DOI 10.1007/s10559-014-9668-7
Cybernetics and Systems Analysis, Vol. 50, No. 5, September, 2014

STOCHASTIC OPTIMAL CONTROL OF RISK PROCESSES WITH LIPSCHITZ PAYOFF FUNCTIONS

B. V. Norkin

UDC 519.21

Abstract. *This paper studies the stochastic optimal control problem of finding optimal dividend policies of an insurance company in discrete time with the use of general Lipschitz payoff functions involving indicators of profitability and risk. To construct positional optimal controls and to evaluate the performance indicators, the dynamic programming method is validated. The convergence rate of the successive approximation method in finding generally unbounded Bellman functions is estimated. The Pareto-optimal set of the problem is numerically approximated by so-called barrier-proportional control strategies.*

Keywords: *risk process, insurance, stochastic optimal control, discrete time, Lipschitz payoff function, optimal dividend policy, dynamic programming, successive approximation method, Pareto optimality, barrier-proportional strategy.*

INTRODUCTION

This work investigates the optimal control problem of finding optimal dividend policies of an insurance company in discrete time with the use of general Lipschitz payoff functions involving profitability and risk indicators. The functioning of such an insurance company is described by a discrete-time stochastic risk process with dividend subtraction. Conceptually, this problem is considered as the following two-criterion problem: maximize the expected profitability indicator (the average total discounted dividends) and minimize the risk (probability) of ruin over a given planning horizon. The objective of this work consists of the construction of Pareto-optimal boundaries in the space of the control quality indicators "profitability–risk."

In a one-criterion statement of the problem on an infinite time interval, the problem of finding optimal dividends was studied in [1–4] and in other works (reviews [5, 6]). In the capacity of the main optimization criterium, average total discounted dividends were used. In particular, the following paradox (de Finetti paradox) is described in [1]: using an optimal control strategy (on an infinite time interval), an insurance company ruins itself with probability one (see also a discussion of the paradox in [2–6]). This result shows that this problem statement is not completely satisfactory. It is desirable to find an efficient dividend strategy under which the probability of ruin is small. Note that a ruin can occur in the very distant future and, hence, it is expedient to consider the process on a finite time interval. Moreover, it is important to explicitly take into account risk indicators in the problem statement.

Special cases of the problem of optimization of a dividend strategy with allowance for risk indicators (the probability of ruin, lifetime of a process, or final state of a process) were studied in [8–17]. In [18–20], this problem was solved by the optimization of the aggregation of profitability and risk indicators. However, in the general case, the problem of multicriterion optimization of dividend strategies for various profitability and risk criteria neither theoretically nor practically (numerically) is not completely solved. The difficulty arises from the fact that the optimal value of the functional of the problem is not a uniformly bounded above function on the set of initial states of the risk process. This does not allow one to use the standard

V. M. Glushkov Institute of Cybernetics, National Academy of Sciences of Ukraine, Kyiv, Ukraine, *bogdan.norkin@gmail.com.* Translated from Kibernetika i Sistemnyi Analiz, No. 5, pp. 139–154, September–October, 2014. Original article submitted December 20, 2013.

sup-norm for Bellman functions and the principle of contracting mappings for finding a solution. Moreover, the construction of a Pareto-optimal set presumes the computation of various profitability and risk indicators for some collection of controls or other, which requires the solution of a large number of integral equations and is not a simple problem.

In principle, a Pareto-optimal boundary can be found as the envelope of the set of all possible pairs "average dividends–probability of ruin" that correspond to all admissible control strategies. The set of such strategies is infinite and, therefore, it is important to exaustively search only for "good" (efficient) strategies. As is mentioned in [1–4], the following barrier (threshold) strategies are optimal for a one-criterion statement: if the current capital is less that some barrier, then dividends are not paid and, otherwise, the amount of the increase in the capital value over the threshold is paid. Thus, in the barrier control strategy, the process does not exceed some barrier. However, it is easy to see that, under the barrier strategy, the probability of ruin tends to one with increasing the planning horizon [1] (see also [4]). Therefore, it is also expedient to consider other types of strategies under which the risk process can be not bounded. One of them is the proportional strategy when a definite share of the excess of the capital value over a given threshold (barrier) is paid in the capacity of dividends. In [3], optimal nonlinear barrier strategies are studied in which barrier parameters are optimized. A combination of barrier and proportional strategies can also be used. A review of results on optimal dividend strategies in one-criterion problems is presented in [5–7].

One more possibility of construction of efficient control strategies consists of the solution of the stochastic optimal control problem for the aggregation of some criteria. In [18, 19], the aggregation of average discounted dividends (with a varied coefficient) and the average discounted lifetime (this indicator was also considered in [2, 16]) were investigated. Discounting means that, in making current decisions, a year of life in the distant future is less significant than a year of life in the near future. Discounting also allows one to avoid the formal problem of a possible infinite average lifetime. In [20], the aggregation of average discounted dividends and the average discounted borrowed capital necessary for the prevention of ruin was considered. It is shown in [18–20] that, for simple aggregations of criteria and a simple set of admissible controls, for the obtained aggregated optimal control problems, the principle of dynamic programming is valid, Bellman equations are satisfied, and there are positional optimal dividend control strategies. However, these results are not valid for general integral Lipschitz risk process control criteria and require a generalization. This article present new linear estimates for Bellman functions and new estimates of the convergence rate for the successive approximation method for solving Bellman equations. New conditions of existence and uniqueness of solutions of Bellman equations are obtained that are based on not the general principle of contracting mappings but on the use of the specificity of a problem.

When a dividend policy (control) is already determined, it is still necessary to compute the corresponding indicators of profitability (expected total dividends) and risk (probability of ruin and average capital deficit). The question of efficiency of computation of these complicated indicators becomes basic. They can be found by solving some integral equations [2, 21, 22] or by estimating with the help of the Monte Carlo method [23]. This article shows that arising integral equations are efficiently solved by the successive approximation method.

MODEL OF A CONTROLLED RISK PROCESS WITH DIVIDEND SUBTRACTION

A risk process describes a stochastic evolution of reserves of an insurance company that are destined for satisfying insurance claims. A mathematical model of the evolution of reserves in discrete time $t = 0, 1, \dots$ is of the form [1–7]

$$X_{t+1} = f(X_t, U_t, Y_t) = \begin{cases} X_t - U_t + Y_t, & X_t \ge 0, \\ X_t, & X_t < 0, \end{cases} \tag{1}$$

$$X_0 = x \ge 0,\ U_t \in \mathbf{U}(X_t),\ t = 0, 1, \dots, \tag{2}$$

where $\{Y_t \ge 0\}$ are equally distributed (as some random quantity Y) independent random quantities (aggregate premiums minus claims) with a common distribution function F; U_t are dividends subtracted from an admissible set $\mathbf{U}(X_t) \subseteq [0, X_t]$; x is the initial state of process (1). We denote by $\tau = \sup\{t \ge 0 \colon \min_{0 \le k < t} X_k \ge 0\}$ ($\tau = 0$ when $X_0 < 0$) the ruin time of the process and by $U = \{U_t \in \mathbf{U}(X_t), t = 0, 1, \dots\}$ a sequence of admissible controls. We assume that a mapping $x \to \mathbf{U}(x) \subseteq [0, x]$ is compact-valued and upper semicontinuous.

Trajectories of risk process (1), (2) are estimated from payoff and risk indicators. In the capacity of payoff indicators, for example, the average life time of a process, average total discounted dividends, average discounted capital value at the ruin time, etc. can be used. In the capacity of risk indicators, the probability of ruin, average capital deficit along a trajectory or at the ruin time, etc. are used.

