Cybernetics and Systems Analysis, Vol. 48, No. 4, July, 2012

# SOLVING WEIGHTED MAX-CUT PROBLEM BY GLOBAL EQUILIBRIUM SEARCH

**V. P. Shylo,**[a†] **O. V. Shylo,**[b] **and V. A. Roschyn**[a‡]

UDC 519.854

**Abstract.** *A new algorithm based on the global equilibrium search (GES) is developed to solve the weighted max-cut problem and is compared with currently the best solution algorithms. The advantages of the GES algorithm both in the performance and in the possibility of finding the best solutions are shown.*

**Keywords:** *maximum weighted cut of a graph, global equilibrium search, computing experiment, algorithm efficiency.*

The problem of the maximum weighted cut of a graph (weighted max-cut problem) has numerous practical applications [1]. As it is a classical discrete optimization problem, it is addressed in many publications, for example [1–6].

Recently, increasing attention has been given to the method of global equilibrium search (GES) [7, 8]. To solve the max-cut problem, special GES Tabu and GES Pr_LocS algorithms were developed based on this method and specific features of the problem [9, 10]. As the analysis of experimental studies has shown, they outperform well-known algorithms. For example, in solving 44 test problems, 12 new records were found; for all the other tests (except for one), already known records were obtained. Noteworthy, the GES method is the most practically efficient among discrete optimization methods.

The present paper proposes a modification of the GES Tabu algorithm to solve the problem under study and presents the results of experimental studies on an extended set of test problems that confirm the advantages of the GES method.

Assume that an undirected graph $G = G(V, E)$ with the sets of vertices $V$ and of edges $E$ is specified. Each edge $(i, j) \in E$ of the graph is associated with a weight $w_{ij} > 0$. A cut of the graph $G$ is a partition $(V_1, V_2)$ of the set of its vertices $V$ into two disjoint subsets $V_1$ and $V_2$ such that $i \in V_1$ and $j \in V_2$. It is obvious that each such partition generates a cut of the graph.

The problem of the maximum weighted cut (max-cut) of the undirected graph $G$ is to find a cut of the maximum total weight

$$w(V_1, V_2) = \sum_{i \in V_1,\, j \in V_2,\, (i,\, j) \in E} w_{ij} .$$

The problem is *NP*-hard even if all the edges are of unit weight. The main difficulty in its solution is that computational costs exponentially grow as the problem dimension increases. Therefore, only approximate methods are efficient for the analysis of this class of high-dimensional problems.

The weighted max-cut problem can be presented as an unconstrained binary quadratic programming (UBQP) problem. We will associate each vertex of the partition $(V_1, V_2)$ of the graph $G$ with a Boolean variable $x_i$. If $i \in V_1$, then $x_i = 0$; otherwise $x_i = 1$, $i \in V_2$. Then the problem becomes: find

$$\max_{x_i \in \{0,1\}} \left\{ f(x) = \sum_{(i,\, j) \in E} w_{ij} (x_i - x_j)^2 \right\}. \tag{1}$$

---

[a]V. M. Glushkov Institute of Cybernetics, National Academy of Sciences of Ukraine, Kyiv, Ukraine, [†]*v.shylo@gmail.com*; [‡]*v.roshin@gmail.com*. [b]University of Pittsburgh, Pittsburgh, USA, *olegio@gmail.com*. Translated from Kibernetika i Sistemnyi Analiz, No. 4, July–August, 2012, pp. 101–105. Original article submitted October 5, 2011.

1060-0396/12/4804-0563

However, reducing the original problem to model (1) and applying the well-known algorithms to it [11, 12] would not be so effective. Therefore, as was mentioned above, special GES algorithms were developed for the max-cut problem; we will elaborate them in the present paper.

The key points of the GES method are generating the solution and searching for a local maximum near this solution. Let us consider the singularities of applying this method at the first stage.

Assume that

$$S_j^1 = \{x \in S \,|\, x_j = 1\},\ S_j^0 = \{x \in S \,|\, x_j = 0\},\ j = 1, \ldots, n,$$

where $S$ is a subset of the set of admissible solutions of problem (1) found by the GES method. Since the "temperature" cycle is the core of the GES method, the following parameters should be set for it: the index $K$ of the temperature cycle and the vector of temperature parameters $\mu = (\mu_0, \ldots, \mu_K)$, $\mu_0 < \mu_1 < .. < \mu_K$, whose indices of components correspond to the numbers of the temperature cycle. The vector $\mu$ allows controlling the difference between the generated initial solution and the best solution $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ from the set $S$ since this solution can be obtained by a random variation of components of the vector $\tilde{x}$ by the following rule: if $\tilde{x}_j = 0$, then this component varies with probability $p_j(\mu_k)$; otherwise, with probability $1 - p_j(\mu_k)$.

