2}| + \| M^{(3,i)}_{j,2} \|_{C(0,T)}) |\psi_j|) \Bigg]$$

$$\le C \left[\left(\frac{\hat{\alpha}}{\lambda_k \exp \lambda_k^2 T} + \frac{\hat{\beta}_1}{\lambda_k} + \frac{\hat{\beta}_2}{\lambda_k^3} \right) |\varphi_{2k-1}| + \left(\frac{\hat{\alpha}}{\lambda_k^2 \exp \lambda_k^2 T} + \frac{\hat{\beta}_1}{\lambda_k} + \frac{\hat{\beta}_2}{\lambda_k^4} \right) |\varphi_{2k}| + \hat{\alpha} \left(\frac{|\psi_{2k-1}|}{\lambda_k^2} + \frac{|\psi_{2k}|}{\lambda_k} \right) \right]. \tag{14}$$

When deriving estimate (14) $\forall k > 0$, we did not take into account the terms $\langle \theta_k, A_k \theta_k \rangle_6 \ge 0$ in case of the boundedness of components of the vector $\theta_k(\cdot)$. Such an operation is correct if the series $\sum_{k=1}^{\infty} \langle \theta_k, A_k \theta_k \rangle_6$ converges.

Indeed, from estimates (13) we obtain

$$\sum_{k=1}^{\infty} \langle \theta_k, A_k \theta_k \rangle_6 \le C \sum_{k=1}^{\infty} \sum_{j,i=2k-1}^{2k} (\| K^{(1,i)}_{j,1} \|_{C(-\alpha,0)\times C(-\alpha,0)} + \| K^{(1,i)}_{j,2} \|_{C(-\alpha,0)}$$

$$+ \| K^{(1,i)}_{j,3} \|_{C(-\alpha,0)\times C(0,T)} + \| K^{(2,i)}_{j,1} \|_{C(-\alpha,0)} + |K^{(2,i)}_{j,2}| + \| K^{(2,i)}_{j,3} \|_{C(0,T)}$$

$$+ \| K^{(3,i)}_{j,1} \|_{C(0,T)\times C(-\alpha,0)} + \| K^{(3,i)}_{j,2} \|_{C(0,T)} + \| K^{(3,i)}_{j,3} \|_{C(0,T)\times C(0,T)}) \le C(\hat{\alpha} + \hat{\beta}_1 + \hat{\beta}_2) \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} < \infty.$$

From the first two equations of system (12) we get

$$\| \upsilon_i \|_{C(-\alpha,0)} \le C \sum_{j=2k-1}^{2k} (\| K^{(1,i)}_{j,1} \|_{C(-\alpha,0)\times C(-\alpha,0)} + \| K^{(1,i)}_{j,2} \|_{C(-\alpha,0)}$$

$$+ \| K^{(1,i)}_{j,3} \|_{C(-\alpha,0)\times C(0,T)}) \|\theta_k\|_6 + \sum_{j=2k-1}^{2k} (\| M^{(1,i)}_{j,1} \|_{C(-\alpha,0)} |\varphi_j| + \| M^{(1,i)}_{j,2} \|_{C(-\alpha,0)} |\psi_j|), \ i = \overline{2k-1, 2k}.$$

From here and from estimates (14) and (13) we obtain the inequalities

$$\| \upsilon_i \|_{C(-\alpha,0)} \le C \Bigg[\left(\frac{\hat{\alpha}}{\lambda_k \exp(\lambda_k^2 T)} + \frac{\hat{\beta}_1}{\lambda_k} + \frac{\hat{\beta}_2}{\lambda_k^3} \right) |\varphi_{2k-1}|$$

$$+ \left(\frac{\hat{\alpha}}{\lambda_k^2 \exp(\lambda_k^2 T)} + \frac{\hat{\beta}_1}{\lambda_k} + \frac{\hat{\beta}_2}{\lambda_k^4} \right) |\varphi_{2k}| + \hat{\alpha} \left(\frac{|\psi_{2k-1}|}{\lambda_k^4} + \frac{|\psi_{2k}|}{\lambda_k^3} \right) \Bigg], \ i = \overline{2k-1, 2k}.$$

Components $|u_i(0)|$, $\|\xi_i\|_{L_2(0,T)}$, $i=\overline{2k-1,2k}$, of the vector $\theta_k(\cdot)$ satisfy inequality (14). Since $\|u_i\|_{C(0,T)} \le |u_i(0)| + T\|\xi_i\|_{L_2(0,T)}$, $i=\overline{2k-1,2k}$, for $\|u_i\|_{C(0,T)}$, $i=\overline{2k-1,2k}$, estimate (14) is true. Thus, the following theorem has been proved.

THEOREM 1. Let functions $\varphi(x)$ and $\psi(x)$ in the optimal control problem (1)–(3), (4) satisfy the conditions

$$\sum_{k=1}^{\infty} \lambda_k^2 \, (|\varphi_{2k-1}| + |\varphi_{2k}|) < \infty, \quad \sum_{k=1}^{\infty} \left(\frac{|\psi_{2k-1}|}{\lambda_k} + |\psi_{2k}| \right) < \infty.$$

Then the continuous functions

$$v(x,t) = v_0(t)\, X_0(x) + \sum_{k=1}^{\infty} (v_{2k-1}(t)\, X_{2k-1}(x) + v_{2k}(t)\, X_{2k}(x)),$$

$$u(x,t) = u(x,0) + \int_0^t \xi(x,\tau)\, d\tau,$$

where

$$u(x,0) = u_0(0) X_0(x) + \sum_{k=1}^{\infty} (u_{2k-1}(0)\, X_{2k-1}(x) + u_{2k}(0)\, X_{2k}(x)),$$

$$\xi(x,t) = \xi_0(t)\, X_0(x) + \sum_{k=1}^{\infty} (\xi_{2k-1}(t)\, X_{2k-1}(x) + \xi_{2k}(t)\, X_{2k}(x)),$$

the coefficients of these representations are defined as solutions of the systems of equations (11) and (12), respectively, are optimal controls in the problem (1)–(3), (4).

CONCLUSIONS

In the paper, we have derived the conditions to find the distributed optimal control for the parabolic–hyperbolic equation with nonlocal boundary conditions and general quadratic performance criterion in a special norm. For the constructed control, we have proved the lemmas about the uniqueness and the theorem about the existence of the found solution.

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