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LAKATOS QUEUING SYSTEMS, THEIR GENERALIZATION AND APPLICATION

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Abstract. *Queuing systems with cyclic waiting time are considered. The results of the Hungarian mathematician L . Lakatos are presented and generalized and possible application fields are specified.*

Keywords: *queuing system, retrial queuing systems, orbit, Lakatos systems, cyclic-waiting queuing systems.*

INTRODUCTION

L. Lakatos, a Hungarian mathematician, professor of the E. Lorand University of Budapest, was the first to consider a special type of retrial queuing systems [1], so-called queuing systems with cyclic waiting time (cyclic-waiting systems).

The Lakatos model [1] is a single-channel queuing system with constant time T of orbit cycle, lossless, without waiting (without classical queue), with unlimited orbit and FCFS service discipline (First Come, First Served). The author considered such a system as a model that appeared in the analysis of the landing of aircraft in connection with testing of a simulation model. The FCFS discipline for cyclic-waiting systems is as follows: if the channel is free and there are no calls in the orbit, an arrival is served immediately, otherwise (the channel is busy and/or there are calls in the orbit) it goes to the orbit and will be served in a time multiple of the orbit cycle.

Retrial systems are classified in [2], and such queuing systems are called *L*-type queuing systems or Lakatos systems.

Since the source of our mathematical models were Lakatos' studies, we present below a detailed review of these studies.

A REVIEW OF LAKATOS' STUDIES

Lakatos System $M/M/1$. In [1], a queuing system is considered, where the flow of arrivals is Poison with parameter λ; service time is exponentially distributed with parameter μ; the service of a call starts immediately when it appears in the system or at the moments that distinct in a time multiple of some T, according to the FCFS discipline. The behavior of such queuing system was analyzed by the method of embedded Markov chains whose states corresponded to the number of calls in the system at the moments $t_k - 0$, where t_k is the moment the service of the kth call starts. The matrix of transition probabilities of states of the chain, generating functions of its elements, and ergodic distribution of probabilities were found (both in the considered system and in all the systems under study), as well as the existence condition for the ergodic distribution, which has the form

$$\frac{\lambda}{\mu} < \frac{e^{-\lambda T}(1-e^{-\mu T})}{1-e^{-\lambda T}}.$$

Lakatos System $Geom/Geom/1$. In [3, 4], Lakatos modifies the system from [1] and proves the corresponding ergodicity theorem and considers T-retrial systems, where calls arrive in intervals multiple of T/n, where n is an integer. In each of these intervals, a new call can appear with probability r or not appear with probability $1-r$. If the call service have started, then in an arbitrary (T/n)th interval service can end with probability q and not end with probability $1-q$.

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Hence, if ξ is interarrival interval and Y is service time, then

$$P\left\{\xi = k\frac{T}{n}\right\} = q(1-q)^{k-1},\ k \geq 1;\ P\left\{Y = k\frac{T}{n}\right\} = r(1-r)^{k-1},\ k \geq 1.$$

Lakatos calls the random variables ξ and Y geometrically distributed; however, strictly speaking, the geometrically distributed quantities are $\xi / \frac{T}{n}$ and $Y / \frac{T}{n}$.

Thus, a discrete queuing system with a cyclic waiting time is considered, where both the arrival flow and service time are geometrically distributed with parameters r and q, respectively. The embedded Markov chain of the system is determined similarly to the system $M/M/1$.

The existence condition for the ergodic distribution is the inequality

$$\left(\frac{rq}{1-q^n}\right)\left(\frac{1-q^n(1-r)^n}{1-q(1-r)}\right) < (1-r)^n.$$

Lakatos System *M* / *Unif* / 1. This model [5, 6] has appeared in the verification of the results of modeling an aircraft landing. In Lakatos' opinion, a cyclic-waiting analytic model $M/M/1$ does not describe all the details of a real system; however, it produces exact analytic results.

Lakatos has generalized the problem and described the model of a real system as follows: a queuing system has Poisson's input flow with parameter λ; service time is a uniformly distributed quantity on the interval $[c, d]$; the values c and d are multiple of the orbit cycle time T; the service of a call starts immediately at the time t it arrives at the system or at the moments multiple of T, according to the FCFS discipline.