We denote by $r(x,u,y)$ a function of reward per one time step (by definition, $r(x,u,y)=0$ when $x<0$). For example,

$$r_0(x,u,y)=\begin{cases} u, & x\geq 0,\\ 0, & x<0,\end{cases}\qquad r_1(x,y)=\begin{cases} 1, & x\geq 0,\\ 0, & x<0,\end{cases}$$

$$r_2(x,u,y)=\begin{cases} \min\{0,x-u+y\}, & x\geq 0,\\ 0, & x<0,\end{cases}\qquad r_3(x,u,y)=\begin{cases} \mathbf{1}_{\{x-u+y<0\}}, & x\geq 0,\\ 0, & x<0,\end{cases}$$

where $\mathbf{1}_{\{\cdot\}}$ is an indicator function of an event in braces (it is equal to one if the condition in braces is fulfilled and is equal to zero otherwise). The discounted payoff function during $(t+1)$ periods under a control $U=\{U_t,\ t=0,1,\dots\}$ and with the initial state x is of the form

$$V_t(x,U)=\mathbf{E}\left[\sum\nolimits_{k=0}^{t}\gamma^k\, r(X_k,U_k,Y_k)\right]=\mathbf{E}\left[\sum\nolimits_{k=0}^{\min\{t,\tau-1\}}\gamma^k\, r(X_k,U_k,Y_k)\right],\tag{3}$$

where γ is a discounting factor, $0<\gamma\leq 1$. In particular, the following indicators are of interest: average total discounted dividends (before a moment $\min\{t,\tau-1\}$)

$$V_t^0(x,U)=\mathbf{E}\left[\sum\nolimits_{k=0}^{t}\gamma^k\, r_0(X_k,U_k,Y_k)\right]=\mathbf{E}\left[\sum\nolimits_{k=0}^{\min\{t,\tau-1\}}\gamma^k\, U_k\right];\tag{4}$$

the average (discounted when $\gamma<1$) life time before the moment $\min\{t,\tau-1\}$

$$V_t^1(x,U)=\mathbf{E}\left[\sum\nolimits_{k=0}^{t}\gamma^k\, r_1(X_k,U_k,Y_k)\right]=\mathbf{E}\left[\sum\nolimits_{k=0}^{\min\{t,\tau-1\}}\gamma^k\right]=\frac{1-\mathbf{E}\gamma^{\min\{t+1,\tau\}}}{1-\gamma};\tag{5}$$

the average discounted capital deficit at the ruin time

$$V_t^2(x,U)=\mathbf{E}\left[\sum\nolimits_{k=0}^{t}\gamma^k\, r_2(X_k,U_k,Y_k)\right]=\mathbf{E}\left[\sum\nolimits_{k=0}^{\min\{t,\tau-1\}}\gamma^k\, \min\{0,X_k-U_k+Y_k\}\right];\tag{6}$$

the probability (discounted when $\gamma<1$) of ruin in time t

$$V_t^3(x,U)=\mathbf{E}\left[\sum\nolimits_{k=0}^{t}\gamma^k\, r_3(X_k,U_k,Y_k)\right]=\mathbf{E}\left[\sum\nolimits_{k=0}^{\min\{t,\tau-1\}}\gamma^k\, \mathbf{1}_{\{X_k-U_k+Y_k<0\}}\right].\tag{7}$$

Comment 1. The average life time $\mathbf{E}\tau$ is not a very convenient risk indicator since it can be equal to infinity. Therefore, along with $\mathbf{E}\tau$, the following so-called average discounted life time (5) is considered in [2, 24]: $V_t^1(x,U)=(1-\mathbf{E}\gamma^{\min\{t+1,\tau\}})/(1-\gamma)\leq 1/(1-\gamma)$. It is obvious that $0\leq V_T^1(x,U)\leq \mathbf{E}\tau$. Discounting can be interpreted as the presence of some (binomially distributed) random factor that can stop a risk process irrespective of its current state [25].

We define (Bellman) payoff functions as follows:

$$V_t(x)=\sup\nolimits_{\{U:\,U_k\in\mathbf{U}(x)\}}V_t(x,U).\tag{8}$$

By definition, we consider that $V_t(x)=0$ for $x<0$. Similarly, $V_t^i(x)=\sup_{\{U:\,U_k\in\mathbf{U}(x)\}}V_t^i(x,U)$, where $V_t^i(x,U)$ for $i=0,1,2,3$ are defined in formulas (4)–(7). For the case when $t=+\infty$, we introduce the special notations

$$V_\infty(x,U)=\mathbf{E}\left[\sum\nolimits_{k=0}^{\tau-1}\gamma^k\, r(X_k,U_k,Y_k)\right]=V(x,U),$$

$$V_\infty(x) = \sup_{\{U: U_k \in \mathbf{U}(x)\}} V(x,U) = V(x), \tag{9}$$

$$V_\infty^i(x,U) = \mathbf{E}\left[\sum_{k=0}^{\tau-1} \gamma^k \, r_i(X_k, U_k, Y_k)\right] = V^i(x,U),$$

$$V_\infty^i(x) = \sup_{\{U: U_k \in \mathbf{U}(x)\}} V^i(x,U) = V^i(x).$$

PROPERTIES OF BELLMAN FUNCTIONS

If the function of reward per time period $r(x,u,y)$ is uniformly bounded, i.e., $r(x,u,y) \le A < +\infty$ for all (x,u,y), then it is obvious that functions (3) and (8) are also uniformly bounded,

$$V_t(x,U) \le V_t(x) \le A(t+1), \ \gamma = 1,$$

$$V_t(x,U) \le V_t(x) \le A/(1-\gamma), \ 0 < \gamma < 1.$$

Otherwise, for example, when $r_0(x,u,y) = u$, the boundedness and finiteness of Bellman functions should be proved.

LEMMA 1 (boundaries for values of payoff functions (4)–(7)). Assume that $\mathbf{U}(x) = [0,x]$, $\mathbf{E}\max\{0,Y\} < +\infty$, $\mathbf{E}\min\{0,Y\} > -\infty$, and $\gamma < 1$. Then the Bellman functions $V_t^i(x) = \sup_{\{U: U_t \in \mathbf{U}(x)\}} V_t^i(x,U)$, where $V_t^i(x,U)$ are defined in formulas (4)–(7), satisfy the following constraints:

$$x \le V_t^0(x) \le V^0(x) \le x + \gamma \mathbf{E}\max\{0,Y\}/(1-\gamma), \tag{10}$$

$$1 \le V_t^1(x) \le V^1(x) \le 1/(1-\gamma), \tag{11}$$

$$\mathbf{E}\min\{0,Y\}/(1-\gamma) \le V^2(x) \le V_t^2(x) \le 0, \tag{12}$$

$$0 \le V_t^3(x) \le V^3(x) \le 1. \tag{13}$$

Proof. Inequalities (10) are given in [26; 4, Lemma 1.8)]. Since

$$1 \le V_t^1(x,U) \le V^1(x,U) = \mathbf{E}\sum_{k=0}^{\tau-1}\gamma^k = (1-\mathbf{E}\gamma^\tau)/(1-\gamma) \le 1/(1-\gamma),$$

we have $1 \le V_t^1(x) = \sup_U V_t^1(x,U) \le \sup_U V^1(x,U) = V^1(x) \le 1/(1-\gamma)$. For $V^2(x,U)$, we obtain

$$0 \ge V_t^2(x,U) \ge V^2(x,U) \ge \mathbf{E}\sum_{k=0}^{\tau-1}\gamma^k \min\{0,Y_k\} = \mathbf{E}\min\{0,Y\}\sum_{k=0}^{\infty}\gamma^k \ge \mathbf{E}\min\{0,Y\}/(1-\gamma),$$

which implies inequalities (12). Since $0 \le V_t^3(x,U) \le V^3(x,U) = \gamma^{\tau-1} \le 1$, inequalities (13) are fulfilled. The lemma is proved.

Comment 2. Together with functions $V_t^i(x,U)$, we will consider their aggregates of the form

$$V_t(x,U) = V_t^0(x,U) + \lambda V_t^i(x,U), \tag{14}$$

where $\lambda \ge 0$ is an aggregating factor. To them correspond the following functions of payoff per step:

$$r(x,u,y) = r_0(x,u,y) + \lambda r_i(x,u,y) = \begin{cases} u + \lambda r_i(x,u,y), & x \ge 0, \\ 0, & x < 0. \end{cases}$$

In this case, lower and upper estimates for the Bellman function $V_t(x) = \sup_U V_t(x,U)$ are easily obtained from estimates (10)–(13) for constituent functions $V_t^i(x,U)$. For example,

$$x \le \sup_U V_t^0(x,U) \le V_t(x) = \sup_U (V_t^0(x,U) + \lambda V_t^1(x,U)) \le \sup_U V_t^0(x,U) + \lambda \sup_U V_t^1(x,U) \le x + \frac{\lambda + \gamma \mathbf{E}\max\{0,Y\}}{1-\gamma}.$$

Similarly, taking into account that $\mathbf{E}\min\{0,Y\}/(1-\gamma)\le V_t^2(x,U)\le 0$, we obtain

$$x+\frac{\lambda\mathbf{E}\min\{0,Y\}}{1-\gamma}\le V_t(x)=\sup_U(V_t^0(x,U)+\lambda V_t^2(x,U))\le\sup_U V_t^0(x,U)\le x+\frac{\gamma\mathbf{E}\max\{0,Y\}}{1-\gamma}.$$

Let the following assumptions be valid:

(1) $\mathbf{E}|Y|<\infty$ for a random quantity Y distributed in exactly the same way as all $\{Y_k\}$;

(2) a mapping $\mathbf{U}(x)\subseteq[0,x]$ is closed-valued and upper semicontinuous;

(3) a function $\bar{r}(x,u)=\mathbf{E}r(x,u,Y)$ is upper semicontinuous with respect to (x,u);

(4) $|r(x,u,y)|\le C_0+C_1|y|+C_2u+C_3x$ for some numbers $C_0,C_1,C_2,C_3\ge 0$ and all $(x\ge 0,u\in\mathbf{U}(x)\subseteq[0,x],y)$.

