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SYSTEMS SIMULATION ANALYSIS AND OPTIMIZATION OF INSURANCE BUSINESS[1]

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Abstract. *Problems of computational actuarial mathematics, dynamic financial analysis, and optimization of insurance business and the possibility of their solution by means of parallel computing on graphics accelerators are discussed. The ruin probability and other performance criteria of an insurance company are estimated by the Monte Carlo method. In many cases, it is the only applicable method. Since the ruin probability is small enough, to achieve an acceptable estimate accuracy, an astronomical number of simulations may be required. Parallelization of the Monte Carlo method and the use of graphical accelerators allow us getting the desired result in a reasonable time. The results of numerical experiments on the developed system of actuarial modeling are presented, allowing the use of graphical accelerator that supports Nvidia CUDA 1.3 and higher.*

Keywords: *computational actuarial mathematics, dynamic financial analysis, simulation modeling, optimization of insurance business, risk process, ruin probability, efficient frontier, parallel computing, Monte Carlo method, GPGPU, CUDA.*

INTRODUCTION

The essence of insurance business is to maximize net profit under sufficient insurance reserves for covering insurance claims. The probability of ruin or insolvency of the company can be taken as a risk measure, and return can be estimated based on the distribution of the obtained dividends and remaining capital at the end of a planned time period. It is important to observe delicate risk–returns balance. For example, the de Finetti paradox [1; 2, Theorem 1.10 (vi)] implies that the optimal strategy that maximizes net average discounted profit (dividends) of the company leads to ruin with probability one (on the infinite time interval).

To formally describe the activity of an insurance company, the classical random risk process (the Cramer–Lundberg model) that models the stochastic evolution of the capital of the insurance company is often used [1–7]. In this model, on the one hand, the capital monotonically and linearly increases with time due to continuously arriving premiums, on the other hand, at random instants of time (of the arrival of insurance claims) it decreases by a random variable (claim). The company is ruined if the capital becomes less than zero.

It is obvious that this process does not reflect many aspects of insurance company's activity, for example, reinsurance, investments, loans, dividend payment, possible catastrophic claims, etc. Therefore, to model a real insurance company, stochastic simulation models of dynamic financial analysis are used [8–10], which consider the influence of many control and random factors on the dynamics of company's reserves. Shares of deductions to insurance reserve, deductions for investment activity, dividends, reinsurance, operational expenditures, and other strategic control parameters are taken as control parameters, and the level of insurance payments (with respect to insurance premiums), investment yield, inflation rate, reinsurance contract payments, etc. are important random factors. The results of modeling for preset values of the

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control parameters are ambiguous and are random variables. A single-shot modeling of the trajectory of the time variation of performance indices of the company does not provide a complete idea of the control quality. It is necessary to model a considerable number of trajectories in order to represent the probability distribution of the results of the company functioning. Then it is possible to calculate any deterministic characteristics of the chosen control strategy: ruin probability on a given time interval, expected dividends, expected residual reserves, etc. However, the problem is that estimating these indices, for example, ruin probability, may require an astronomical number of simulations, which cannot be fulfilled in a reasonable time even on modern computers. This problem is recognized in actuarial mathematics; therefore, for example, various formulas for the approximate estimate of ruin probability are obtained for classical process of risk [3–7]. However, they do not work for more general models, and especially for real models of insurance activity.

The Monte Carlo method of statistical simulation modeling is a universal and often a unique method to model complex dynamic stochastic systems. But it may need a huge number of simulations for adequate accuracy since ruin probability is normally small [6]. For example, if a ten-year insurance (quarter) statistics and ten-year planned horizon are used, the number of possible scenarios of evolution of the capital is of order 10^{64} and, respectively, the probability of each separate scenario is 10^{-64}. The exhaustive search of such number of scenarios, exact calculation of averaged performance parameters of the company over all the scenarios and ruin probabilities are impossible on modern computers. However, escalating the computer power and paralleling allow us to solve the problem approximately by the Monte-Carlo method with a comprehensible accuracy. In [11–13], the parallel version of the Monte-Carlo method (along with the parallel method of successive approximations) is implemented on a cluster of several personal computers and is applied to find the ruin probability as a function of the seed capital of an insurance company. In [14, 15], a program system is announced, which is intended for parallel simulation modeling of the evolution of reserves of an insurance company (the parallel Monte-Carlo method) and for the solution of various actuarial computing problems. The system allows us to carry out calculations both on the central (multi-core) processor and using the graphic accelerator with program architecture of Nvidia CUDA [16], and also ten millions of simulations in real time, to model promptly the operation of the insurance company for various values of control parameters, to estimate the ruin probability, to predict expected results, to analyze the dependence of performance parameters of the company on any control parameters, and to correlate risk and returns. The system uses data of the real insurance statistics. In the present paper, we will describe the operation principles of a new version of the system and present the results of numerical experiments.