This queuing system was analyzed in [5, 6] by the method of embedded Markov chains, the generating functions of elements of the matrix of transition probabilities were found, and the existence condition for the ergodic distribution was established in the form

$$\frac{\lambda(c+d+T)}{2} < 1.$$

Limiting Distributions for Lakatos Systems *M* / *M* / 1 and *M* / *Unif* / 1. Lakatos noticed in [7] that during cyclic expectation there may be a situation where the channel is already free and the call is not ready for the repeated service yet. Thus, the service of a sequence of calls is not a continuous process, there are time intervals no greater than T (queue cycle) where the channel is idle. Clearly, the influence of the idle time of the channel on the system operation becomes less as $T \to 0$ and the process of service becomes continuous in the limiting case.

Analytic expressions for limiting distributions were derived in [7]. The generating function of the limiting distribution for a cyclic-waiting system with exponentially distributed service time as $T \to 0$ has the form

$$P^*(z) = \frac{1-\rho}{1-\rho z},\quad \rho = \frac{\lambda}{\mu}$$

if $\rho < 1$, and if the service time is uniformly distributed, as $T \to 0$ it has the form

$$P^*(z) = \left(1 - \frac{\lambda(c+d)}{2}\right)\frac{(1-z)[e^{-\lambda(1-z)c} - e^{-\lambda(1-z)d}]}{e^{-\lambda(1-z)c} - e^{-\lambda(1-z)d} - \lambda z(d-c)(1-z)}.$$

Generalized Lakatos System *M* / *M* / 1 — System with Failures. For the retrial model $M/M/1$ from [1], generalizations were made, namely: the retrial system is treated as a system that accepts calls of two types. Two Poisson flows with parameters λ_1 and λ_2 arrive at the input of such a queuing system [8], the service time of the calls of both types is exponentially distributed with parameters μ_1 and μ_2, respectively. There are no constraints for calls of the second type. Calls of the first type can be accepted only in the free system; if there is one call of the first type in it, then all other calls of this type are refused and they leave the system.

The existence condition for the ergodic distribution reduces to the inequality

$$\frac{\lambda_2}{\mu_2} < \frac{e^{-\lambda_2 T}(1-e^{-\mu_2 T})}{1-e^{-\lambda_2 T}}.$$

GENERALIZED LAKATOS MODEL, *T*-RETRIAL SYSTEM $GI/G/1$

Let us consider a single-channel queuing system with a recurrent input flow and continuous distribution function $A(x)$ of interarrival time, general distribution function $B(x)$ of service time, constant time T of on-orbit wait, and FCFS discipline. Thus, the Lakatos model $M/M/1$ from [1] can be generalized according to the input flow and service time [9].

Let t_n be the arrival time of the nth call, $t_n + Tk_n$ be the time its service starts. Note that k_n is always an integer nonnegative number that is equal to the number of cycles of the nth calls on the orbit. Let also $\xi_n = t_{n+1} - t_n$ and Y_n be the service time of the nth call.

Let us establish the relationship between k_n and k_{n+1}. Let $k_n = i$. If $(k-1)T < Ti + Y_n - \xi_n < kT$, where $k \ge 1$ is an integer, then $k_{n+1} = k$; if $Ti + Y_n - \xi_n < 0$, then $k_{n+1} = 0$. Thus, k_n is a homogeneous Markov chain with transition probabilities p_{ik}, where

$$p_{ik} = P\{(k-i-1)T < Y_n - \xi_n < (k-i)T\}$$

for $k \ge 1$;

$$p_{i0} = P\{Y_n - \xi_n < -Ti\}.$$

Denote $f_j = P\{(j-1)T < Y_n - \xi_n < jT\}$. We have

$$f_j = \int_0^\infty [B(x+jT) - B(x+(j-1)T)]\, dA(x). \tag{1}$$

Then the transition probabilities can be expressed as

$$p_{ik} = f_{k-i} \quad \text{if } k \ge 1; \tag{2}$$

$$p_{i0} = \sum_{j=-\infty}^{-i} f_j. \tag{3}$$

THEOREM 1. If series $\sum_{j=-\infty}^{\infty} jf_j$ absolutely converges and $\sum_{j=-\infty}^{\infty} jf_j < 0$, then the Markov chain (k_n) is ergodic.