Note that condition 3 is fulfilled for a function $r(x,u,y)$ that is upper semicontinuous with respect to (x,u) and satisfies Assumptions 1 and 4. Condition 4 is fulfilled for Lipschitz functions $r(x,u,y)$.

LEMMA 2 (boundary for common payoff functions (8)). If Assumptions 1–4 are valid, then we have

$$|V_t(x)|\le|V(x)|\le(C_0+C_1\mathbf{E}|Y|+(C_2+C_3)x)/(1-\gamma)+(C_2+C_3)\gamma\mathbf{E}\max\{0,Y\}/(1-\gamma)^2.$$

Proof. It is obvious that, for $k<\tau$, by virtue of model (1), (2) and Assumption 2, we have

$$0\le U_k\le X_k\le X_{k-1}+\max\{0,Y_{k-1}\}\le x+\sum_{s=0}^{k-1}\max\{0,Y_s\}$$

and, hence, in view of Assumption 4, we obtain

$$|r(X_k,U_k,Y_k)|\le C_0+C_1|Y_k|+C_2U_k+C_3X_k\le C_0+C_1|Y_k|+(C_2+C_3)X_k$$

$$\le C_0+C_1|Y_k|+(C_2+C_3)\left(x+\sum_{s=0}^{k-1}\max\{0,Y_s\}\right).$$

Then

$$|V_t(x)|\le|V(x)|\le\sup_{\{U:\,U_k\in\mathbf{U}(X_k)\}}\mathbf{E}\left[\sum_{k=0}^{\tau-1}\gamma^k|r(X_k,U_k,Y_k)|\right]$$

$$\le\sup_{\{U:U_k\in\mathbf{U}(X_k)\}}\mathbf{E}\left[\sum_{k=0}^{\tau-1}\gamma^k\left(C_0+C_1|Y_k|+(C_2+C_3)\left(x+\sum_{s=0}^{k-1}\max\{0,Y_s\}\right)\right)\right]$$

$$\le\left[\sum_{k=0}^{\infty}\gamma^k(C_0+C_1\mathbf{E}|Y|+(C_2+C_3)(x+k\mathbf{E}\max\{0,Y\}))\right].$$

Taking into account that $\sum_{k=0}^{\infty}\gamma^k=1/(1-\gamma)$ and $\sum_{k=0}^{\infty}k\gamma^k=\gamma/(1-\gamma)^2$, we obtain the statement of the lemma.

Let us consider the (Bellman) recurrence relations

$$\tilde{V}_t(x)=\sup_{u\in\mathbf{U}(x)}\{\mathbf{E}r(x,u,Y)+\gamma\mathbf{E}\tilde{V}_{t-1}(x-u+Y)\},\ \tilde{V}_{-1}(x)\equiv 0,\ 0\le t\le T<\infty. \tag{15}$$

The following lemma establishes boundaries for functions $\tilde{V}_t(x)$.

LEMMA 3. Let Assumptions 1–4 be valid. Then functions $\tilde{V}_t(x)$ (15) satisfy the following inequalities:

$$|\tilde{V}_0(x)|\le C_0+C_1\mathbf{E}|Y|+(C_2+C_3)x,$$

$$|\tilde{V}_t(x)|\le(1+\gamma+\ldots+\gamma^t)(C_0+C_1\mathbf{E}|Y|+(C_2+C_3)x)$$

$$+[(\gamma+\ldots+\gamma^t)+(\gamma^2+\ldots+\gamma^t)+\ldots+\gamma^t](C_2+C_3)\mathbf{E}\max\{0,Y\},\ t\ge 1,$$

and, hence, when $0<\gamma<1$, we have

$$|\tilde{V}_t(x)|\le(C_0+C_1\mathbf{E}|Y|+(C_2+C_3)x)/(1-\gamma)+\gamma(C_2+C_3)\mathbf{E}\max\{0,Y\}/(1-\gamma)^2.$$

We prove the lemma by induction. For $t=0$, the statement of the lemma is true,

$$|\widetilde{V}_0(x)| \le \sup_{u\in \mathbf{U}(x)} \mathbf{E}|r(x,u,Y)| \le \sup_{u\in[0,x]} \mathbf{E}(C_0 + C_1|Y| + C_2 u + C_3 x)$$

$$\le C_0 + C_1 \mathbf{E}|Y| + (C_2 + C_3)x .$$

Assume that the statement of the lemma is true for $t-1$, and prove it for t. In fact, we have

$$|\widetilde{V}_t(x)| \le \sup_{u\in \mathbf{U}(x)} \{\mathbf{E}|r(x,u,Y)| + \gamma \mathbf{E}|\widetilde{V}_{t-1}(x-u+Y)|\} \le \sup_{u\in[0,x]} \{\mathbf{E}(C_0 + C_1|Y|$$

$$+ C_2 u + C_3 x) + \gamma(1+\gamma+\ldots+\gamma^{t-1})(C_0 + C_1\mathbf{E}|Y| + (C_2+C_3)\mathbf{E}\max\{0, x-u+Y\})$$

$$+\gamma[(\gamma+\ldots+\gamma^{t-1}) + (\gamma^2+\ldots+\gamma^{t-1}) + \ldots + \gamma^{t-1}](C_2+C_3)\mathbf{E}\max\{0,Y\}\} \le (C_0 + C_1\mathbf{E}|Y|$$

$$+ (C_2+C_3)x) + \gamma(1+\gamma+\ldots+\gamma^{t-1})(C_0 + C_1\mathbf{E}|Y| + (C_2+C_3)(x + \mathbf{E}\max\{0,Y\}))$$

$$+\gamma[(\gamma+\ldots+\gamma^{t-1}) + (\gamma^2+\ldots+\gamma^{t-1}) + \ldots + \gamma^{t-1}](C_2+C_3)\mathbf{E}\max\{0,Y\},$$

which is what had to be proved.

The following theorem formulates sufficient conditions for the coincidence of functions $V_t(x)$ (8) and $\widetilde{V}_t(x)$ (15).

THEOREM 1 (properties of functions (15) and the existence of optimal controls over a finite time horizon $T<\infty$). Let Assumptions 1–4 be valid. Then functions $\widetilde{V}_t(x)$ (15) coincide with functions $V_t(x)$ (8) and are upper semicontinuous with respect to *x*. Moreover, functions $\mathbf{E}\widetilde{V}_{t-1}(x-u+Y)$ are upper semicontinuous with respect to (x,u) and $U_t = u^*_{T-t}(x^t)$, where

$$u^*_t(x) = \max\{v \in \arg\max_{u\in\mathbf{U}(x)} \{\mathbf{E}r(x,u,Y) + \gamma\mathbf{E}\widetilde{V}_{t-1}(x-u+Y)\}\} \tag{16}$$

are correctly defined and Borel measurable and are optimal controls (solutions) for problems (8).

Proof. Under Assumptions 3 and 4, the function $\mathbf{E}r(x,u,Y)$ is upper semicontinuous with respect to (x,u) by virtue of the Fatou lemma [29, Ch. II, Sec. 6, Theorem 2(b)]. Then, in view of the upper semicontinuity of the mapping $\mathbf{U}(\cdot)$, the maximum function

$$\widetilde{V}_0(x) = \sup_{u\in\mathbf{U}(x)} \mathbf{E}r(x,u,Y) = \max_{u\in\mathbf{U}(x)} \mathbf{E}r(x,u,Y) < +\infty$$

is upper semicontinuous with respect to $x \ge 0$ [28, Ch. 3, Sec. 1, sentence 21], the mapping $U^*_0(x) = \arg\max_{u\in\mathbf{U}(x)} \mathbf{E}r(x,u,Y)$ is closed-valued and measurable, and the selector $u^*_0(x) = \max\{u \in U^*_0(x)\}$ is a Borel measurable function [30, Sec. 14].