The probabilities $p_j(\mu_k)$ of generating the initial solutions can be calculated based on the concepts borrowed from the annealing method [13] and depend on the current temperature $\mu_k$ and on the already found subset $S$ of the set of admissible solutions. They are calculated for $j = 1, \ldots, n$, $k = 1, \ldots, K$ by one of the formulas:

$$p_j(\mu_k) = \frac{1}{1 + \dfrac{1 - p_j(\mu_0)}{p_j(\mu_0)} \exp\left\{ \dfrac{1}{2} \sum_{i=0}^{k-1} (\mu_{i+1} - \mu_i)(E_{ij}^0 + E_{i+1j}^0 - E_{ij}^1 - E_{i+1j}^1) \right\}},$$

where

$$E_{kj}^u = \begin{cases} 0 \text{ if } S_j^u = \varnothing,\ u \in \{0, 1\}, \\ \dfrac{\sum\limits_{x \in S_j^u} f(x) \exp\{\mu_k (f(x) - f(\tilde{x}))\}}{\sum\limits_{x \in S_j^u} \exp\{\mu_k (f(x) - f(\tilde{x}))\}} & \text{if } S_j^u \neq \varnothing, \end{cases}$$

or

$$p_j(\mu_k) = \frac{\sum\limits_{x \in S_j^1} \exp(-\mu_k f(x))}{\sum\limits_{x \in S} \exp(-\mu_k f(x))},$$

or

$$p_j(\mu_k) = \frac{1}{1 + \dfrac{1 - p_j(\mu_0)}{p_j(\mu_0)} \exp\{(\mu_k - \mu_0)\}(f_j^0 - f_j^1)}, \tag{2}$$

where $f_j^u = \max\limits_{x \in S_j^u} f(x)$, $u \in \{0, 1\}$.

In the GES algorithm modification under study, components of the probability vector can be found by formula (2).

The values of $\mu_k$, $k = 0, \ldots, K$, that define the annealing curve are calculated by the formulas $\mu_0 = 0$, $\mu_{k+1} = \alpha \mu_k$, $k = 1, \ldots, K - 1$. The values of $\mu_1$ and $\alpha > 1$ should be selected so that to make the probability vector $p(\mu_K)$ for the last temperature cycle approximately equal to the record solution from the set $S$. The annealing curve is universal and is used in solving all the problems. Only coefficients of the objective function are scaled so that to equate the value of the expected record to the given value. Noteworthy is that the tabu algorithm is taken as a search one at the second stage of the GES method and the specificity of the problem is taken into account.

To compare the developed algorithm to available ones, experimental calculations in solving test problems from [14] were carried out and their number was increased. Such problems are used for test calculations and include toroidal, planar, and random graphs. As in [6], the first 54 problems were considered.