Majorizing the General Process of Service. A Lakatos system can be applied to estimate more complex systems with not necessarily the FCFS service. In many situations, models with the FCFS service discipline are not adequate to real systems. In particular, this can be true in aircraft landing, where another aircraft is landing while the aircraft is in the air and is sent around. If, however, we consider the time W_n from the moment t_n till the beginning of landing of the last arrived aircraft, there is always $W_n \le Tk_n$, hence, the condition of Theorem 1 guarantees the ergodicity of a W_n-sequences of more complex structure.

Equations for Stationary Distribution. Let $\pi_k = \lim_{n\to\infty} P\{k_n = k\}$, and the condition of Theorem 1 be satisfied. We have the system of equations

$$\pi_k = \sum_{i=0}^{\infty} \pi_i p_{ik}, \quad j \ge 0;$$

$$\sum_{k=0}^{\infty} \pi_k = 1. \tag{4}$$

It follows from Eqs. (2) and (3) that

$$\pi_k = \sum_{i=0}^{\infty} \pi_i f_{k-i}, \quad k \ge 1;$$

$$\pi_0 = \sum_{i=0}^{\infty} \pi_i \sum_{j=-\infty}^{-i} f_j. \tag{5}$$

This system can be solved recursively. We assume that call service time is always less than or equal to T, i.e., $B(T) = 1$. We assume also that $f_{-1} > 0$. In this case, it follows from formula (1) that $f_j = 0$ for all $j \ge 2$. We have

$$\pi_k = \sum_{j=0}^{k+1} \pi_i f_{k-i}, \quad k \ge 1.$$

In the system of equations described above it is possible to denote all the unknowns $\pi_2, \pi_3, \ldots$ by π_0 and π_1, for example, $\pi_2 = \frac{1}{f_{-1}}(\pi_1(1-f_0) - \pi_0 f_1)$. Thus, we have

$$\pi_k = a_k \pi_0 + b_k \pi_1, \ k \ge 0, \tag{6}$$

where a_k and b_k are known constants ($a_0 = 1$, $b_0 = 0$, $a_1 = 0$, $b_1 = 1$). Substituting (6) into (5), we obtain a linear relationship between π_0 and π_1, which allows designating all the π_k by π_0. To find the latter, it will suffice to use the normalization condition (4).

GENERALIZED LAKATOS MODEL $GI/G/1$ WITH ARBITRARY ORBIT

Let us consider a single-channel queuing system. Denote by t_n the nth moment of call arrival, $n \ge 0$; $\xi_n = t_n - t_{n-1}$, $n \ge 1$. Assume that ξ_n are independent equally distributed random variables with the distribution function $A(x) = P\{\xi_n \le x\}$.

Denote by Y_n service time of the nth call and put $U_n = Y_{n-1} - \xi_n$, $n \ge 1$. Assume that U_n are independent equally distributed random variables with the distribution function $C(x) = P\{U_n \le x\}$, $-\infty < x < \infty$.

If the nth call arrives at a free system (i.e., calls are absent both in the channel and on the orbit), then it is forwarded to the channel immediately; otherwise, the nth call is forwarded to the orbit and comes back trying to be served. If the channel is busy, the call goes to the orbit. Thus, calls can arrive for service at the moments $t_n + \gamma_{n1}$, $t_n + \gamma_{n1} + \gamma_{n2}$, where γ_{nk} are independent equally distributed random variables with the distribution function $D(x) = P\{\gamma_{nk} \le x\}$. The nth call arrives at the channel immediately after returning from the orbit but not earlier than the $(n-1)$th call has been served. Assume that the random sequences (ξ_n), (U_n), and (γ_{nk}) are statistically independent.

Thus, the Lakatos model $GI/G/1$ is generalized according to the input flow, service time, and on-orbit time [10].

To analyze the described queuing system, let us consider the Markov chain $(W_n, n \ge 0)$, where W_n is waiting time of the nth call to the beginning of service. In the general case, W_n can take any nonnegative value. If γ_{nk} are discrete random variables, then W_n is also discrete. If the system starts its operation at the time t_0, then $W_0 = 0$.