Let us show that all $\widetilde{V}_t(x)$ are upper semicontinuous with respect to *x* and that $u^*_t(x)$ are measurable optimal controls for problems (8). As is obvious, the statement is true when $t=0$. By induction, assume that the function $\widetilde{V}_{t-1}(x)$ is upper semicontinuous. Then the function $\widetilde{V}_{t-1}(x-u+y)$ is upper semicontinuous with respect to (x,u) for each y and, by virtue of Assumption 4 and Lemma 2, $\widetilde{V}_{t-1}(x') \le A_{t-1} + B_{t-1}x'$, where $A_{t-1}>0$ and $B_{t-1}>0$, are some constants, and, hence,

$$\widetilde{V}_{t-1}(x-u+y) \le A_{t-1} + B_{t-1}\max\{0, x-u+y\}$$

$$\le A_{t-1} + B_{t-1}\max\{0, x - \inf_{u\in\mathbf{U}(x)} u\} + B_{t-1}\max\{0,y\},$$

$$\mathbf{E}\widetilde{V}_{t-1}(x-u+Y) \le A_{t-1} + B_{t-1}\max\{0, x-\inf_{u\in\mathbf{U}(x)} u\} + B_{t-1}\mathbf{E}\max\{0,Y\}.$$

By virtue of the Fatou lemma [29, Ch. II, Sec. 6, Theorem 2(b)], the function $\mathbf{E}\widetilde{V}_{t-1}(x-u+Y)$ is upper semicontinuous with respect to (x,u). This implies [28, Ch. 3, Sec. 1, sentence 21] that the maximum function $\widetilde{V}_t(x) = \sup_{u\in\mathbf{U}(x)}\{\mathbf{E}r(x,u,Y) + \gamma\mathbf{E}\widetilde{V}_{t-1}(x-u+Y)\}$ is upper semicontinuous with respect to *x*, the mapping

$$U^*_t(x) = \arg\max_{u\in\mathbf{U}(x)}\{\mathbf{E}r(x,u,Y) + \gamma\mathbf{E}\widetilde{V}_{t-1}(x-u+Y)\}$$

is closed-valued and measurable, and the function $u_t^*(x)=\max\{u\in U_t^*(x)\}$ is Borel measurable (which follows from the statements [30, Sec. 14.3, 14.31, 14.32, and 14.37]). Hence, according to [4, Corollary 1.2], functions $V_t(x)$ (8) coincide with functions $\widetilde{V}_t(x)$ (15) and the sequence $U^*=\{u_{T-t}^*(X_t), 0\le t\le T\}$ is an optimal control for problems (8),

$$V_t(x)=V_t(x,U^*)=\mathbf{E}\sum_{k=0}^{\tau^*-1}\gamma^k r(x,u_{T-t}^*(X_k),Y_k),\ 0\le t\le T,$$

where τ^* is the ruin time of process (2) with $U^*=\{U_k^*=u_{T-t}^*(X_k)\}$.

The theorem is proved.

We will investigate the case of an infinite time horizon $T=\infty$. Let us consider the (Bellman) equation

$$V(x)=\sup_{u\in\mathbf{U}(x)}\{\mathbf{E}r(x,u,Y)+\gamma\mathbf{E}V(f(x,u,Y))\}=\sup_{u\in\mathbf{U}(x)}\{\mathbf{E}r(x,u,Y)+\gamma\mathbf{E}V(x-u+Y)\}. \tag{17}$$

The following statements establish the existence and uniqueness of a linearly bounded upper semicontinuous solution $V(\cdot)$ to Bellman equation (17) and also the existence of optimal positional controls $u^*(x)$ for the case when the reward function $r(x,u,y)$ and the Bellman function $V(x)$ itself are not uniformly but linearly bounded. Then the standard norm $||V||=\sup_{x\ge 0}V(x)=+\infty$ becomes pointless and fixed point theorems in the Banach space of bounded functions are inapplicable to the substantiation of the existence and uniqueness of a solution to Eq. (17).

Equation (17) is usually solved by successive approximation method (15), and, therefore, we first note in the lemma formulated below that a sequence $\{\widetilde{V}_k(\cdot)\}$ converges to some function $V(\cdot)$ and then, in the next theorem (about properties of a Bellman function), show that this limiting function is the unique solution to Eq. (17).

LEMMA 4. Under Assumptions 1–4, sequence (15) uniformly converges on each interval $[0,x_{\max}]$ to some function $V(x)$, and the following estimates take place:

$$\sup_{x\in[0,x_{\max}]}|V(x)-\widetilde{V}_t(x)|\le\frac{C_0+C_1\mathbf{E}|Y|+(C_2+C_3)x_{\max}}{1-\gamma}\gamma^{t+1}+\frac{(C_2+C_3)\mathbf{E}\max\{0,Y\}(\gamma+(1-\gamma)(t+1))}{(1-\gamma)^2}\gamma^{t+1}. \tag{18}$$

Proof. In fact, under the conditions of Theorem 1, by virtue of the optimality of controls, we have

$$\widetilde{V}_{t+1}(x)=\max_{u\in\mathbf{U}(x)}\{\mathbf{E}r(x,u,Y)+\gamma\mathbf{E}\widetilde{V}_t(x-u+Y)\}$$

$$=\mathbf{E}r(x,u_{t+1}^*(x),Y)+\gamma\mathbf{E}\widetilde{V}_t(x-u_{t+1}^*(x)+Y),$$

$$\widetilde{V}_t(x)=\max_{u\in\mathbf{U}(x)}\{\mathbf{E}r(x,u,Y)+\gamma\mathbf{E}\widetilde{V}_{t-1}(x-u+Y)\}$$

$$\ge\mathbf{E}r(x,u_{t+1}^*(x),Y)+\gamma\mathbf{E}\widetilde{V}_{t-1}(x-u_{t+1}^*(x)+Y);$$

therefore,

$$\widetilde{V}_{t+1}(x)-\widetilde{V}_t(x)\le\gamma\mathbf{E}[\widetilde{V}_t(x-u_{t+1}^*(x)+Y)-\widetilde{V}_{t-1}(x-u_{t+1}^*(x)+Y)]$$

$$\le\gamma\mathbf{E}\sup_{0\le x'\le x+\max\{0,Y\}}|\widetilde{V}_t(x')-\widetilde{V}_{t-1}(x')|.$$

Similarly, we have

$$\widetilde{V}_{t+1}(x)=\max_{u\in\mathbf{U}(x)}\{\mathbf{E}r(x,u,Y)+\gamma\mathbf{E}\widetilde{V}_t(x-u+Y)\}$$

$$\ge\mathbf{E}r(x,u_t^*(x),Y)+\gamma\mathbf{E}\widetilde{V}_t(x-u_t^*(x)+Y),$$

$$\widetilde{V}_t(x)=\max_{u\in\mathbf{U}(x)}\{\mathbf{E}r(x,u,Y)+\gamma\mathbf{E}\widetilde{V}_{t-1}(x-u+Y)\}=\mathbf{E}r(x,u_t^*(x),Y)+\gamma\mathbf{E}\widetilde{V}_{t-1}(x-u_t^*(x)+Y);$$

hence,

$$\widetilde{V}_{t+1}(x)-\widetilde{V}_t(x)\ge\gamma\mathbf{E}[\widetilde{V}_t(x-u_t^*(x)+Y)-\widetilde{V}_{t-1}(x-u_t^*(x)+Y)]=-\gamma\mathbf{E}\sup_{0\le x'\le x+\max\{0,Y\}}|\widetilde{V}_t(x')-\widetilde{V}_{t-1}(x')|.$$

Therefore,

$$|\widetilde{V}_{t+1}(x)-\widetilde{V}_t(x)|\le\gamma\mathbf{E}\sup_{0\le x'\le x+\max\{0,Y\}}|\widetilde{V}_t(x')-\widetilde{V}_{t-1}(x')|$$

and, iterating with respect to t, we obtain the estimate

$$|\widetilde{V}_{t+1}(x)-\widetilde{V}_t(x)|\le\gamma^{t+1}\mathbf{E}\sup_{0\le x'\le x+(t+1)\max\{0,Y\}}|\widetilde{V}_0(x')|.$$