TABLE 1

| Problem | Number of vertices of the graph | BKS | Best GES | BFS | Value | $t_{\min}$ | $t_{\mathrm{av}}$ | $\tilde{t}_{\mathrm{av}}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| G1 | 800 | 11624 | 11624 | 11624(20) | 11624,00 | 0,47 | 3,25 | 3,25 |
| G2 | 800 | 11620 | 11620 | 11620(20) | 11620.00 | 1.98 | 8.04 | 8.04 |
| G3 | 800 | 11622 | 11622 | 11622(20) | 11622.00 | 2.09 | 4.91 | 4.91 |
| G4 | 800 | 11646 |  | 11646(20) | 11646.00 | 0.39 | 7.83 | 7.83 |
| G5 | 800 | 11631 |  | 11631(20) | 11631.00 | 0.44 | 5.39 | 5.39 |
| G6 | 800 | 2178 |  | 2178(20) | 2178.00 | 2.89 | 5.76 | 5.76 |
| G7 | 800 | 2003 |  | ***2006***(20) | 2006.00 | 2.86 | 7.07 | 7.07 |
| G8 | 800 | 2003 |  | ***2005***(20) | 2005.00 | 4.02 | 12.15 | 12.15 |
| G9 | 800 | 2048 |  | ***2054***(20) | 2054.00 | 1.95 | 6.22 | 6.22 |
| G10 | 800 | 1994 |  | ***2000***(20) | 2000.00 | 2.19 | 10.84 | 10.84 |
| G11 | 800 | 564 | 564 | 564(20) | 564.00 | 0.61 | 0.98 | 0.98 |
| G12 | 800 | 556 | 556 | 556(20) | 556.00 | 0.78 | 1.53 | 1.53 |
| G13 | 800 | 582 | 582 | 582(20) | 582.00 | 0.88 | 1.39 | 1.39 |
| G14 | 800 | 3063 | ***3064*** | ***3064***(20) | 3064.00 | 7.58 | 337.00 | 337.00 |
| G15 | 800 | 3050 | 3050 | 3050(20) | 3050.00 | 2.25 | 16.33 | 16.33 |
| G16 | 800 | 3052 | 3052 | 3052(20) | 3052.00 | 2.22 | 18.91 | 18.91 |
| G17 | 800 | 3043 |  | ***3047***(20) | 3047.00 | 7.25 | 97.68 | 97.68 |
| G18 | 800 | 988 |  | ***992***(20) | 992.00 | 1.86 | 127.40 | 127.40 |
| G19 | 800 | 903 |  | ***906***(20) | 906.00 | 1.88 | 17.10 | 17.10 |
| G20 | 800 | 941 |  | 941(20) | 941.00 | 1.17 | 1.59 | 1.59 |
| G21 | 800 | 931 |  | 931(20) | 931.00 | 1.88 | 6.79 | 6.79 |
| G22 | 2000 | 13358 | ***13359*** | ***13359***(20) | 13359.00 | 21.81 | 92.60 | 92.60 |
| G23 | 2000 | 13354 | 13342 | 13342(20) | 13342.00 | 15.53 | 266.42 | 266.42 |
| G24 | 2000 | 13331 | ***13337*** | ***13337***(20) | 13337.00 | 36.09 | 391.13 | 391.13 |
| G25 | 2000 | 13326 |  | ***13340***(20) | 13340.00 | 20.22 | 306.81 | 306.81 |
| G26 | 2000 | 13314 |  | ***13328***(19) | 13327.90 | 17.69 | 1069.62 | 1026.01 |
| G27 | 2000 | 3318 |  | ***3341***(20) | 3341.00 | 38.75 | 393.68 | 393.68 |
| G28 | 2000 | 3285 |  | ***3298***(20) | 3298.00 | 39.66 | 630.14 | 630.14 |
| G29 | 2000 | 3389 |  | ***3405***(20) | 3405.00 | 17.27 | 189.53 | 189.53 |
| G30 | 2000 | 3403 |  | ***3413***(20) | 3413.00 | 15.77 | 140.97 | 140.97 |
| G31 | 2000 | 3288 |  | ***3310***(20) | 3310.00 | 42.31 | 336.01 | 336.01 |
| G32 | 2000 | 1402 | ***1410*** | ***1410***(20) | 1410.00 | 5.2 | 46.38 | 46.38 |
| G33 | 2000 | 1376 | ***1382*** | ***1382***(20) | 1382.00 | 26.95 | 239.26 | 239.26 |
| G34 | 2000 | 1372 | ***1384*** | ***1384***(20) | 1384.00 | 4.91 | 56.65 | 56.65 |
| G35 | 2000 | 7672 | ***7686*** | ***7686***(12) | 7685.55 | 186.78 | 996.26 | 1066.24 |
| G36 | 2000 | 7670 | ***7677*** | ***7680***(1) | 7676.65 | 3100.63 | 3100.63 | 1242.00 |
| G37 | 2000 | 7681 | ***7691*** | ***7691***(3) | 7690.10 | 611.14 | 1803.06 | 1561.45 |
| G38 | 2000 | 7681 |  | ***7687***(20) | 7687.00 | 13.05 | 381.45 | 381.45 |
| G39 | 2000 | 2395 |  | ***2408***(20) | 2408.00 | 21.83 | 191.86 | 191.86 |
| G40 | 2000 | 2387 |  | ***2400***(11) | 2399.55 | 465.92 | 1738.83 | 1033.28 |
| G41 | 2000 | 2398 |  | ***2405***(20) | 2405.00 | 10.42 | 43.45 | 43.45 |
| G42 | 2000 | 2469 |  | ***2481***(20) | 2481.00 | 6.02 | 212.21 | 212.21 |
| G43 | 1000 | 6660 | 6660 | 6660(20) | 6660.00 | 7.69 | 18.61 | 18.61 |
| G44 | 1000 | 6650 | 6650 | 6650(20) | 6650.00 | 4.95 | 10.89 | 10.89 |
| G45 | 1000 | 6654 | 6654 | 6654(20) | 6654.00 | 10.02 | 62.29 | 62.29 |
| G46 | 1000 | 6645 |  | ***6649***(20) | 6649.00 | 6.52 | 27.68 | 27.68 |
| G47 | 1000 | 6656 |  | ***6657***(20) | 6657.00 | 7.55 | 26.52 | 26.52 |
| G48 | 3000 | 6000 | 6000 | 6000(20) | 6000.00 | 0.01 | 0.05 | 0.05 |
| G49 | 3000 | 6000 | 6000 | 6000(20) | 6000.00 | 0 | 0.16 | 0.16 |
| G50 | 3000 | 5880 | 5880 | 5880(20) | 5880.00 | 0.02 | 0.08 | 0.08 |
| G51 | 1000 | 3846 |  | ***3848***(20) | 3848.00 | 3.25 | 117.45 | 117.45 |
| G52 | 1000 | 3849 |  | ***3851***(20) | 3851.00 | 12.61 | 158.16 | 158.16 |
| G53 | 1000 | 3846 |  | ***3850***(18) | 3849.90 | 73.76 | 908.81 | 827.05 |
| G54 | 1000 | 3846 |  | ***3852***(20) | 3852.00 | 19.58 | 329.19 | 329.19 |

TABLE 1, continued

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| sg3dl101000 | 1000 | 896 | 896 | 896(20) | 896.00 | 2.15 | 8.09 | 8.09 |
| sg3dl102000 | 1000 | 900 | 900 | 900(20) | 900.00 | 1.2 | 2.13 | 2.13 |
| sg3dl103000 | 1000 | 892 | 892 | 892(20) | 892.00 | 1.58 | 6.33 | 6.33 |
| sg3dl104000 | 1000 | 898 | 898 | 898(20) | 898.00 | 1.59 | 7.96 | 7.96 |
| sg3dl105000 | 1000 | 886 | 886 | 886(20) | 886.00 | 2.17 | 27.99 | 27.99 |
| sg3dl106000 | 1000 | 888 | 888 | 888(20) | 888.00 | 1.34 | 2.04 | 2.04 |
| sg3dl107000 | 1000 | 900 | 900 | 900(20) | 900.00 | 1.92 | 68.04 | 68.04 |
| sg3dl108000 | 1000 | 882 | 882 | 882(20) | 882.00 | 2.04 | 15.10 | 15.10 |
| sg3dl109000 | 1000 | 902 | 902 | 902(20) | 902.00 | 1.72 | 7.84 | 7.84 |
| sg3dl1010000 | 1000 | 894 | 894 | 894(20) | 894.00 | 1.25 | 3.07 | 3.07 |
| sg3dl141000 | 2744 | 2446 | 2446 | 2446(19) | 2445.90 | 16.74 | 577.25 | 573.96 |
| sg3dl142000 | 2744 | 2458 | 2458 | 2458(20) | 2458.00 | 9.95 | 93.52 | 93.52 |
| sg3dl143000 | 2744 | 2442 | 2442 | 2442(20) | 2442.00 | 149.01 | 1125.69 | 1125.69 |
| sg3dl144000 | 2744 | 2450 | 2450 | 2450(15) | 2449.50 | 43.58 | 1135.02 | 998.35 |
| sg3dl145000 | 2744 | 2446 | 2446 | 2446(20) | 2446.00 | 8.94 | 104.75 | 104.75 |
| sg3dl146000 | 2744 | 2450 | 2450 | ***2452***(15) | 2451.50 | 54.98 | 1372.25 | 1069.78 |
| sg3dl147000 | 2744 | 2444 | 2444 | 2444(20) | 2444.00 | 14.54 | 354.47 | 354.47 |
| sg3dl148000 | 2744 | 2446 | ***2448*** | ***2448***(15) | 2447.50 | 15.51 | 1186.50 | 986.67 |
| sg3dl149000 | 2744 | 2424 | ***2426*** | ***2426***(20) | 2426.00 | 9.66 | 208.53 | 208.53 |
| sg3dl1410000 | 2744 | 2458 | 2458 | 2458(19) | 2457.90 | 12.56 | 1109.46 | 1060.13 |