To derive the main stochastic relation, let us introduce a random variable

$$Z_n(y) = \begin{cases} \min\{\gamma_{n1} + \ldots + \gamma_{nk} \ge y\}, & y > 0, \\ 0, & y \le 0. \end{cases}$$

Thus, $Z_n(y)$ is the first value of random walk, which attains or exceeds the level y (Fig. 1).

If the dependence on n is insignificant, we will use $Z(y), \gamma_k$ instead of $Z_n(y), \gamma_{nk}$.

Thus, the main stochastic relation has the form

$$W_n = Z_n(W_{n-1} + U_n), \ n \ge 1. \tag{7}$$

Indeed, the $(n-1)$th call will arrive at the channel at the time $t_{n-1} + W_{n-1}$; its service will continue up to the time $t_{n-1} + W_{n-1} + Y_{n-1}$. If $t_{n-1} + W_{n-1} + Y_{n-1} \le t_n$, then we have $W_n = 0$ or, which is equivalent, $W_{n-1} + U_n \le 0$ since $U_n = Y_{n-1} - (t_n - t_{n-1})$. If $W_{n-1} + U_n = y > 0$, then the $(n-1)$th call will be served at the time $t_n + y$, whence it follows that waiting time of the nth call equals $Z_n(y)$, which is the first sum of orbit cycles, which is no less than y.

Assume that the Markov chain (W_n) has stationary distribution function of $F(w)$. Considering (7), we can write the equations for this function

$$F(w) = \begin{cases} \int_0^\infty C(-x)\, dF(x), & w \le 0, \\ \iint_{0 < x+u \le w} \Phi(w, x+u)\, dF(x)\, dC(u), & w > 0, \end{cases}$$

where $\Phi(w, y) = P\{Z(y) \le w\}$. The normalization equation also holds in this case and has the form $F(\infty) = 1$.

Thus, it is possible to receive the stability condition for the Markov chain (W_n). As usually in queuing systems theory, the stability is understood as the statistical boundedness $\liminf_n F_n(w) \to 1$ as $w \to \infty$.

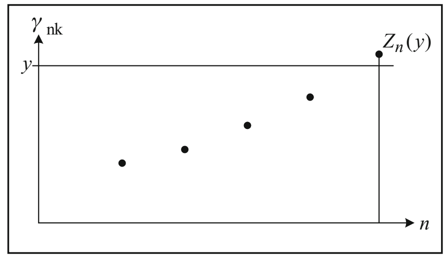

Fig. 1. Setting up $Z_n(y)$ from the dependence of γ_{nk} on *n*.

Ergodicity Condition for a Lattice Distribution Function of On-Orbit Time. Assume that the random variables γ_{nk} are lattice random variables with the maximum step $\Delta > 0$, i.e., γ_{nk} / Δ are integers and at the same time for any $\delta > \Delta$ the quantities γ_{nk} / δ are not integer with positive probability.

Put $d(j) = P\{\gamma_{nk} = j\Delta\}, j \ge 0,$ assuming that $d(0) = 0,$ i.e., on-orbit time is a positive quantity, and let

$$d_1 = \mathrm{E}\{\gamma_{nk}\} = \Delta \sum_{j=0}^{\infty} jd(j), \qquad d_2 = \mathrm{E}\{\gamma_{nk}^2\} = \Delta^2 \sum_{j=0}^{\infty} j^2 d(j)$$

provided that d_2 is a bounded quantity.

THEOREM 2. If the inequality

$$\Delta \sum_{j=-\infty}^{\infty} jP\{j\Delta < U_n \le (j+1)\Delta\} + \frac{d_2 + \Delta d_1}{2d_1} < 0 \tag{8}$$

holds, then the Markov chain (W_n) is stochastically bounded.

Ergodicity Condition for Nonlattice Distribution Function of On-Orbit Time. Let $D(x)$ be a nonlattice distribution function; $D(0) = P\{\gamma_{nk} = 0\} = 0$; and d_1 and $d_2 < \infty$ be the first and second moments of this distribution.

THEOREM 3. If the inequality

$$\int_{-\infty}^{\infty} xdC(x) + \frac{d_2}{2d_1} < 0, \tag{9}$$

holds, then the sequence (W_n) is stochastically bounded.

Theorems 2 and 3 are proved in [10].