The use of the graphic accelerator (Nvidia GeForce GTX 560 2Gb) has reduced the computing time by an order of magnitude as compared with parallel computing on the four-core Intel Core i5 3570K central processor. Moreover, the rate of calculations makes it possible to carry out serial (coordinatewise) optimization of the company operation with respect to parameters taking into account return and risk. The control strategies included in the system are parametric, i.e., depend on a finite set of numerical parameters. All the variations of the control parameters are mapped in the space of performance parameters of the company, namely, in the "return–risk" plane, which makes it possible to perform a purposeful enumeration of the values of parameters being optimized. Simultaneous control of return and risk makes it possible to overcome in some extent the de Finetti paradox. Note that this paradox was overcome in [17, 18] by introducing an explicit constraint for ruin probability. Problems of stochastic optimal control of an insurance company (which are more complex for the computational point of view) are considered in [2].

The use of parallel computing, in particular, multi-core graphic accelerators, is a new promising direction in computing actuarial mathematics. Applying parallel computing for the optimization of expectations in stochastic programming was considered as far back as by Dantzig [19, 20]. Related problems of parallel modeling of random walks on graphic accelerators are considered in [21]. In [22, 23], graphic accelerators are used to solve some optimization problems.

RISK PROCESSES AS MODELS OF THE EVOLUTION OF RESERVES OF AN INSURANCE COMPANY

Aggregated Models. Insurance company should sustain some level of insurance reserves to cover current random insurance claims. The mathematical model of the stochastic evolution of reserves x^t of insurance company has the form

$$x^t = u + \int_0^t (c - d(x^s))\,ds - S_t, \quad 0 \le t \le T, \tag{1}$$

where t is a time parameter $(0 \le t \le T)$, $x_0 = u \ge 0$ is a seed capital (insurance reserve), $S_t = \sum_{k=1}^{N_t} z_k$ are aggregated random insurance claims, z_k are random claims at the moments $(t_1 + \ldots + t_k)$, where times t_i have the distribution function $\overline{F}_{t_i}(\cdot)$, N_t is the number of random claims arrived by time t, c is aggregated insurance premium per unit time, $d(\cdot)$ is some function that expresses the intensity of payments of dividends and other deductions depending on current reserves, and $0 \le d(\cdot) \le c$. The function $d(\cdot)$ has the meaning of positional control of dividends. For example,

$$d(x) = \begin{cases} \gamma(x - b(x)), & x \ge b(x), \\ 0, & x < b(x), \end{cases}$$

where $b(\cdot)$ is some monotonically increasing function called dividend barrier, γ is a capital share that is not used for business dealing (for example, dividends). The minimum dividend barrier is governed by legislation [24, art. 30]. In the Cramer–Lundberg–de Finetti classical model [1–7] (with subtraction of constant dividends) d, $0 \le d \le c$, the capital evolution equation has the form

$$x^t = u + (c - d)t - \sum_{k=1}^{N_t} z_k, \ 0 \le t \le T, \tag{2}$$

where $\{z_k, k = 1, 2, \ldots\}$ are independent equally distributed random variables of claims with the common distribution function $F(\cdot)$ and average μ, N_t is an integer random variable with Poisson distribution with the parameter λ (time intensity of the arrival of claims in the exponential distribution).

Reinsurance of risks is an important aspect of insurance activity. On the one hand, company's income decreases by the value of the premium paid to the reinsuring companies, on the other hand, the value of actual insurance claims to the company decreases due to the partial covering of the claims by the reinsuring companies. In the insurance statistics, reinsurance is mapped by the following parameters: the volume r and share α of (outcoming) reinsurance in the total amount of premiums, the volume q of incoming reinsurance from residents and non-residents (foreign reinsurers), the level β of reinsurance payments (reinsurance payments to reinsurance cost ratio).

Taking into account the reinsurance, the reserve evolution equation (1) becomes

$$x^t = u + \int_0^t (c - r - d(x^s))ds - (S_t - Q_t(r)), \ 0 \le t \le T,$$

where $r = \alpha c$ is the part of the premium paid to the reinsurer and $Q_t(r)$ are resultant aggregated payments according to reinsurance contracts till the moment t.

In practice, the financial state of the company is registered at discrete times, for example, quarterly [25, 26]. In this case, the mathematical model of the stochastic evolution of reserves x^t of the insurance company has the form

$$x^t = x^{t-1} + c^t - r^t - (s^t - q^t) - d^t, \tag{3}$$

where $t = 0, 1, \ldots, T-1$ is a discrete time parameter, $x^0 = u \ge 0$ is the seed capital (reserve), x^t is the current insurance reserve at time t, $\{c^t, \ r^t, \ d^t, \ s^t, q^t\}$ are, respectively, total quarter premiums, expenditures for reinsurance contracts, dividend payments, random insurance claims, and payments according to reinsurance contracts for the period t.

As the aggregated model of insurance payments in (3), the following relation can be used [18]:

$$s^t - q^t = c^t \xi^t - r^t \zeta^t = c^t (\xi^t - \alpha^t \zeta^t), \tag{4}$$

where $\alpha^t = r^t / c^t$ is the level of expenditures for reinsurance, (ξ^t, ζ^t) is the realization of a random vector (ξ, ζ) whose distribution can be found from the insurance statistics. For example, let the historical statistics $\{c^\tau, r^\tau, s^\tau, q^\tau, \tau = 1, \ldots, m\}$ be known. Then as the empirical distribution of the vector (ξ, ζ), it is possible to take the frequency of vectors $\{(s^\tau / c^\tau, q^\tau / r^\tau), \ \tau = 1, \ldots, m\}$ appearing in the Borel subsets of the space $\mathbb{R}^2$.