The GES algorithm is implemented in C++, all the computing experiments involved a PC with Intel® Core QUAD CPU Q9550 2.83GHz and 8.0GB RAM. Each problem was solved 20 times with one-hour time limit. The parameters of the GES algorithm were defined as follows. At the beginning of each temperature cycle, $p_j(\mu_0) = 1/2$, $j = 1, \ldots, n$. The following values were used for the temperature schedule: $\mu_0 = 0$, $\mu_1 = 7 * 10^{-7}$ / coef, and $\mu_k = \mu_{k-1} \frac{\log(10/\text{coef}) - \log \mu_1}{48}$ for $k = 2, \ldots, K$, $K = 50$, $\text{coef} = 1.8 * 10^8 / f(x^{BKS})$, and $x^{BKS}$ being a known record solution of the problem.

Some results of these studies are summarized in Table 1. As each problem was solved 20 times for 20 different initial approximations, the average values concerning the objective function and the problem solution time should be considered. The following notation is used in the table: BKS is a record for the problem known from the literature, Best GES and BFS are the best results obtained by the GES algorithms [9,10] and the algorithm under study, respectively (the number of the revealed records for 20 solution attempts is specified in brackets), Value is the mean value of the objective function; $t_{\min}$ and $t_{\text{av}}$ are, respectively, the minimum and average time (in sec) of finding a record, and $\tilde{t}_{\text{av}}$ is the average solution time (in sec) for one problem. The record known from the literature and enhanced by the GES algorithm is bolded and italicized.

An analysis of the results of experimental calculations allows concluding that the GES method is highly efficient and competitive in solving this class of problems. With the use of this method, records were enhanced for 37 problems and the known records were found for the remaining ones. The performance of the GES method also exceeds that of the well-known ones. Currently, it is undoubtedly the best method to solve max-cut problems.

## REFERENCES

1. S. Poljak and Z. Tuza, "Maximum cuts and large bipartite subgraphs," DIMACS Series in Discrete Mathematics and Theoretical Computer Science, **20**, 181–244 (1995).
2. S. Burer, R. D. C. Monteiro, and Y. Zhang, "Rank-two relaxation heuristics for MAX-CUT and other binary quadratic programs," SIAM J. on Optimization, **12**, 503–521 (2002).
3. P. Festa, P. M. Pardalos, M. G. C. Resende, and C.C. Ribeiro, "Randomized heuristics for the maxcut problem," Optimization Methods and Software, **17**, 1033–1058 (2002).
4. M. X. Goemans and D. P. Williamson, "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming," J. of ACM, **42**, 1115–1145 (1995).
5. G. Palubeckis and V. Krivickiene, "Application of multistart tabu search to the MAX-CUT problem," in: Information Technology and Control, **2(31)**, Technologija, Kaunas (2004), pp. 29–35.

6. R. Marti, A. Duarte, and M. Laguna, “Advanced scatter search for the MAX-CUT problem,” INFORMS J. on Computing, **21**, No. 1, 26–38 (2009).
7. V. P. Shilo, “The method of global equilibrium search,” Cybern. Syst. Analysis, **35**, No. 1, 68–73 (1999).
8. I. V. Sergienko and V. P. Shylo, Discrete Optimization Problems: Challenges, Methods of Solution, and Analysis [in Russian], Naukova Dumka, Kyiv (2003).
9. V. P. Shylo and O. V. Shylo, “Solving the maxcut problem by the global equilibrium search,” Cybern. Syst. Analysis, **46**, No. 5, 744–754 (2010).
10. V. P. Shylo and O. V. Shylo, “Path relinking scheme for the MAX-CUT problem within global equilibrium search,” Intern. J. of Swarm Intelligence Research (IJSIR), **2**, No. 2, 42–51 (2011).
11. P. Pardalos, O. Prokopyev, O. Shylo, and V. Shylo, “Global equilibrium search applied to the unconstrained binary quadratic optimization problem,” Optimization Methods and Software, **23**, 129–140 (2008).
12. V. P. Shylo and O. V. Shylo, “Solving unconstrained binary quadratic programming problem by global equilibrium search,” Cybern. Syst. Analysis, **47**, No. 6, 889–897 (2011).
13. S. Kirkpatrick, C. D. Gelatti, and M. P. Vecchi, “Optimization by simulated annealing,” Science, **220**, 671–680 (1983).
14. C. Helmberg and F. Rendl, “A spectral bundle method for semidefinite programming,” SIAM J. on Optimization, **10**, 673–696 (2000).