Recall that $|\widetilde{V}_0(x')|=\mathbf{E}|r(x,u,Y)|\le C_0+C_1\mathbf{E}|Y|+(C_2+C_3)x'$ and, hence,

$$|\widetilde{V}_{t+1}(x)-\widetilde{V}_t(x)|\le\gamma^{t+1}(C_0+C_1\mathbf{E}|Y|+(C_2+C_3)(x+(t+1)\mathbf{E}\max\{0,Y\})).$$

Thus, the sequence $\{\widetilde{V}_t(x)\}$ is fundamental, converges to some limit $V(x)$, and the following estimate takes place:

$$|V(x)-\widetilde{V}_t(x)|\le\sum_{k=t}^{\infty}|\widetilde{V}_{k+1}(x)-\widetilde{V}_k(x)|$$

$$\le\sum_{k=t}^{\infty}\gamma^{k+1}(C_0+C_1\mathbf{E}|Y|+(C_2+C_3)(x+(k+1)\mathbf{E}\max\{0,Y\}))$$

$$\le\frac{C_0+C_1\mathbf{E}|Y|+(C_2+C_3)x}{1-\gamma}\gamma^{t+1}+(C_2+C_3)\mathbf{E}\max\{0,Y\}\sum_{k=t}^{\infty}(k+1)\gamma^{k+1}.$$

Taking into account that $\sum_{k=t}^{\infty}(k+1)\gamma^{k+1}=\left(\frac{\gamma}{(1-\gamma)^2}+\frac{t+1}{1-\gamma}\right)\gamma^{t+1}$, we finally obtain the estimate

$$|V(x)-\widetilde{V}_t(x)|\le\frac{C_0+C_1\mathbf{E}|Y|+(C_2+C_3)x}{1-\gamma}\gamma^{t+1}+(C_2+C_3)\mathbf{E}\max\{0,Y\}\left(\frac{\gamma}{(1-\gamma)^2}+\frac{t+1}{1-\gamma}\right)\gamma^{t+1}$$

$$=\frac{C_0+C_1\mathbf{E}|Y|+(C_2+C_3)x}{1-\gamma}\gamma^{t+1}+\frac{(C_2+C_3)\mathbf{E}\max\{0,Y\}(\gamma+(1-\gamma)(t+1))}{(1-\gamma)^2}\gamma^{t+1},$$

which implies uniform estimate (18). The lemma is proved.

THEOREM 2 (properties of a Bellman function and the existence of optimal controls in the case of an infinite time horizon $T=\infty$). Under Assumptions 1–4, the limit of $V(x)=\lim_{t\to\infty}\widetilde{V}_t(x)$ exists, where $\widetilde{V}_t(x)$ are defined in relations (15), and is an upper semicontinuous function. The function $V(x)$ is the unique upper semicontinuous solution to Eq. (17); it satisfies the condition $|V(x)|\le A+Bx$ for all $x\ge 0$, where $A\ge 0$, $B\ge 0$ are arbitrary constants. The function $\varphi_x(u)=\mathbf{E}V(x-u+Y)$ is upper semicontinuous, the extremal mapping

$$U^*(x)=\arg\max_{u\in\mathbf{U}(x)}\{\mathbf{E}r(x,u,Y)+\gamma\mathbf{E}V(x-u+Y)\}\tag{19}$$

is upper semicontinuous, and the function

$$u^*(x)=\max\{u\in U^*(x)\}\tag{20}$$

is correctly defined, Borel measurable, and is a solution to optimal control problem (9).

Proof. We prove that $V(x)$ is upper semicontinuous. By virtue of Lemma 4, the sequence of functions $\{\widetilde{V}_t(\cdot)\}$ uniformly converges on each compact to some limiting function $V(\cdot)$. Since, according to Theorem 1, all functions $\widetilde{V}_t(\cdot)$ are upper semicontinuous, their uniform limit $V(\cdot)$ is also an upper semicontinuous function.

Let us show that the limit $V(\cdot)$ satisfies Bellman equation (17). For each fixed x, we consider the following functions;

$$v_{t-1,x}(u)=\mathbf{E}r(x,u,Y)+\gamma\mathbf{E}\widetilde{V}_{t-1}(x-u+Y),\ u\in\mathbf{U}(x),$$

$$v_x(u)=\mathbf{E}r(x,u,Y)+\gamma\mathbf{E}V(x-u+Y),\ u\in\mathbf{U}(x).$$

By virtue of Theorems 1, the functions $v_{t-1,x}(u)$ are upper semicontinuous with respect to u. Moreover, they uniformly converge to the (upper semicontinuous) function $v_x(u)$ on the compact $\mathbf{U}(x)$, namely, by virtue of estimates (18), for $u \in \mathbf{U}(x)$, we have

$$|v_x(u)-v_{t-1,x}(u)| \le \gamma \mathbf{E}|V(x-u+Y)-\widetilde{V}_{t-1}(x-u+Y)| \le \frac{C_0+C_1\mathbf{E}|Y|+(C_2+C_3)(x+\mathbf{E}\max\{0,Y\})}{1-\gamma}\gamma^{t+1}$$

$$+\frac{(C_2+C_3)\mathbf{E}\max\{0,Y\}(\gamma+(1-\gamma)(t+1))}{(1-\gamma)^2}\gamma^{t+1};$$

therefore,

$$V(x)=\lim_{t\to\infty}\widetilde{V}_t(x)=\lim_{t\to\infty}\sup_{u\in\mathbf{U}(x)}v_{t-1,x}(u)=\sup_{u\in\mathbf{U}(x)}\lim_{t\to\infty}v_{t-1,x}(u)$$

$$=\sup_{u\in\mathbf{U}(x)}v_x(u)=\sup_{u\in\mathbf{U}(x)}[\mathbf{E}r(x,u,Y)+\gamma\mathbf{E}V(x-u+Y)].$$

Thus, the upper semicontinuous function $V(x)$ satisfies Bellman equation (17) and there is measurable [30, Sec. 14] function $u^*(x)$ (20).

Let us prove the uniqueness of the constructed solution $V(x)$ to Eq. (17). Assume that there are one more solution $\widetilde{V}(\cdot)$ such that $|\widetilde{V}(x)| \le A+Bx$ and the corresponding optimal control $\widetilde{u}(\cdot)$. We have

$$V(x)=\sup_{u\in\mathbf{U}(x)}\{\mathbf{E}r(x,u,Y_0)+\gamma\mathbf{E}V(x-u+Y_0)\}=\mathbf{E}r(x,u^*(x),Y_0)+\gamma\mathbf{E}V(x-u^*(x)+Y_0),$$

$$\widetilde{V}(x)=\sup_{u\in\mathbf{U}(x)}\{\mathbf{E}r(x,u,Y_0)+\gamma\mathbf{E}\widetilde{V}(x-u+Y_0)\}=\mathbf{E}r(x,\widetilde{u}(x),Y_0)+\gamma\mathbf{E}\widetilde{V}(x-\widetilde{u}(x)+Y_0).$$

From this we obtain

$$V(x)-\widetilde{V}(x) \le \gamma\mathbf{E}(V(x-u^*(x)+Y_0)-\widetilde{V}(x-u^*(x)+Y_0))$$

$$\le \gamma\mathbf{E}\sup_{0\le x'\le x+\max\{0,Y_0\}}|V(x')-\widetilde{V}(x')|,$$

$$\widetilde{V}(x)-V(x) \le \gamma\mathbf{E}(\widetilde{V}(x-\widetilde{u}(x)+Y_0)-V(x-\widetilde{u}(x)+Y_0)) \le \gamma\mathbf{E}\sup_{0\le x'\le x+\max\{0,Y_0\}}|\widetilde{V}(x')-V(x')|,$$

$$|V(x)-\widetilde{V}(x)| \le \gamma\mathbf{E}\sup_{0\le x'\le x+\max\{0,Y_0\}}|V(x')-\widetilde{V}(x')|.$$

For any t, we have

$$|V(x)-\widetilde{V}(x)| \le \gamma\mathbf{E}\sup_{0\le x'\le x+\max\{0,Y_0\}}|V(x')-\widetilde{V}(x')|$$

$$\le \gamma^t\mathbf{E}\sup_{0\le x'\le x+\sum_{k=0}^{t-1}\max\{0,Y_k\}}|V(x')-\widetilde{V}(x')|$$

$$\le \gamma^t\mathbf{E}\sup_{0\le x'\le x+\sum_{k=0}^{t-1}\max\{0,Y_k\}}|V(x')|+\gamma^t\mathbf{E}\sup_{0\le x'\le x+\sum_{k=0}^{t-1}\max\{0,Y_k\}}|\widetilde{V}(x')|.$$

Taking into account that, according to Lemma 3, we have

$$|V(x')| \le (C_0+C_1\mathbf{E}|Y|+(C_2+C_3)(x'+\mathbf{E}\max\{0,Y\}))/(1-\gamma)$$

$$+(C_2+C_3)\gamma\mathbf{E}\max\{0,Y\}/(1-\gamma)^2,$$

and, under the Assumption, $|\widetilde{V}(x')| \le A+Bx'$, we obtain

$$|V(x)-V'(x)| \le \gamma^t(C_0+C_1\mathbf{E}|Y|+(C_2+C_3)(x+t\mathbf{E}\max\{0,Y\}))/(1-\gamma)$$

$$+\gamma^t(C_2+C_3)\gamma\mathbf{E}\max\{0,Y\}/(1-\gamma)^2+(A+B(x+t\mathbf{E}\max\{0,Y\}))\gamma^t \to 0,$$

which is what had to be proved.