Let us consider special cases and remarks.

Case 1. Let us consider a Lakatos system [1] with on-orbit time $T > 0$.

In this case,

$$\frac{d_2 + \Delta d_1}{2d_1} = T.$$

Case 2. In the proof of Theorem 3, a Markov chain (V_n) is constructed, whose terms are multiple of some $\Delta > 0$ and satisfy the property $W_n \le V_n, n \ge 0$.

It is obvious that if (V_n) is proved to be stochastically bounded, the same can be said about the sequence (W_n).

Let us construct the Markov chain (V_n). Select $\Delta > 0$ and consider the function $\psi(x) = \min\{n\Delta \ge x,\ n \in \{1, 2, \ldots\}\}$. We have $x \le \psi(x) \le x + \Delta$.

Suppose $V_0 = \psi(W_0)$ and define V_n recursively as

$$V_n = \psi(Z_n(V_{n-1} + U_n)),\ n \ge 1.$$

Obviously, both functions $\psi(x)$ and Z_n are nondecreasing, whence it follows that

$$\{V_{n-1} \ge W_{n-1}\} \Rightarrow \{V_n \ge W_n\}.$$

Let $D(x) = 1 - e^{-\nu x}, x \geq 0$; $\frac{d_2}{2d_1} = \frac{1}{\nu}$. If $\nu \to \infty$, then the condition $W_n \leq V_n, n \geq 0$, is transformed to the well-known stationarity condition $\int_{-\infty}^{\infty} x dC(x) < 0$ for a queuing system $GI/G/1$.

Remark 1. The stationarity conditions (8) and (9) cannot be improved essentially: if we replace the sign < with > in each of these conditions, then the system will not be statistically bounded.

Remark 2. Both condition (8) and condition (9) are not sufficient for the Markov chain (W_n) to be ergodic. However, this sequence is ergodic if the additional condition $C(x) > 0$ is satisfied for all negative values of x.

STATIONARY CHARACTERISTICS OF *T*-RETRIAL LAKATOS SYSTEM *GI / G / 1*

Let us consider a single-channel Lakatos queuing system with generally distributed interarrival time, generally distributed service time, and constant time of one on-orbit cycle [11].

Let $A(t)$ be interarrival time distribution function, $B(t)$ be the distribution function of the service time Y_n of the nth call, t_n be the time of arrival of the nth call, $t_n + V_n$ be the time of the end of its service, and T be the time of one on-orbit cycle.

Markov Chain. Considering the queue formation order and service discipline in the system, we have: if $t_n \geq t_{n-1} + V_{n-1}$, then the nth call arrives for service at the time t_n and then $V_n = Y_n$; if the inequality $(k-1)T < t_{n-1} - t_n + V_{n-1} \leq kT$ holds for some integer $k \geq 1$, then the nth call arrives for service at the time $t_n + kT$, i.e., $V_n = kT + Y_n$.

Thus, the service of the nth call starts at the time $t_n + k_n T$, where k_n is the minimum integer for which the system is currently free from all the previous calls ($k_n T$ is on-orbit time).

A random sequence (k_n) is a homogeneous Markov chain. In the general case, the difference $k_n - k_{n-1}$ can be as much as large. However, in the problems known to the authors of the paper (for example, in some systems of air field service) it is natural to accept the condition whereby service time is always $Y_n \leq T$, or

$$B(T) = 1. \tag{10}$$

Condition (10) provides the inequality $k_{n+1} - k_n \leq 1$. Indeed, if $k_{n-1} = k$, then the nth call will be accepted no later than $t_{n-1} + (k+1)T$, i.e., $k_n \leq k+1$.

Let

$$f_k = P\{(k-1)T < Y_{n-1} - (t_n - t_{n-1}) \leq kT\} = \int_0^T (A(y-(k-1)T) - A(y-kT))dB(y).$$

Then under condition (10) $\sum_{k=-\infty}^{1} f_k = 1$.

Let p_{ij} be the transition probabilities of the Markov chain (k_n), then

$$p_{ij} = f_{j-i} \text{ for } 1 \leq j \leq i+1; \tag{11}$$

$$p_{i0} = \sum_{k=-\infty}^{-i} f_k. \tag{12}$$

The Markov chain (k_n) is a discrete random walk with a jump limited on the right. The condition of its ergodicity is known in queuing theory [12]. Due to condition (10), it can be rearranged as

$$\sum_{k=-\infty}^{1} k f_k < 0. \tag{13}$$

Stationary Distribution of the Markov Chain. Assume that condition (13) is satisfied and denote by $\upsilon = (\upsilon_j)$ the stationary distribution of the Markov chain (k_n). Equations (11) and (12) yield the system of equations

$$v_j = \sum_{i=j-1}^{\infty} v_i f_{j-i} \text{ for } j \geq 1; \tag{14}$$

$$v_0 = \sum_{i=0}^{\infty} v_i \sum_{k=-\infty}^{-i} f_k. \tag{15}$$

Equations (14) and (15) are supplemented with the normalization condition

$$\sum_{j=0}^{\infty} v_j = 1.$$

We can show by direct substitution that system (14), (15) under condition (13) has a probabilistic solution

$$v_j = (1-z)z^j, \; j \geq 1,$$

where z is the root of the equation

$$\sum_{k=-\infty}^{1} f_k z^{-k} = 1, \tag{16}$$

which lies within the interval (0, 1).

The left-hand side of Eq. (16) is a function convex in the half-interval (0, 1], which tends to infinity as $z \to 0$ since $f_1 > 0$ and is equal to unity for $z = 1$. Moreover, the left derivative of this function at the point $z = 1$ is positive due to condition (13).

Thus, there is a unique root of Eq. (16) in the interval $(0, 1)$. This also follows from the theory of right-continuous random walks [12].

Let us consider the case where the service time $Y_n = \tau$, $t_n - t_{n-1}$ is exponentially distributed with parameter λ. Then $f_1 = 1 - e^{-\lambda\tau}$; $f_k = e^{-\lambda\tau + \lambda kT}(1 - e^{-\lambda T})$ for $k \leq 0$. The ergodicity condition (13) becomes $z < 1$, where $z = e^{\lambda T}(1 - e^{-\lambda\tau})$.

Let the flow of calls that arrive at the system be group Poison, and the number ζ of calls in one group be a geometrically distributed random variable $P\{\zeta = k\} = (1-\theta)\theta^{k-1}$, $k \geq 1$, and intervals between groups be exponentially distributed random variables with parameter λ. Then we have $f_k = \theta 1_{\{k=1]} + (1-\theta) f_k^0$, where f_k^0 is the value of f_k for ordinary (not group) Poisson flow with parameter λ; $1_{\{k=1]}$ is equal to unity if $k = 1$, otherwise it is zero; $f_k^0 = e^{-\lambda(\tau - kT)}(1 - e^{-\lambda T})$ for $k \leq 0; f_1^0 = 1 - e^{-\lambda\tau}$.

For the parameter z, there is an equation

$$\frac{\theta}{z} + (1-\theta)\left(\frac{1-a}{z} + \frac{a(1-b)}{1-bz}\right) = 1$$

or

$$\theta + (1-\theta)\left(1 - a + az\frac{1-b}{1-bz}\right) = z,$$

where we assume for simplicity $a = e^{-\lambda\tau}$ and $b = e^{-\lambda T}$. Solving this equation yields a formula for z provided that $z < 1$:

$$z = \frac{1 - a + a\theta}{b} = \frac{1 - e^{-\lambda\tau} + e^{-\lambda\tau}\theta}{e^{-\lambda T}}.$$

If $\theta \to 0$, then $z \to \frac{1-a}{b}$, which corresponds to the case of a Poison flow of calls.

Average Number of On-Orbit Calls. Let $N(t)$ be the number of calls on the orbit at the time t. Then the integral $\int_0^T N(t)\,dt$ is the total on-orbit time of the calls that arrived at the system in the interval $(0, T)$, except for the residual waiting time of calls that have not been accepted at the time T. From ergodic considerations, for large s

$$\int_0^s \mathrm{E}[N(t)]\,dt \sim \lambda s \mathrm{E}[KT], \; \lambda = \frac{1}{\int_0^{\infty} x\,dA(x)},$$

where K is the stationary version of k_n, whence the ergodic average number of calls on the orbit is

$$\lim_{s\to\infty}\frac{1}{s}\int_0^s \mathrm{E}[N(t)] = \lambda T\sum_{k=0}^{\infty} k(1-z)z^k = \frac{\lambda Tz}{1-z}.$$

The stationary average number $\overline{K}$ of cycles the call stays on the orbit is determined by the formula $\overline{K} = \dfrac{z}{1-z}$.

ERGODICITY CONDITION FOR THE LAKATOS SYSTEM $SM / SM / 1$

In [13], the ergodicity conditions are generalized for the case where ξ_n, Y_n, and γ_{nk} can be dependent and have the same sense as earlier and W_n denotes the residual time till the $(n-1)$th call abandons the system at the time of arrival of the nth call.

Assume that the dependence of the above-mentioned quantities can be modeled by an ergodic Markov chain $(\omega_n, n \ge 0)$ with states $1, 2, \dots, r$, transition matrix $G = (g_{ij})$, and ergodic distribution $\pi = (\pi_1, \dots, \pi_r)$. Let us assume the following:

- if the values of $\omega_0, \dots, \omega_n$ are known, then random variables $\xi_1, \dots, \xi_{n+1}$; $Y_0, \dots, Y_n$ $(0 \le j \le n$ and $k > 0)$ are independent in the aggregate;
- the relations

$$P\{\xi_{n+1} \le x \,|\, \omega_n = i\} = A_i(x),$$

$$P\{Y_n \le x \,|\, \omega_n = i\} = B_i(x),$$

$$P\{\gamma_{nk} \le x \,|\, \omega_n = i\} = D_i(x)$$

hold irrespective of the values of ω_s, $s \ne n$.

Let us assume

$$a_i = \int_0^\infty x dA_i(x),\ b_i = \int_0^\infty x dB_i(x),\ d_{is} = \int_0^\infty x^s dD_i(x);\quad a = \sum_{i=1}^r a_i\pi_i,\ b = \sum_{i=1}^r b_i\pi_i$$

(the values $s = 1, 2$ are only used).

THEOREM 4. If $A_i(x) < 1\ \forall x > 0$, $D_i(x)$ are nonlattice distribution functions, and the inequality

$$b - a + \frac{1}{2}\sum_{i=1}^r \frac{d_{i2}\pi_i}{d_{i1}} < 0$$

holds, then the Markov chain (ω_n, W_n) is ergodic.

Theorem 4 is proved in [13].

Remark 3. Deriving the above results requires a technique of probabilistic estimates in specific cases; nevertheless, all the proofs are based on some common ideas. In all the cases a Markov chain (X_n, Z_n) is considered, where $X_n \ge 0$ and Z_n is the vector of additional variables. The inequalities $E\{X_n - X_0 \mid X_0 = x, Z_0 = z\} \le -\varepsilon, x > x_0$ are established in a certain way, for some n and $\varepsilon > 0$ and $E\{X_1 \mid X_0 = x, Z_0 = z\} \le c,\ x \le x_0$. Finally, the existence of the moments of clearing and updating is used [14].

APPLICATION OF LAKATOS SYSTEMS

As already mentioned, Lakatos models underlie the mathematical models of aircraft landing process, where the input flow is the aircraft arriving at the airport, on-orbit cycle is a go-around or rectangular route of an aircraft in the holding area, and the service channel is the runway. Note that the holding area is several circles one above another; the arrangement of the area and motion in it obey all the echelon standards. The reasons for go-around (in the holding area) may be meteorological conditions, unprepared runway, incorrect position of the approaching aircraft, infringement of echelon standards, etc. The FCFS discipline can be used because overtaking is prohibited when both circling and flying a hold. In the general case, an aircraft can fly the circle several times before landing. Thus, after the arrival in the airdrome zone, an aircraft can be served (landed) at once or in a time multiple of some amount T.

The Lakatos model cannot be said to be completely adequate to the real landing system. If the runway is ready, an air traffic controller can land a newly arrived aircraft when there are other aircraft circling or flying a hold. The air traffic controller makes a decision on landing order in each specific case individually: runway condition, fuel range, aircraft position, and many other factors are taken into account. Note that a queue reorganization does not always provide more optimal service [15], because each airdrome has individual dimensions of the holding area and circle (rectangular route), and individual arrival routes. Therefore, general conclusions should not be made on the order of service with occupied holding areas.

However, the Lakatos model is the only one to analytically describe the landing system with an occupied holding area and provide exact analytic expressions for evaluating the airdrome handling capacity (runway capacity in this case).

Note that some airports have several holding areas (as in Heathrow) and several runways (as in Borispol), which requires new Lakatos-type models, namely multiorbital and multichannel models.

Lakatos systems with repetitions can also be used for modeling telecommunication systems such as simple call centers (phone inquiry services) [16–20].

Let us consider the functioning of a simple call center providing consulting services. Calls from telephone subscribers arrive at the call center with some intensity. If there is a free operator at the call center, a subscriber is serviced. If all channels are busy, the subscriber will be queued, assigned a number, and offered to listen to a melody. Subscribers are queued and serviced according to the FCFS discipline. Such a system can be modelled by multichannel Lakatos-type queuing system with repetitions: a multichannel queuing system with queued calls being in its orbit.

Lakatos systems with repetitions and orbit can also be used in computer systems and networks [21]. For example, light signals in optical buffers with ring resonators (IBM, 2006) [22] used in chip interconnections are delayed as follows. If the optical path is ring-shaped, light is circling at resonant frequencies, thus increasing the signal delay. The optical buffer with ring resonator consists of many sequential lightguide rings. This device can be modeled by a Lakatos system with repetitions because light signals cannot "overtake" each other and always follow one another. The serving device is the processor memory, RAM, etc.

Another type of optical buffers is used in optical computer networks intended for delaying light signals to transform them into electric signals. Modern optical networks use Dense Wavelength Division Multiplexing, i.e., transmission of several optical signals with different wave lengths through one optical fiber, which allows data transmission at 10 Tbit/sec. Nevertheless, batch switching in such optical lines requires transformation of transfer rates in lines so that they match the equivalent switching properties of network nodes. Since batch switches process slower electronic signals, switching becomes problematic in terms of network transfer rates. Using new batch-oriented technologies such as OPS (Optical Packet Switching) and OBS (Optical Burst Switching) will help to solve this problem by switching optical signals. Optical buffers provide a solution for external signal blocking, which occurs each time when two or more data batches arrive at the same address at the same time. Since light cannot be stored in a place, data are buffered by being transferred along a FDL (Fiber Delay Line) of appropriate length which is selected from the set of FDL-lines by means of a switching matrix.

CONCLUSIONS

We have reviewed the studies of Prof. Lakatos related to cyclic-waiting queuing systems and represented the analytic results for different modifications of such queuing systems. Lakatos derived the existence conditions for the ergodic distribution for systems $M/M/1$, $Geom/Geom/1$, and $M/Unif/1$; for queuing systems $M/M/1$ and $M/Unif/1$ the generating functions of limiting distributions were found; for systems with failures $M/M/1$ the existence condition for ergodic distribution and limiting distribution of generating functions was obtained.

We have generalized the Lakatos models, analyzed a queuing system $GI/G/1$ with constant orbit and FCFS service discipline. The existence condition for ergodic distribution and the ergodic average number of calls on orbit have been found for such a system. For the case where the service time does not exceed a constant on-orbit time T, the stationary distribution of the corresponding Markov chain has been found.

For retrial queuing systems $GI/G/1$ with general distribution function of on-orbit time and FCFS service discipline, the existence condition for the ergodic distribution of the corresponding Markov chain has been derived for two cases, namely: when on-orbit time distribution function is lattice and when it is continuous.

The ergodicity condition for an embedded Markov chain for $SM/SM/1$ Lakatos queuing system has been investigated for the total nonlattice distribution of the time of returning from the orbit.

Thus, the Lakatos model has been considerably generalized, namely: all the three distributions $A(x), B(x)$, and $D(x)$ can be arbitrary; the system $SM / SM / 1$ has also been considered. The ergodicity conditions for an embedded Markov chain and some stationary characteristics of generalized systems have been found.

The application fields of the Lakatos model have been specified and new problems have been stated.

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REFERENCES

1. L. Lakatos, "On a simple continuous cyclic-waiting problem," Annales Univ. Sci., No. 14, 105–113 (1994).
2. E. V. