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AN EXTRAGRADIENT ALGORITHM FOR MONOTONE VARIATIONAL INEQUALITIES[1]

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Abstract. *A new iterative algorithm is proposed for solving the variational inequality problem with a monotone and Lipschitz continuous mapping in a Hilbert space. The algorithm is based on the following two well-known methods: the Popov algorithm and so-called subgradient extragradient algorithm. An advantage of the algorithm is the computation of only one value of the inequality mapping and one projection onto the admissible set per one iteration. The weak convergence of sequences generated by the proposed algorithm is proved.*

Keywords: *variational inequality, monotone mapping, extragradient method, convergence.*

PROBLEM STATEMENT

Let C be a nonempty subset in a real Hilbert space H, and let A be a mapping in H. Let us consider the variational inequality

$$\text{find } x \in C : (Ax, y - x) \geq 0 \ \ \forall y \in C. \tag{1}$$

We denote the solution set of variational inequality (1) by $VI(A, C)$.

Assume that the following conditions are fulfilled:

- a set $C \subseteq H$ is convex and closed;
- a mapping $A : H \to H$ is monotone and Lipschitz with a constant $L > 0$;
- $VI(A, C) \neq \varnothing$.

Many problems in operations research (search for saddle points, finding a Nash equilibrium in noncooperative games, and minimization) and mathematical physics can be written in the form of variational inequalities [1–6], and a great number of methods [7–23], in particular, gradient methods are offered to date to solve them. It is well known that, for the convergence of simplest gradient methods in the case of non-optimization statements, strengthened monotonicity conditions should be fulfilled [7]. There are several approaches to overcome this difficulty. One of them is the regularization of the initial problem with a view to imparting a required property to it [8, 21]. Convergence without problem modification is provided in iterative methods of extragradient type, which were first proposed by A. S. Antipin [24] and G. M. Korpelevich [25].

For variational inequality (1), the extragradient Korpelevich algorithm is of the form

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases}$$

where $\lambda \in (0, 1/L)$ and P_C is an operator of metric projection onto the set C. It is well known that the sequences (x_n) and (y_n) weakly converge to some point $z \in VI(A, C)$. This algorithm is generalized and investigated in many publications [26–30, etc.].

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Among drawbacks of the Korpelevich algorithm is the computation of values of the mapping A at two different points (this is in the forefront in models of optimum control and especially in distributed parameter systems [6]) and the necessity of two projections onto the admissible set C (that are resource-consuming when the structure of the set C is complicated) to pass to the next iteration.

The first drawback is absent in the Popov extragradient algorithm [31] proposed as a modification of the Arrow–Hurwicz algorithm of searching for saddle points of convex-concave functions. For variational inequality (1), the Popov algorithm assumes the form

$$\begin{cases} x_0, y_0 \in C, \\ x_{n+1} = P_C(x_n - \lambda A y_n), \\ y_{n+1} = P_C(x_{n+1} - \lambda A y_n), \end{cases}$$

where $\lambda \in (0, 1/3L)$. In [31], for the case when $H = \mathbb{R}^d$, the convergence of the sequences (x_n) and (y_n) generated by this method is proved.

For variational inequalities [32] and problems of equilibrium programming [33], modifications of the Korpelevich algorithm with one metric projection onto the admissible set are proposed. In these so-called subgradient extragradient algorithms, the first stage of iteration coincides with the first stage of iteration in the Korpelevich algorithm, and then, for obtaining x_{n+1}, instead of projecting a point $x_n - \lambda A y_n$ onto the admissible set C, the point $x_n - \lambda A y_n$ is projected onto some semispace supporting for C. For variational inequality (1), the subgradient extragradient algorithm is of the form

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A x_n), \\ H_n = \{z \in H : (x_n - \lambda A x_n - y_n, z - y_n) \le 0\}, \\ x_{n+1} = P_{H_n}(x_n - \lambda A y_n), \end{cases}$$

where $\lambda \in (0, 1/L)$. The weak convergence of the sequences (x_n) and (y_n) generated by this algorithm to some point $z \in VI(A, C)$ is proved in [32, 33].

This work proposes a new algorithm for solving variational inequalities (1) with a monotone and Lipschitz mapping combining positive features of the Popov extragradient algorithm and subgradient extragradient algorithm. The weak convergence theorem for the algorithm is proved.

AUXILIARY STATEMENTS

Let H be a real Hilbert space with the scalar product $(\cdot, \cdot)$ and a generated norm $\|\cdot\|$, and let $C \subseteq H$ be a nonempty convex and closed set. Let P_C be a metric projection onto the set C, i.e., $P_C x$ is a unique element of the set C with the property $\|P_C x - x\| = \min_{z \in C} \|z - x\|$. The following characterizations of the element $P_C x$ are useful [2, 34]:

$$y = P_C x \Leftrightarrow y \in C, \ (y - x, z - y) \ge 0 \ \ \forall z \in C, \tag{2}$$

$$y = P_C x \Leftrightarrow y \in C, \ \|y - z\|^2 \le \|x - z\|^2 - \|y - x\|^2 \ \ \forall z \in C. \tag{3}$$

It follows from inequality (2) that $x \in VI(A, C)$ if and only if $x = P_C(x - \lambda A x)$, where $\lambda > 0$ [2, 34].

If a mapping $A : H \to H$ is monotone and continuous and a set $C \subseteq H$ is convex and closed, then $x \in VI(A, C)$ if and only if $x \in C$ and $(Ay, y - x) \ge 0$ for all $y \in C$ [1, 2, 34]. In particular, the set $VI(A, C)$ is convex and closed.

In proving the convergence of sequences of elements in a Hilbert space, we will use the well-known Opial lemma.

LEMMA 1 [35]. Let a sequence (x_n) of elements of the Hilbert space H weakly converges to an element $x \in H$. Then, for all $y \in H \setminus \{x\}$, we have $\varliminf_{n \to \infty} \|x_n - x\| < \varliminf_{n \to \infty} \|x_n - y\|$.

ALGORITHM DESCRIPTION

To solve problem (1), the following algorithm is proposed.

Algorithm 1

1. Specify $x_0, y_0 \in C$, and $\lambda > 0$.
2. Compute
$$\begin{cases} x_1 = P_C(x_0 - \lambda A y_0), \\ y_1 = P_C(x_1 - \lambda A y_0). \end{cases}$$
3. Given x_n, y_n, and y_{n-1}, construct a semispace
$$H_n = \{z \in H : (x_n - \lambda A y_{n-1} - y_n, z - y_n) \le 0\}$$
and compute
$$\begin{cases} x_{n+1} = P_{H_n}(x_n - \lambda A y_n), \\ y_{n+1} = P_C(x_{n+1} - \lambda A y_n). \end{cases}$$
4. If $x_{n+1} = x_n$ and $y_{n+1} = y_n = y_{n-1}$, then complete the computation and, otherwise, put $n := n+1$ and pass to step 3.

We first note that $C \subseteq H_n$. In fact, if we assume that an element $z \in C \setminus H_n$ exists, then the inequality $(x_n - \lambda A y_{n-1} - y_n, z - y_n) > 0$ will contradict the fact that $y_n = P_C(x_n - \lambda A y_{n-1})$.

Let us show that the completion of algorithm 1 leads to the solution of variational inequality (1).

LEMMA 2. If $x_{n+1} = x_n$ and $y_{n+1} = y_n = y_{n-1}$ in Algorithm 1, then $y_n \in VI(A, C)$.

Proof. If we have $x_{n+1} = x_n$ in Algorithm 1, then characterization (2) implies
$$(Ay_n, x - x_n) \ge 0 \ \ \forall x \in H_n. \tag{4}$$
Taking into account that $x_{n+1} \in H_n$ and $y_n = y_{n-1}$, we obtain $(x_n - \lambda A y_n - y_n, x_n - y_n) \le 0$, whence we conclude that $(Ay_n, x_n - y_n) \ge 0$. Then we represent inequality (4) in the form $(Ay_n, x - y_n) - (Ay_n, x_n - y_n) \ge 0 \ \forall x \in H_n$. Hence, we have $(Ay_n, x - y_n) \ge (Ay_n, x_n - y_n) \ge 0 \ \forall x \in H_n$. Since $y_n \in C \subseteq H_n$, we obtain that $y_n \in VI(A, C)$.

We pass to the proof of the weak convergence of the algorithm.

MAIN RESULT

We first prove an important inequality relating the distances from the points generated by the algorithm to the set $VI(A, C)$.

LEMMA 3. Let sequences (x_n) and (y_n) be generated by algorithm 1, $z \in VI(A, C)$. Then the following inequality holds:
$$\begin{aligned} \| x_{n+1} - z \|^2 \le \| x_n - z \|^2 - (1 - 2\lambda L) \| x_{n+1} - y_n \|^2 \\ - (1 - \lambda L) \| x_n - y_n \|^2 + \lambda L \| x_n - y_{n-1} \|^2. \end{aligned} \tag{5}$$

Proof. Since $z \in VI(A, C) \subseteq H_n$, $x_{n+1} = P_{H_n}(x_n - \lambda A y_n)$ and characterization (3) imply
$$\begin{aligned} \| x_{n+1} - z \|^2 &\le \| x_n - \lambda A y_n - z \|^2 - \| x_n - \lambda A y_n - x_{n+1} \|^2 \\ &= \| x_n - z \|^2 - \| x_n - x_{n+1} \|^2 - 2\lambda (Ay_n, x_{n+1} - z). \end{aligned} \tag{6}$$
It follows from the monotonicity of A and the membership $z \in VI(A, C)$ that $(Ay_n, y_n - z) \ge 0$. Adding a nonnegative addend $2\lambda (Ay_n, y_n - z)$ to the right side of inequality (6), we obtain
$$\| x_{n+1} - z \|^2 \le \| x_n - z \|^2 - \| x_n - x_{n+1} \|^2 - 2\lambda (Ay_n, x_{n+1} - y_n)$$
$$= \| x_n - z \|^2 - \| x_n - y_n \|^2 - \| x_{n+1} - y_n \|^2 - 2(x_n - y_n, y_n - x_{n+1})$$

$$-2\lambda(Ay_n, x_{n+1}-y_n) = \|x_n - z\|^2 - \|x_n - y_n\|^2 - \|x_{n+1}-y_n\|^2$$

$$+2\lambda(Ay_{n-1}-Ay_n, x_{n+1}-y_n) + 2(x_n - \lambda Ay_{n-1} - y_n, x_{n+1}-y_n). \quad (7)$$

The membership $x_{n+1} \in H_n$ implies the inequality $(x_n - \lambda Ay_{n-1} - y_n, x_{n+1} - y_n) \le 0$. We estimate the addend $2\lambda(Ay_{n-1} - Ay_n, x_{n+1} - y_n)$ in inequality (7) as follows:

$$2\lambda(Ay_{n-1}-Ay_n, x_{n+1}-y_n) \le 2\lambda L\|y_{n-1}-y_n\|\,\|x_{n+1}-y_n\|$$

$$\le 2\lambda L(\|y_{n-1}-x_n\| + \|x_n - y_n\|)\|x_{n+1}-y_n\|$$

$$\le \lambda L(\|y_{n-1}-x_n\|^2 + 2\|x_{n+1}-y_n\|^2 + \|x_n - y_n\|^2).$$

In view of the previous calculations, we obtain inequality (5).

Let us formulate the main result of this work.

THEOREM 1. Let a set $C \subseteq H$ be convex and closed, let the mapping $A: H \to H$ be monotone and Lipschitz with a constant $L > 0$, let $VI(A, C) \neq \varnothing$, and let $\lambda \in \left(0, \frac{1}{3L}\right)$. Then sequences (x_n) and (y_n) generated by algorithm 1 weakly converge to some point $z \in VI(A, C)$.

Proof. We first show the boundedness of the sequence (x_n). We fix a number $N \in \mathbb{N}$ and consider inequalities (5) for all numbers N, $N+1$, …, M, where $M > N$. Summing up them, we obtain

$$\|x_{M+1} - z\|^2 \le \|x_N - z\|^2 - (1-3\lambda L)\sum_{n=N}^{M}\|x_{n+1}-y_n\|^2$$

$$-(1-\lambda L)\sum_{n=N}^{M}\|x_n - y_n\|^2 + \lambda L\|x_N - y_{N-1}\|^2. \quad (8)$$

This implies the boundedness of the sequence (x_n).

From inequality (8), we obtain the convergence of the series $\sum_n \|x_{n+1} - y_n\|^2$ and $\sum_n \|x_n - y_n\|^2$. Thus, we have

$$\lim_{n\to\infty}\|x_{n+1}-y_n\| = \lim_{n\to\infty}\|x_n - y_n\| = 0. \quad (9)$$

Let us consider the subsequence (x_{n_k}) weakly convergent to some point $z \in H$. Then $y_{n_k} \to z$ weakly and $z \in C$. Let us show that $z \in VI(A, C)$. We have

$$(y_{n_k+1} - x_{n_k+1} + \lambda Ay_{n_k}, x - y_{n_k+1}) \ge 0 \quad \forall x \in C.$$

Using the monotonicity of the mapping A, we infer

$$0 \le (y_{n_k+1} - x_{n_k+1} + \lambda Ay_{n_k}, x - y_{n_k+1}) + (y_{n_k+1} - x_{n_k+1}, x - y_{n_k+1})$$

$$+\lambda(Ay_{n_k}, y_{n_k} - y_{n_k+1}) + \lambda(Ay_{n_k}, x - y_{n_k}) \le (y_{n_k+1} - x_{n_k+1}, x - y_{n_k+1})$$

$$+\lambda(Ay_{n_k}, y_{n_k} - y_{n_k+1}) + \lambda(Ax, x - y_{n_k}).$$

Passing to the limit with allowance for equalities (9), we obtain $(Ax, x - z) \ge 0$ $\forall x \in C$. Hence, $z \in VI(A, C)$.

Let us show that $x_n \to z$ weakly (then $\|x_n - y_n\| \to 0$ implies $y_n \to z$ weakly). Using proof by contradiction, we assume that there is a subsequence (x_{m_k}) such that $x_{m_k} \to z'$ weakly and $z \neq z'$. It follows from inequality (5) and the inequality $0 < 3\lambda L < 1$ that

$$\|x_{n+1} - x\|^2 + \lambda L\|x_{n+1} - y_n\|^2 \le \|x_n - x\|^2 + \lambda L\|x_n - y_{n-1}\|^2 \quad \forall x \in VI(A, C).$$

Thus, for all $x \in VI(A, C)$, there is

$$\lim_{n\to\infty}(\|x_n - x\|^2 + \lambda L\|x_n - y_{n-1}\|^2) \in \mathbb{R}.$$

Applying the Opial lemma twice,

$$\lim_{n\to\infty}(\|x_n - z\|^2 + \lambda L\|x_n - y_{n-1}\|^2) = \lim_{k\to\infty}(\|x_{n_k} - z\|^2 + \lambda L\|x_{n_k} - y_{n_k-1}\|^2)$$

$$= \underline{\lim}_{k\to\infty}\|x_{n_k} - z\|^2 < \underline{\lim}_{k\to\infty}\|x_{n_k} - z'\|^2 = \lim_{k\to\infty}(\|x_{n_k} - z'\|^2 + \lambda L\|x_{n_k} - y_{n_k-1}\|^2)$$

$$= \lim_{n\to\infty}(\|x_n - z'\|^2 + \lambda L\|x_n - y_{n-1}\|^2) = \lim_{k\to\infty}(\|x_{m_k} - z'\|^2 + \lambda L\|x_{m_k} - y_{m_k-1}\|^2)$$

$$= \underline{\lim}_{k\to\infty}\|x_{m_k} - z'\|^2 < \underline{\lim}_{k\to\infty}\|x_{m_k} - z\|^2$$

$$= \lim_{k\to\infty}(\|x_{m_k} - z\|^2 + \lambda L\|x_{m_k} - y_{m_k-1}\|^2) = \lim_{n\to\infty}(\|x_n - z\|^2 + \lambda L\|x_n - y_{n-1}\|^2),$$

we obtain an inconsistent inequality. Thus, $z = z'$.

Note that a weak limit $z \in VI(A, C)$ of the sequence (x_n) generated by Algorithm 1 possesses the property

$$P_{VI(A,C)}x_n \to z \text{ strongly.} \tag{10}$$

In fact, we have $(P_{VI(A,C)}x_n - x_n, z - P_{VI(A,C)}x_n) \geq 0$. If we prove that $P_{VI(A,C)}x_n \to \bar{z}$ strongly, then, after passage to the limit, we obtain $(\bar{z} - z, z - \bar{z}) \geq 0$, i.e., $z = \bar{z}$ and convergence (10) holds. Let us prove the strong convergence of $(P_{VI(A,C)}x_n)$. We have

$$\|x_{n+1} - P_{VI(A,C)}x_{n+1}\|^2 \leq \|x_{n+1} - P_{VI(A,C)}x_n\|^2$$

$$\leq \|x_n - P_{VI(A,C)}x_n\|^2 + \lambda L\|x_n - y_{n-1}\|^2.$$

The summability of the series $\sum_n \|x_{n+1} - y_n\|^2$ implies the existence of $\lim_{n\to\infty}\|x_n - P_{VI(A,C)}x_n\| \in \mathbb{R}$. Applying inequality (3) and Lemma 2, we obtain

$$\|P_{VI(A,C)}x_m - P_{VI(A,C)}x_n\|^2 \leq \|x_m - P_{VI(A,C)}x_n\|^2 - \|P_{VI(A,C)}x_m - x_m\|^2$$

$$\leq \|x_{m-1} - P_{VI(A,C)}x_n\|^2 - \|P_{VI(A,C)}x_m - x_m\|^2 + \lambda L\|x_{m-1} - y_{m-2}\|^2 \leq$$

$$\ldots \leq \|x_n - P_{VI(A,C)}x_n\|^2 - \|P_{VI(A,C)}x_m - x_m\|^2 + \lambda L\sum_{k=n}^{m}\|x_{k-1} - y_{k-2}\|^2, \; m > n.$$

This implies the fundamentality of the sequence $(P_{VI(A,C)}x_n)$.

Comment 1. The statement (on convergence) similar to Theorem 1 will also take place for the following iterative process:

$$\begin{cases} x_{n+1} = P_{H_n}(x_n - \lambda_n Ay_n), \\ y_{n+1} = P_C(x_{n+1} - \lambda_n Ay_n), \end{cases}$$

where $H_n = \{z \in H\colon (x_n - \lambda_{n-1}Ay_{n-1} - y_n, z - y_n) \leq 0\}$, provided that $0 < \inf \lambda_n \leq \sup \lambda_n < \frac{1}{3L}$.

Comment 2. An obvious drawback of the studied algorithm is the assumption that the Lipschitz constant L is known. It is necessary to develop a version of the algorithm without using this information with a step control similar to the well-known Armijo rule [9, 10].

This work considers the problem of obtaining an approximate solution for variational inequalities with monotone and Lipschitz mappings in a Hilbert space. A new iterative algorithm is proposed on the basis of two methods, namely, the Popov algorithm [31] and subgradient extragradient algorithm [32, 33]. A weak convergence theorem is proved for sequences generated by the algorithm. Note that the computational expenditures required for the execution of an iteration step of Algorithm 1 are almost equal to the expenditures for implementation of a step in the classical gradient projection method.

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