Note that, for any control $U'=\{U'_t\in\mathbf{U}(x)\}$ and the corresponding trajectory $\{X'_t=X'_{t-1}-U'_{t-1}+Y_t, X'_0=x, U'_{t-1}\in\mathbf{U}(X'_{t-1}), t\ge 0\}$, for conditional expectations $\bar r(X'_k,U'_k)=\mathbf{E}\{r(X'_k,U'_k,Y_k)|X'_k,U'_k\}$, the following condition [4, (1.3), Corollary 1.3] is satisfied: $\lim_{t\to\infty}\sup_{U'}\mathbf{E}\sum_{k=t}^{\infty}|\bar r(X'_k,U'_k)|\gamma^k=0$.

In fact,

$$\sup_{U'}\mathbf{E}\sum_{k=t}^{\infty}|\mathbf{E}\{r(X'_k,U'_k,Y_k)|X'_k,U'_k\}|\gamma^k\le\sup_{U'}\mathbf{E}\sum_{k=t}^{\infty}|r(X'_k,U'_k,Y_k)|\gamma^k$$

$$=\sup_{U'}\mathbf{E}\sum_{k=t}^{\tau-1}|r(X'_k,U'_k,Y_k)|\gamma^k=\gamma^t\sup_{U'}\mathbf{E}\sum_{k=t}^{\tau-1}|r(X'_k,U'_k,Y_k)|\gamma^{k-t}$$

$$\le\gamma^t\sup_{U'}\mathbf{E}\sum_{k=t}^{\tau-1}(C_0+C_1|Y_k|+C_2U'_k+C_3X'_k)\gamma^{k-t}$$

$$\le\gamma^t\sup_{U'}\mathbf{E}\sum_{k=t}^{\tau-1}(C_0+C_1|Y_k|+(C_2+C_3)X'_k)\gamma^{k-t}$$

$$\le\gamma^t\sup_{U'}\mathbf{E}\sum_{k=t}^{\tau-1}(C_0+C_1|Y_k|+(C_2+C_3)(X'_t+\sum_{s=0}^{k-1}\max\{0,Y_s\}))\gamma^{k-t}$$

$$\le\frac{C_0+C_1\mathbf{E}|Y|}{1-\gamma}\gamma^t+\gamma^t(C_2+C_3)\sup_{U'}\mathbf{E}X'_t\sum_{k=t}^{\infty}\gamma^{k-t}+(C_2+C_3)\,\mathbf{E}\max\{0,Y\}\sum_{k=t}^{\infty}k\gamma^k\le\frac{C_0+C_1\mathbf{E}|Y|}{1-\gamma}\gamma^t$$

$$+\frac{(C_2+C_3)(x+t\,\mathbf{E}\max\{0,Y\})}{1-\gamma}\gamma^t+(C_2+C_3)\mathbf{E}\max\{0,Y\}\sum_{k=t}^{\infty}k\gamma^k\to 0.$$

Therefore, by virtue of [4, Corollary 1.3], $u^*(x)$ is an optimal control of problem (9), and this control is such that

$$V(x)=\sup_{\{U:U_k\in\mathbf{U}(X_k)\}}V(x,U)=V(x,U^*)=\mathbf{E}\sum_{t=0}^{\tau^*-1}\gamma^t r(X_t^*,u^*(X_t^*),Y_t),$$

where $\{X_{t+1}^*=X_t^*-u^*(X_t^*)+Y_t, X_0^*=x, 0\le t<\tau^*\}$ and τ^* is the ruin time of this process.

The theorem is proved.

OTHER (NON-DIVIDEND) BELLMAN PAYOFF FUNCTIONS

In [1–4], the barrier type of optimal controls is established for optimal dividend control problem (4), i.e., $u^*(x)=\max\{0,x-b\}$ for some $b\ge 0$. Note that, under the condition $\Pr\{Y<0\}>0$, for any barrier strategy $\tilde u(x)=\max\{0,x-b\}$, the probability of ruin of a process $\{\tilde x_{t+1}=\tilde x_t-\tilde u(\tilde x_t)+Y_t, \tilde x_0\in[0,b]\}$ is equal to one. Therefore, along with a barrier strategy, it is expedient to consider other types of strategies, for example, barrier-proportional strategies $\tilde u(x)=\max\{0,\alpha(x-b)\}$, where $b\ge 0$ and $0<\alpha\le 1$. In this case, the trajectory $\{X_t\}$ does not exceed the piecewise-linear barrier $B(x,\alpha,b)=\min\{x,\alpha b+(1-\alpha)x\}$. In [31, 32], a nonlinear dividend barrier $B(x)$ corresponding to the control $u(x)=\max\{0,x-B(x)\}$ is considered. However, in the general case, optimal dividend strategies can have a more complicated structure (a sequence of barriers and others [5, Sec. 1.5; 6]). If the parametric form of a dividend barrier is chosen, then the problem can consist of searching for optimal parameters of the barrier.

For any fixed continuous control $\tilde u(x)=\mathbf{U}(x)\subseteq[0,x]$, the corresponding values of average discounted dividends $\tilde V^0(x)=\mathbf{E}\sum_{t=0}^{\tilde\tau-1}\gamma^t\tilde u(x_t)$ and the average discounted lifetime $\tilde V^1(x)=\mathbf{E}\sum_{y=0}^{\tilde\tau(x)-1}\gamma^t=(1-\mathbf{E}\gamma^{\tilde\tau(x)})/(1-\gamma)$ can be found from the following equations by virtue of Theorems 2:

$$\tilde V^0(x)=\tilde u(x)+\gamma\mathbf{E}\tilde V^0(f(x,\tilde u(x),Y))=\tilde u(x)+\gamma\mathbf{E}\tilde V^0(x-\tilde u(x)+Y),\tag{21}$$

$$\tilde V^1(x)=1+\gamma\mathbf{E}\tilde V^1(f(x,\tilde u(x),Y))=1+\gamma\mathbf{E}\tilde V^1(x-\tilde u(x)+Y).\tag{22}$$

One more risk indicator for an insurance company is the average (discounted) reserve deficit at the time of ruin,

$$\tilde V^2(x)=\mathbf{E}\sum_{k=0}^{\tilde\tau(x)-1}\gamma^k r_2(\tilde x_k,\tilde u(\tilde x_k),Y_k)=-\mathbf{E}\sum_{k=0}^{\tilde\tau(x)-1}\gamma^k\min\{0,\tilde x_k-\tilde u(\tilde x_k)+Y_k\}.$$

When $u(x) \le x$, we have $0 \le r_2(\tilde{x}_k, \tilde{u}(\tilde{x}_k), Y) = -\min\{0, \tilde{x}_k - \tilde{u}(\tilde{x}_k) + Y\} \le -\min\{0, Y\}$. This indicator satisfies the following equation (when $\mathbf{E}\min\{0, Y\} > -\infty$):

$$\tilde{V}^2(x) = \mathbf{E}(-\min\{0, x - \tilde{u}(x) + Y\}) + \gamma \mathbf{E}\tilde{V}^2(x - \tilde{u}(x) + Y). \tag{23}$$

An important indicator of the performance of an insurance company is the (discounted when $\gamma < 1$) probability of ruin $\tilde{V}^3(x) = \mathbf{E}\sum_{k=0}^{\tilde{\tau}-1} \gamma^k \mathbf{1}_{\{\tilde{x}_k - \tilde{u}(\tilde{x}_k) + \tilde{x}_k < 0\}}$ considered as a function of the initial capital x under controls $\tilde{u}(\cdot)$, where

$$\{\tilde{x}_t = \tilde{x}_{t-1} - \tilde{u}(\tilde{x}_{t-1}) + Y_t,\ \tilde{x}_0 = x,\ t = 0, 1, \ldots, \tilde{\tau}(x)\},$$

$$\tilde{\tau}(x) = \sup\{t \in [0, \infty) : \min_{0 \le t' < t} \tilde{x}_{t'} \ge 0\}.$$

The function $\tilde{V}^3(x)$ satisfies the equation

$$\tilde{V}^3(x) = \mathbf{E}\mathbf{1}_{\{x - \tilde{u}(x) + Y < 0\}} + \gamma \mathbf{E}\tilde{V}^3(x - \tilde{u}(x) + Y). \tag{24}$$

If a process is considered on a finite time interval, then functions $\tilde{V}_t^3(x)$ of probability of ruin in t time intervals should be introduced at the initial state of the process x. These functions are connected by the relationships

$$\tilde{V}_t^3(x) = \mathbf{E}\mathbf{1}_{\{x - \tilde{u}(x) + Y < 0\}} + \gamma \mathbf{E}\tilde{V}_{t-1}^3(x - \tilde{u}(x) + Y),\ \tilde{V}_{-1}^3(x) = 0,\ t = 0, 1, \ldots .$$

Since the stochastic optimal dividend control problem being considered is multicriterion, for this x, it is expedient to construct sets of points $\{(\tilde{V}^0(x, \lambda), \tilde{V}^i(x, \lambda)), \lambda \ge 0\}$, $i = 1, 2, 3$, in the "profitability–risk" coordinates, where the parameter λ plays the role of the weight coefficient of aggregation of criteria in aggregates (14). To this end, for each $\lambda \ge 0$, it is necessary to solve integral Bellman equation (17) with $r(x, u, y) = u + \lambda r_i(x, u, y)$ and to find the corresponding optimal control function $\tilde{u}(x, \lambda)$ (20); then, for the found control $\tilde{u}(x, \lambda)$, to solve integral equations for dividends $\tilde{V}^0(x)$ (21), lifetime $\tilde{V}^1(x)$ (22), deficit at the ruin time $\tilde{V}^2(x)$ (23), and probability of ruin $\tilde{V}^3(x)$ (24). To implement this plan, integral equations (17) and (21)–(24) should be efficiently solved.

SUCCESSIVE APPROXIMATION METHOD FOR SOLVING BELLMAN EQUATIONS

Approximate numerical methods for solving one-criterion stochastic optimal control problems (with bounded Bellman functions) are studied in [33].

Equations (17) and (21)–(23) can be solved numerically by the successive approximation method

$$V_k(x) = \max_{u \in \mathbf{U}(x)} \{\mathbf{E}r(x, u, Y) + \gamma \mathbf{E}V_{k-1}(x - u + Y)\},\ V_{-1}(x) = 0,\ k = 0, 1, \ldots, \tag{25}$$

$$V_k^0(x) = \tilde{u}(x) + \gamma \mathbf{E}V_{k-1}^0(x - \tilde{u}(x) + Y),\ V_{-1}^0(x) = 0,\ k = 0, 1, \ldots, \tag{26}$$

$$V_k^1(x) = 1 + \gamma \mathbf{E}V_{k-1}^1(x - \tilde{u}(x) + Y),\ V_{-1}^1(x) = 0,\ k = 0, 1, \ldots, \tag{27}$$

$$V_k^2(x) = \mathbf{E}(-\min\{0, x - \tilde{u}(x) + Y\}) + \gamma \mathbf{E}V_{k-1}^2(x - \tilde{u}(x) + Y),$$

$$V_{-1}^2(x) = 0,\ k = 0, 1, \ldots, \tag{28}$$

$$\tilde{V}_t^3(x) = \mathbf{E}\mathbf{1}_{\{x - \tilde{u}(x) + Y < 0\}} + \gamma \mathbf{E}\tilde{V}_{t-1}^3(x - \tilde{u}(x) + Y),\ V_{-1}^3(x) = 0,\ k = 0, 1, \ldots . \tag{29}$$

Under conditions 1–4 and $\gamma < 1$, iterative method (25) converges by virtue of Theorem 2 and estimate (18) of Lemma 4. Similar estimates and uniform convergence when $\gamma < 1$ take place on any finite interval of values $x \in [0, x_{\max}]$ for sequences $\{V_k^i(x),\ k = 0, 1, \ldots\}$, $i = 0, 1, 2, 3$ (see (26)–(29)), for any fixed continuous control $\tilde{u}(x) \in [0, x]$ by virtue of Theorems 2 (for $\mathbf{U}(x) = \tilde{u}(x)$).

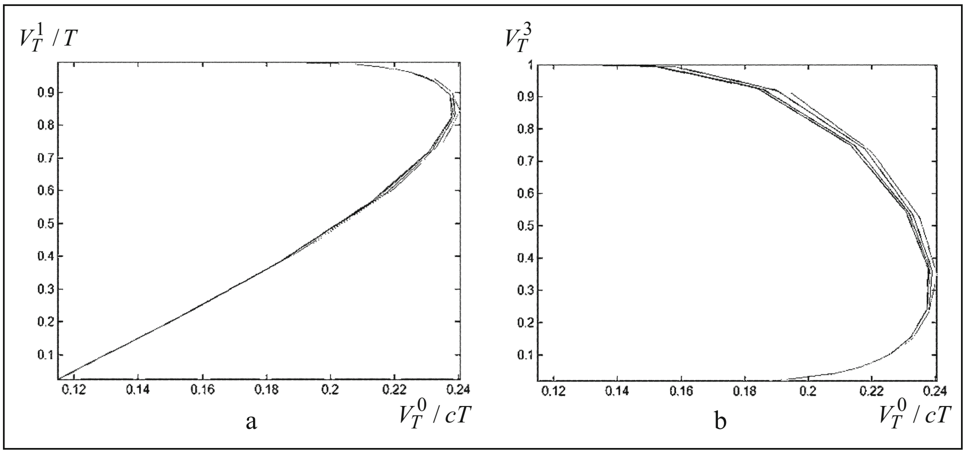

Fig. 1. Plots of approximation of Pareto-optimal sets.

When $\gamma = 1$, sequences $\{V_t^i(\cdot), t = 0, 1, \ldots\}$, $i = 2, 3$, are bounded and monotonically converge to their limits $\{V^i(\cdot)\}$, but the convergence can be slower than that with estimates (18) and nonuniform. In this case, the investigation of convergence requires a more close analysis [21, 22] since operators in the right side of recurrence relations (28) and (29) are not necessarily contracting.

The following lemma establishes the uniform convergence of the sequence of controls (16) to extremal mapping (19).

LEMMA 5 (convergence of a sequence of controls). Under the conditions of Theorem 2, the sequence of controls (16) uniformly converges to extremal mapping (19), namely, for any sequence of points $\{x_t \to x, t = 0, 1, \ldots\}$, the set of all limiting points of the sequence $\{u_t(x_t), t = 0, 1, \ldots\}$ belongs to $U^*(x)$.

Proof. We denote

$$v_{t-1}(x,u) = \mathbf{E}r(x,u,Y) + \gamma \mathbf{E}\widetilde{V}_{t-1}(x-u+Y), \quad v(x,u) = \mathbf{E}r(x,u,Y) + \gamma \mathbf{E}V(x-u+Y).$$

As has been noted in the proof of Theorem 2, functions $v_{t-1}(x,u)$, $\widetilde{V}_t(x)$ uniformly converge to functions $v(x,u)$ and $V(x)$, respectively. Let $\{x_t \to x\}$, and let $\{u_{t_k}(x_{t_k}) \to u, k = 1,2,\ldots\}$. Since $u_{t_k}(x_{t_k}) \in U_{t_k}^*(x_{t_k})$, we have $u_{t_k}(x_{t_k}) \in [0, x_{t_k}]$ and $\widetilde{V}_{t_k}(x_{t_k}) = v_{t_k-1}(x_{t_k}, u_{t_k})$. By virtue of uniform convergence properties, we obtain from this that $u \in [0, x]$ and $\widetilde{V}(x) = v_{t-1}(x,u)$, i.e., $u \in U^*(x)$, which is what had to be proved.

APPROXIMATION OF PARETO-OPTIMAL SETS IN THE DIVIDEND OPTIMIZATION PROBLEM

This section presents some results of numerical experiments on the approximation of the Pareto-optimal set of the optimal dividend control problem with respect to the profitability–risk criteria. The structure of optimal controls of problems (8) and (9) was first investigated experimentally by the successive approximation method for the aggregation of criteria (14). It turned out that, in a wide range of changing the parameters $0 \le \lambda \le 100$ and $0.5 < \gamma < 1$, optimal controls are of barrier type. Then approximations of Pareto-optimal boundaries for the sets $\{(V_T^0(x,\widetilde{u})/(cT), V_T^1(x,\widetilde{u})/T\}$ and $\{(V_T^0(x,\widetilde{u})/(cT), V_T^3(x,\widetilde{u}))\}$ were constructed, where controls are of the barrier-proportional form $\widetilde{u}(x) = \alpha \max\{0, x-b\}$, $\alpha \in (0,1]$, $b \in [0,x]$, with some fixed initial capital x and a planning horizon T. Here, the normalizing factor cT signifies the total insurance premium obtained in a time T. The quantities $V_T^0(x,\widetilde{u})$, $V_T^1(x,\widetilde{u})$, and $V_T^3(x,\widetilde{u})$ are iteratively found according to relations (26), (27), and (29), respectively. Figure 1 presents computational results for the initial capital $x = 10$, planning horizon $T = 100$, insurance premium $c = 1$, and random requirements $Y \in \{1, -1\}$ with the probabilities $\{0.6, 0.4\}$. Questions of constructing the distribution of the random quantity Y from data of insurance statistics are considered in [23, 34].

Computational experiments were performed with the help of the Matlab 8.2 system on a PC with Intel Core i5 3570K (3.4 GHz) and 8Gb of RAM. Plotting required no more than several seconds.

In Fig. 1, examples of approximation of Pareto-optimal sets are presented in the planes "normalized life time V_T^1 / T–normalized dividends V_T^0 / cT" (Fig. 1a) and "probability of ruin V_T^3–normalized dividends V_T^0 / cT" (Fig. 1b).

Numerical experiments show that, on a finite time interval in the presence of a discounting factor, barrier strategies $u(x) = \max\{0, x-b\}$ dominate over barrier-proportional strategies $u(x) = \alpha \max\{0, x-b\}$ when $\alpha < 1$, but, in the absence of discounting, barrier-proportional strategies can dominate over barrier strategies.

CONCLUSIONS

This article considers the stochastic optimal control problem of finding dividend policies of an insurance company with integral criteria uniting profitability and risk indicators. Applicability conditions for the method of dynamic programming are established and estimates for the convergence rate of the successive approximation method (15) for solving the problem are obtained. Numerical experiments revealed that optimal controls in the aggregated one-criterion problem are of the form of a barrier strategy. For constructing an approximation of a Pareto-optimal set, barrier-proportional control strategies are used. Numerical experiments showed that, on a finite time interval in the presence of the discounting factor, barrier strategies dominate over barrier-proportional strategies, but, in the absence of discounting, barrier-proportional strategies can dominate over purely barrier strategies.

REFERENCES

1. B. de Finetti, "Su un' impostazione alternativa della teoria collettiva del rischio," in: Trans. XV-th Intern. Congress of Actuaries, **2**, 433–443 (1957).
2. K. Borch, "The theory of risk (with discussion)," J. Roy. Statist. Soc. Ser. B, **29**, 432–467 (1967).
3. H. U. Gerber, An Introduction to Mathematical Risk Theory, Huebner Foundation Monographs, Philadelphia (1979).
4. H. Schmidli, Stochastic Control in Insurance, Springer, London (2008).
5. B. Avanzi, "Strategies for dividend distribution: A review," North Amer. Actuar. J., **13**, No. 2, 217–251 (2009).
6. H. Albrecher and S. Thonhauser, "Optimality results for dividend problems in insurance," Rev. R. Acad. Cien. Ser. A. Math., **103**, No. 2, 295–320 (2009).
7. S. Asmussen and H. Albrecher, Ruin Probabilities, 2nd Edition, World Sci., London (2010).
8. S. E. Shreve, J. P. Lehoczky, and D. P. Gaver, "Optimal consumption for general diffusions with absorbing and deflecting barriers," SIAM J. Control and Optim., **22**, No. 1, 55–75 (1984).
9. A. B. Piunovsky, Optimal Control of Random Sequences in Problems with Constraints [in Russian], Nauchn. Kniga, Moscow (1996).
10. A. B. Piunovsky, Optimal Control of Random Sequences in Problems with Constraints, Springer, Dordrecht (1997).
11. S. P. Sethi, Optimal Consumption and Investment with Bankruptcy, Kluwer, Boston–Dordrecht–London (1997).
12. J. Paulsen, "Optimal dividend payouts for diffusions with solvency constraints," Finance Stoch., **7**, No. 4, 457–473 (2003).
13. C. Hipp, "Optimal dividend payment under a ruin constraint: Discrete time and state space," Blatter der DGVFM (Blatter der Deutschen Gesellschaft fur Versicherungs-und Finanzmathematik e.V.), **26**, No. 2, 255–264 (2003).
14. D. C. M. Dickson and H. R. Waters, "Some optimal dividend problems," ASTIN Bulletin, **34**, No. 1, 49–74 (2004).
15. H. U. Gerber, E. S. W. Shiu, and N. Smith, "Maximizing dividends without bankruptcy," ASTIN Bull., **36**, No. 1, 5–23 (2006).
16. S. Thonhauser and H. Albrecher, "Dividend maximization under consideration of the time value of ruin," Insurance: Mathematics and Economics, **41**, 163–184 (2007).
17. E. Bayraktar and V. Young, "Maximizing utility of consumption subject to a constraint on the probability of lifetime ruin," Finance and Research Letters, **5**, No. 4, 204–212 (2008).
18. B. V. Norkin, "On numerical solution of the stochastic optimal control problem for the dividend policy of an insurance company," Computer Mathematics, No. 1, 131–139 (2014).

19. B. V. Norkin, “On stochastic optimal control of risk processes in discrete time,” Theory of Optimal Solutions, No. 1, 55–62 (2014).
20. B. V. Norkin, “On stochastic optimal control of discrete-time risk processes,” Problems of Control and Informatics, No. 5, 108–121 (2014).
21. B. V. Norkin, “Necessary and sufficient conditions of existence and uniqueness of solutions to integral equations of actuarial mathematics,” Cybernetics and Systems Analysis, **42**, No. 5, 743–749 (2006).
22. B. V. Norkin, “On the solution of the basic integral equation of actuarial mathematics by the successive approximation method,” Ukrainian Mathematical Journal, **59**, No, 12, 112–127 (2007).
23. B. V. Norkin, “Systems simulation analysis and optimization of insurance business,” Cybernetics and Systems Analysis,” **50**, No. 2, 260–270 (2014).
24. H. U. Gerber and E. S. W. Shiu, “On the time value of ruin,” North Amer. Actuar. J., **2**, 48–78 (1998).
25. Y. Ermoliev, T. Ermolieva, G. Fischer, et al., “Discounting, catastrophic risks management and vulnerability modeling,” Math. and Comput. in Simul., **79**, 917–924 (2008).
26. H. U. Gerber, “Entscheidungskriterien fur den zusammengesetzten Poisso–Prozess,” Schweiz. Verein. Versicherungsmath. Mitt., **69**, 185–228 (1969).
27. D. P. Bertsekas and S. E. Shreve, Stochastic Optimal Control: The Discrete-Time Case, Athena Sci., Belmont (Mass.) (1996).
28. J. P. Aubin and I. Ekeland, Applied Nonlinear Analysis [Russian translation], Mir, Moscow (1988).
29. A. N. Shiryaev, Probability [in Russian], Nauka, Moscow (1980).
30. R. T. Rockafellar and R. J.-B. Wets, Variational Analysis, Springer, Berlin (1998).
31. H. Gerber, “The dilemma between dividends and safety and a generalization of the Lundberg–Cramer formulas,” Scand. Actuar. J., No. 1, 46–57 (1974).
32. H. Albrecher and R. Kainhofer, “Risk theory with a nonlinear dividend barrier,” Computing, **68**, No. 4, 289–311 (2002).
33. K. J. Kushner and P. Dupuis, Numerical Methods for Stochastic Control Problems in Continuous Time, Springer, New York (1992).
34. B. V. Norkin, “On the identification of models of dynamic financial analysis of an insurance company,” Computer Mathematics, No. 2, 24–33 (2013).