Often, the available insurance statistics has the form $\{c^\tau, r^\tau, s^\tau - q^\tau, \tau = 1, \ldots, m\}$ [25, 26]. On the assumption that all reinsurance contracts obey the proportional principle, i.e., $q^\tau = (r^\tau / c^\tau) s^\tau = \alpha^\tau s^\tau$, relation (4) becomes $s^t - q^t = s^t (1 - \alpha^t)$.

Let us consider the ruin (insolvency) probability $\psi(\cdot)=\Pr\{\inf_{0\le t\le T} x^t<0\}$ of insurance company as a function of parameters of the process $\{x^t\}$. This probability can be used as a risk measure in the control of the insurance company. For example, the ruin probability (on an infinite time interval) in the classical model of insurance company (2) with exponential distribution of claims has the form [3–7]

$$\psi(u,c,d,\lambda,\mu)=\begin{cases}\dfrac{1}{1+\rho}\exp\left(-\dfrac{\rho u}{(1+\rho)\mu}\right), & \rho>0,\\ 1, & \rho\le 0,\end{cases} \tag{5}$$

where $\rho=(c-d)/(\lambda\mu)-1$. In this case, the dependence $\psi(u,c,d,\lambda,\mu)$ is known in an explicit form. If the distribution of claims is not exponential, then the following estimate is true for the ruin probability [3–7]:

$$\psi(u,c,d,\lambda,F)\le e^{-Ru}, \tag{6}$$

where R is the positive root (if exists) of the equation

$$\frac{\lambda}{c-d}\int_0^{+\infty} e^{Ry}(1-F(y))dy=1.$$

In the more general model (1), the dependence of ruin probability on the parameters is unknown and can only be obtained by the Monte-Carlo method. Formula (5) is used here to test and adjust the Monte-Carlo method. Along with the ruin probability, of interest is also the distribution of the capital x^T at some time T and of the obtained dividends $\int_0^T d(x^s)ds$, their mean values and variances at this moment, and the dependence of these quantities on various parameters.

Modeling a Portfolio of Insurance Contracts. In insurance, by a portfolio is understood the structure of aggregated insurance premiums obtained from different forms of insurance and the structure of payments for reinsurance of various forms of risks. The dynamics of insurance reserves with regard for the (fixed) structure of insurance portfolio for constant dividends is described by the following relations [18]:

$$x^t=x^{t-1}+\sum_{i=1}^n(y_i-z_i)-\sum_{i=1}^n(\xi_i^t y_i-\zeta_i^t z_i)-d,\ x^0=u,\ t=1,\dots,T,$$

$$0\le z_i\le y_i,\ i=1,\dots,n+1,\ 0\le d\le\sum_{i=1}^n y_i\le c.$$

The vector variable $y=(y_1,\dots,y_n)$ specifies the (fixed) structure of the portfolio of insurance contracts of the company and y_i specifies the premium per unit time from the ith form of insurance; the vector variable $z=(z_1,\dots,z_n)$ specifies the structure of the portfolio of reinsurance contracts, where the variable z_i specifies payment per unit time for reinsurance of risks in the ith form of insurance; variable $x=(x^0,x^1,\dots,x^T)$ describes the stochastic dynamics of insurance reserves of the company at discrete times $t=0,1,\dots,T$ and is a random vector value; variable d specifies deterministic deductions from premiums, not related to insurance payments (for example, taxes, salary, maintenance of infrastructure, dividends, etc.). In the present paper, we do not consider problems of the optimal control of insurance company; therefore, variables y, z, and d do not depend on time. Parameter c specifies planned volume of premiums over all the insurance contracts per unit time; $x^0=u$ is the initial reserve capital; T is planned time horizon (the number of quarters or years considered for the estimate of ruin probability). Thus, $\left(c-d-\sum_{i=1}^n z_i\right)$ specifies deductions per unit time from premiums to insurance reserves of the company. Quantities $(\xi_1^t,\dots,\xi_n^t)$ specify (random) values of unprofitableness of different forms of insurance $i=1,\dots,n$ per unit income of premiums at unit time interval (quarter, year, etc.) with number t, i.e., $\xi_i^t y_i$ are total insurance payments on the ith form of insurance at time $t=1,\dots,T$. The quantity ζ_i^t specifies random return from the reinsurance of risks of the ith form, i.e., payments from reinsurers per unit premiums to the reinsurers with respect to risks of the ith form.

OPTIMIZATION OF INSURANCE BUSINESS

Insurance business assumes obtaining the maximum net profit at sufficient insurance reserves to cover possible insurance claims with a prescribed degree of reliability. In a competitive environment, company tends to save or increase its share in the market.

Optimization in the Classical Model of Insurance Activity. The traditional theory of microeconomics of insurance is based on the concept of expected utility [3]. Economic agents, including insurance companies, select the behavior based on the principle of the maximization of expected utility. Utility functions allow qualitative explanation of many phenomena in economy; however, they are of little use for practical calculations since they cannot be measured by physical or other direct methods.

Another approach to the analysis of insurance activity is based on the analysis of return and risk (ruin probability). Collected dividends [1, 2, 17, 18], the capital of the company at the end of the planned period [6], or average return of the insurance activity [27] can be used as a measure of return, and the variance of capital [6], ruin probability [2, 6], its upper exponential estimates [27, 28], the Lundberg coefficient [2, Ch. 4; 3, Sec. 14.5; 29], expectation of maximum total losses [3, Sec. 13.6, 14.4, 14.5], and other characteristics of ruin event can be used as risk measures. The main problem in this approach is the calculation or estimation of the ruin probability of insurance company. The literature on this problem is quite extensive [3–7].

Following [1], another optimization criterion, the level of net profit (dividends), was used in [17], and a constraint was superimposed on the ruin probability to resolve the de Finetti paradox. Instead of exact ruin probability, its exponential estimate was used (such approach was used earlier in [27]). This approach allows us to obtain an explicit expression for the Lundberg coefficient, to eliminate a complex probability constraint from the problem statement, decompose the problem, and to obtain analytic solutions in many cases. It was applied to solve a number of problems of control of insurance business, namely, to choose the initial insurance reserve, optimize tariff rates, insurance portfolio (the structure of insurance contracts), reinsurance contracts, and perform optimum control of the dividend policy of the company [17]. Both the parameters of reinsurance contracts and other performance characteristics of the insurance company were used as optimization variables, for example, assurance prices or levels of investments into various fields of its activity. The required values were the level of dividend payments and other control parameters calculated as functions of the seed capital. A similar approach for the company in the large is considered in [28]. The obtained functions can be used for the current control of the company, namely: the current random value of the capital is substituted therein and the obtained parameters are applied for operating control of the company.

Optimization of the Structure of Insurance Portfolio for a Finite Number of Scenarios for Random Data. In the notation of the previous section, we can formulate the problem of approximate optimization of the net profit (dividends) of the insurance company under constraints imposed on the ruin probability as follows [18]:

$$d \to \max_{\{x^t\},\,\{y_i\},\,\{z_i\},\,d},$$

$$x^t = x^{t-1} + \sum_{i=1}^{n} (y_i - z_i) - \sum_{i=1}^{n} (\xi_i^t y_i - \zeta_i^t z_i) - d,\ x_0 = u,\ t = 1,\dots,T,$$

$$0 \le z_i \le y_i \le c,\ i = 1,\dots,n,$$

$$\sum_{i=1}^{n} z_i \le c,\ x^T \ge \alpha x^0,\ \Pr\{x^t \ge 0,\ i = 1,\dots,T\} \ge 1-\varepsilon .$$

The objective function is the value of the dividends obtained per unit time. The optimization is carried out with respect to the variables $\{x^t\}, \{y_i\}, \{z_i\}, d,\ i = 1,\dots,n,\ t = 1,\dots,T$. Parameter c denotes planned volume of arriving premiums (minus obligatory current expenditures). The vector quantities $\xi^t = (\xi_1^t,\dots,\xi_n^t)$ specify (random) loss ratio of various forms of insurance $i = 1,\dots,n$ per unit income of premiums at unit time interval (quarter, year, etc.) with number t, i.e.,

$\xi_i^t y_i$ are total insurance payments on the ith form of insurance in time t. Respectively, the quantities $\zeta^t = (\zeta_1^t, \ldots, \zeta_n^t)$ specify (random) returns of risk reinsurance of the form $i = 1, \ldots, n$ at unit time interval with the number t, i.e., $\zeta_i^t z_i$ are total payments on the i form of reinsurance during the time interval t. The constraint $x^T \geq \alpha x^0$ specifies the obligatory claim by the value of residual reserve at the end time T, $\alpha > 0$. Function $f(\{x^t\}, \{y_i\}, d) = \Pr\{x^t \geq 0, i = 1, \ldots, T\}$ describes the probability of non-ruin of the company for the structure (y, z) of the portfolio of insurance and reinsurance contracts and the value of dividends d.

Despite the static nature, this formulation is a complex computing problem since non-ruin probability is a discontinuous nonconvex function of the arguments. This problem pertains to one-step stochastic programming models (in the given context, one-step means that the independent variables of the model $\{y_i, z_i, d\}$ are deterministic values, i.e., they do not depend on random parameters though there can be several time stages). If there is a finite number of scenarios $\{\xi_s^t, \zeta_s^t, t = 1, \ldots, T\}$, $s = 1, \ldots, S$, with probabilities $\{p_s, s = 1, \ldots, S\}$, for random data $\{\xi^t, \zeta^t, t = 1, \ldots, T\}$ the study [18] proposes to reduce this problem to the equivalent mixed integer programming problem:

$$d \to \max_{\{x_s^t\},\, \{y_i\},\, \{z_i\},\, d,\, \{\nu_s\}},$$

$$x_s^t = x_s^{t-1} + \sum_{i=1}^{n} (y_i - z_i) - \sum_{i=1}^{n} (\xi_{is}^t y_i - \zeta_{is}^t z_i) - d, \; x_0 = u, \; t = 1, \ldots, T,$$

$$0 \leq z_i \leq y_i \leq c, \; i = 1, \ldots, n, \; 0 \leq d \leq \sum_{i=1}^{n} y_i \leq c,$$

$$x_s^T \geq \alpha x_0, \; s = 1, \ldots, S,$$

$$x_s^t \geq -M_s \nu_s, \; s = 1, \ldots, S, \; t = 1, \ldots, T,$$

$$\sum_{i=1}^{S} \nu_i \leq \varepsilon, \; \nu_s \in \{0, 1\}, \; s = 1, \ldots, S,$$

where M_s are constants such that $x_s^t \geq -M_s$ for all $t = 1, \ldots, T$ in scenario s, for example, $M = u - (\max_{i,t} \xi_{is}^t) cT - 2cT$. The obtained problem can be solved using the modern discrete optimization software such as IBM ILOG CPLEX [30].

Parallel Simulation Modeling and Optimization of Insurance Business with the Use of Graphic Accelerators of Calculations. The methods of insurance business optimization considered in the previous sections are of limited utility since they use idealized models of the evolution of reserves and simplified formulations of optimization problems. Moreover, it is insufficient to solve the problem for one set of parameters and given distribution of random data, it is much more important to determine how the solution depends on the variation in parameters and distributions. An alternative approach to insurance business optimization is simulation modeling of the processes of evolution of reserves of the insurance company and statistical estimate (by the Monte Carlo method) of its performance parameters for different values of controlled parameters [8–10]. It implies huge amount of computation and requires the application of parallel computing [11–15]. The essence of the method is independent parallel modeling of a great number S of trajectories of the stochastic evolution of reserves x_t of the insurance company on the time interval $[0, T]$ for the given set of parameters (u, c, d, λ, μ) and calculation of the share $p_S(t)$ of nonruin trajectories by the time t, as well as average net profit (obtained dividends)

$$D_S = (1/S) \sum_{s=1}^{S} \int_0^{\tau_s} d(x_s^t)\, dt,$$

where $\{x_s^t, 0 \leq t \leq \tau_s\}$ is the trajectory of the risk process (1), (2) or (3) in the sth trial, τ_s is the moment of ruin or $\tau_s = T$ if there was no ruin till the time T in the sth trial. During parallel modeling, the computing cores do not exchange information, and upon completion of the modeling, information about the trajectories is gathered on one core and functions $p_S(u, c, d, \lambda, \mu, T)$ and $D_S(u, c, d, \lambda, \mu, T)$ are constructed as functions of some parameter. The modeling results are mapped in the plane "variable parameter–ruin probability" and in the plane "risk–income," i.e., in the plane "ruin probability–obtained dividend." The accuracy of the estimate of ruin probability by the Monte Carlo method can be estimated using the exponential Hoeffding inequality [31]

$$\Pr\{|p_S(u,c,d,\lambda,\mu,T)-\psi(u,c,d,\lambda,\mu,T)|\geq\delta\}\leq 2e^{-S\delta^2/2},$$

whence the (10^{-k})-confidence boundary for $|p_N-\psi|$ has the form

$$\delta_k(S)=\sqrt{2(k\ln 10+\ln 2)}/\sqrt{S}. \quad (7)$$

The scattering of values (standard deviation) of dividends can be estimated by standard formulas of mathematical statistics, for example, the estimate of standard deviation for dividends has the form

$$\sigma_S=\left(\frac{1}{S-1}\sum_{s=1}^{S}\left(\int_0^{\tau_s}d(x_s^t)dt-D_S\right)^2\right)^{1/2}.$$

PROGRAM IMPLEMENTATION OF THE PARALLEL MONTE-CARLO METHOD ON GPU FOR THE SOLUTION OF PROBLEMS OF ACTUARIAL COMPUTING MATHEMATICS

To carry out calculations, the program system of insurance modeling was developed [14, 15]. The graphic interface of the system allows analyzing the dependence of any utility function (for example, obtained dividends or a residual reserve) on different parameters of company operation. It is also possible to construct the dependence of ruin probability (used as a measure of risk) on parameters. The system operates in the MS Windows environment; the graphic interface requires .NET 4, the computing part can operate independently on any processor of x86 architecture; however, processor of the Intel Ivy Bridge family is recommended. If in the system there is a graphic accelerator supporting technology CUDA 1.3 and higher, it can be used for the acceleration of calculations (ten and more times).

The following functions are implemented in the system:

- choice of the model (classical model with continuous time or a model with discrete time);
- loading of statistical data and standard data analysis;
- representation of intervals and a mesh for variable parameters of the model;
- representation of the parameter of statistical modeling (number of simulations);
- representation of parameters for parallelizing of calculations (load distribution among the computing cores of GPU);
- saving and loading of the project (data and parameters of the model);
- the possibility of performing calculations on CPU or GPU;
- construction and graphic representation of the dependences of ruin probability, total dividends, and residual reserve on any parameter under study;
- estimate of the errors of the results of calculations;
- mapping of the results of modeling (efficient frontier) in the plane "ruin probability–total dividends;"
- estimate of the rate of calculations;
- help for the user;
- output of the results into storage and printout.

The system allows us to analyze the dependence of the company on the following parameters: time interval, seed capital, total quarter premium, operating expenditures, reinsurance level, profitability of deposits, the value and probability of catastrophic claims, parameters of dividend policy (royalties, absolute deductions, dividend barrier), and parameters of legislative regulation of insurance business. In the current version of the system, only one parameter can vary in one run cycle. The dependence on all the parameters can be analyzed in one minute of system operation.

The system operates on real statistical data obtained for certain companies by processing information from Ukrainian journals "Strakhova Sprava," Insurance TOP, and from the Internet site [25].

For the implementation of the parallel Monte Carlo method, of key importance is the problem of parallel generation of a great number of long independent numerical sequences consisting of independent random numbers. In the CUDA software environment [16], it is solved using the library of generating pseudorandom numbers CUDA CURAND [32]. In solving the problem on a four-core Intel Core i5 3570K, the hardware generator of random numbers (RdRand) embedded in processors of Ivy Bridge family [33] is involved.

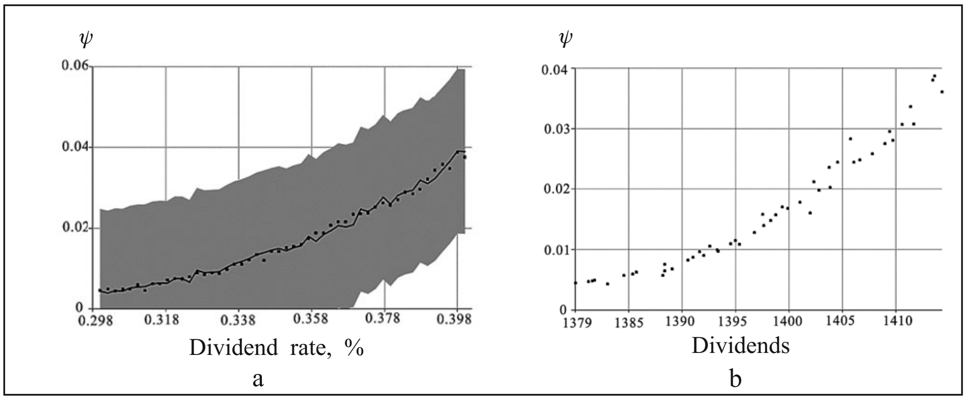

Fig. 1

RESULTS OF NUMERICAL EXPERIMENTS

To carry out test numerical experiments, we used the classical Cramer–Lundberg model (2) with exponentially distributed values of claims and the model with discrete time (3). For the ruin probability $\psi(u, c, d, \alpha, \mu)$ model (2) admits simple analytic solution (5). The numerical experiments were carried out on the Intel Core i5 3570K, Nvidia GeForce GTX 560 personal computer.

Acceleration of Calculations on GPU. We have compared the time of estimation of the ruin probability for the process (3), (4) with the parameters $T = 40$, $u = 140$, and $c = 100$ on CPU and GPU. In all the experiments, the velocity of parallel computing on GPU (Nvidia GeForce GTX 560 on regular frequencies) more than ten times exceeded the velocity of parallel computing on CPU (Intel Core i5 3570K on regular frequencies). Figure 1a shows the dependence of ruin probability ψ with variations in dividend deductions (from 30 to 40%) from the current capital, obtained on GPU (points) and CPU (solid line) for the same number of 25,600 trajectories per one value of the parameter. The shaded area shows the error of calculations (7) for $k = 2$ (99% confidence interval) estimated using the Hoeffding inequality. As is seen from the figure, the Hoeffding inequality is of little use for estimates of the accuracy of probabilities of order 10^{-2} and smaller, as the real accuracy of calculations (the value of oscillations is shown by a solid line in Fig. 1a) is higher by an order of magnitude. Figure 1b shows the dependence of ψ on the value of the obtained dividends (efficient frontier).

Accuracy and Time of the Estimate of Small Ruin Probabilities on GPU. The system allows estimating rather small ruin probabilities in a reasonable time: for example, ruin probability to the order of 10^{-7} as a function of the parameter "deduction of dividends" on a mesh of 30 points is estimated within one minute. For (nearly real) values of the parameters $u = 140$ (initial insurance reserve, million hryvnias), $c = 100$ (total quarter premium, million hryvnias), $T = 10$ years (time interval), $\gamma \in [0.12, 0.15]$ (capital share deducted quarterly for the payment of dividends), the time of calculation of the curves of the dependence of ruin probability on γ with low (Fig. 2a) and high (Fig. 2b) accuracies are 59.6 sec and 4067 sec, respectively. The rate of simulation calculations has amounted 9.34 million trajectories per second.

Analysis of the Dependence of Ruin Probabilities and Net Profit on the Parameters using GPU. The developed system of insurance modeling allows the real-time analysis of the dependence of ruin probability ψ and net profit (total dividends, residual capital) of the insurance company as a function of any parameter of the model. To this end, it is sufficient to specify the minimum and maximum values of the variable parameter and the number of intermediate values in the window of the system interface. In the numerical experiments, parameters varied within the following limits: $T = 40$ quarters, $u \in [0, 140]$, $c = 100$, deductions from the current capital on dividends $\gamma \in [0, 0.4]$, reinsurance level $\alpha \in [0, 0.1]$, large claims $z_{LCS} \in [0.67c, c]$, and probability of large claim $p \in [0, 0.1]$. The computing time is reduced due to a large parallelizing of the method of statistical modeling using the graphic accelerator GPU. Figure 3 shows the dependences of the ruin probability ψ on the following parameters: (a) seed capital $u \in [0, 140]$; (b) reinsurance level $\alpha \in [0, 0.1]$; (c) annual interest rate $\delta \in [0, 0.1]$; (d) values of large claims (in shares of the premium $c = 100$ and for the claim probability $p_{LCS} = 0.01$); (e) the probability of large claims ($z_{LCS} = c = 100$); (f) parameter of the legislative regulation of the obligatory minimum reserve (minimum of the share of previous annual claims and 18% of previous annual claims with the possibility of large claims $z_{LCS} = 0.67c$ with probability $p_{LCS} = 0.01$).

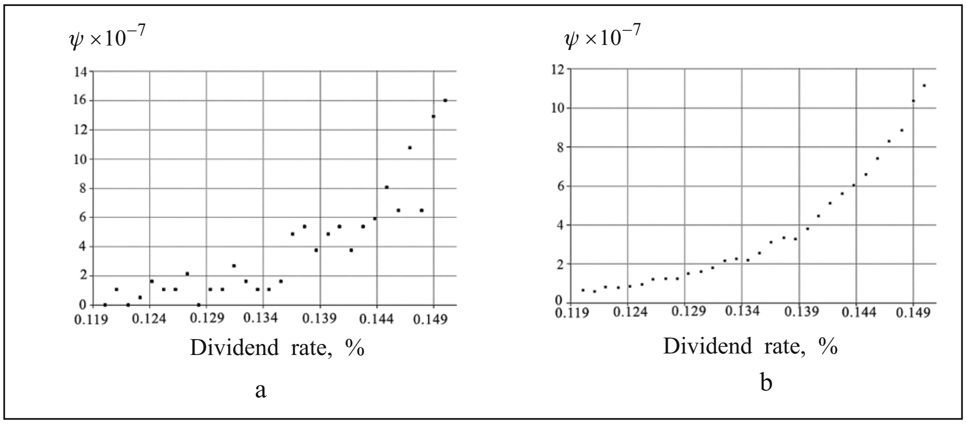
ψ ×10−7
0 2 4 6 8 10 12 16 14
0.119 0.124 0.129 0.134 0.139 0.144 0.149
Dividend rate, %
a
ψ ×10−7
0 2 4 6 8 10 12
0.119 0.124 0.129 0.134 0.139 0.144 0.149
Dividend rate, %
b

Fig. 2

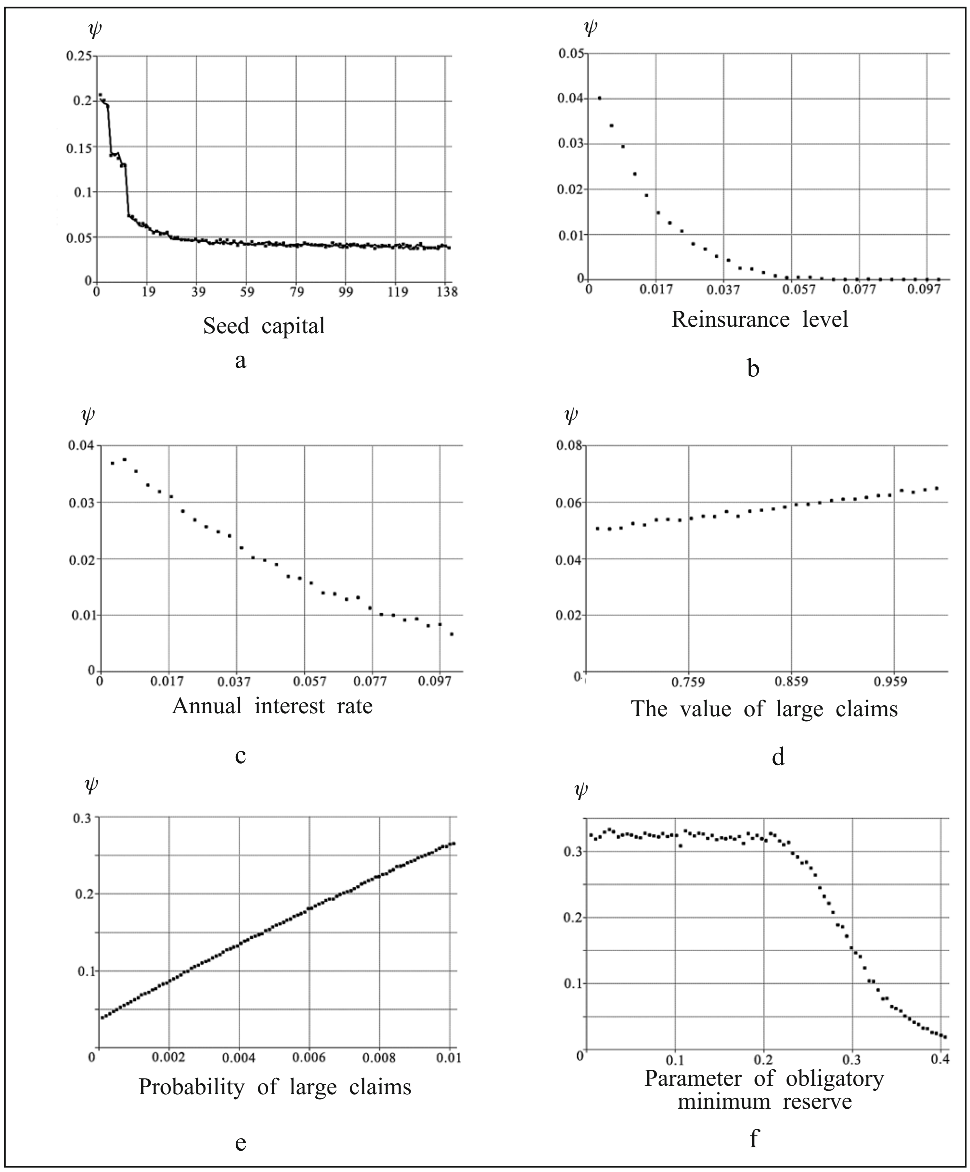
ψ
0 0.05 0.1 0.15 0.2 0.25
0 19 39 59 79 99 119 138
Seed capital
a
ψ
0 0.01 0.02 0.03 0.04 0.05
0 0.017 0.037 0.057 0.077 0.097
Reinsurance level
b
ψ
0 0.01 0.02 0.03 0.04
0 0.017 0.037 0.057 0.077 0.097
Annual interest rate
c
ψ
0 0.02 0.04 0.06 0.08
0.759 0.859 0.959
The value of large claims
d
ψ
0 0.1 0.2 0.3
0.002 0.004 0.006 0.008 0.01
Probability of large claims
e
ψ
0 0.1 0.2 0.3
0.1 0.2 0.3 0.4
Parameter of obligatory
minimum reserve
f

Fig. 3

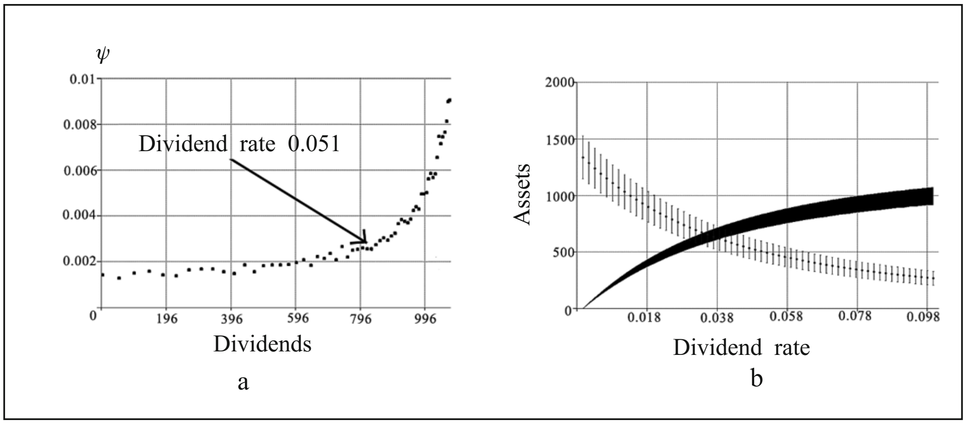

Fig. 4

Optimization of Insurance Business using Efficient Frontier. Efficient frontier is the most important tool of the analysis and optimization of insurance (as well as any other) risk business [8–10]. It relates the return of business (average dividends) and risk (ruin probability) on one graph. The use of efficient frontier allows us to avoid in a certain measure the de Finetti paradox, which implies that the optimal strategy of the distribution of dividends leads to ruin with probability one. In choosing the dividend strategy, the corresponding ruin probability can be taken into account. Figure 4a shows such frontier for the following set of parameters: $T = 40$, $u = 140$, $c = 100$, $z_{LCS} = c$, $p_{LCS} = 0.03$, and $\gamma \in [0, 0.1]$. By hovering the cursor over points of the efficient frontier, the system allows us to map to what values of the variable parameter certain point corresponds. When choosing the optimal value of the parameter, it is necessary to take into account (random) values of the residual capital of the company x^T. Figure 4b shows the corresponding dependences (with indicating the error, a standard deviation) of the average obtained dividends and average residual capital on the level of deductions of dividends $\gamma \in [0, 0.1]$.

CONCLUSIONS

The estimate of ruin probability and other parameters of the operation of an insurance company, as well as the analysis of the dependences of these values on the parameters is a nontrivial computing problem of actuarial mathematics. The use of parallel computing and graphic accelerators allows us to carry out numerical calculations in complex general actuarial models by the Monte Carlo method; in this case, a comprehensible relative accuracy on the ruin probabilities of order up to $10^{-6} - 10^{-7}$ is attained practically in real time.

The developed system of actuarial simulation allows the real-time analysis (due to the acceleration of calculations on GPU) of the dependence of ruin probability and expected net profit of the insurance company as a function of any parameter of control of the company and thus makes it possible to analyze the influence of control factors on the operation of the insurance company. The system also allows the real-time comparison of return and risk in the choice of control parameters of the company and provides a practical alternative to the analytic methods of the optimization of insurance business.

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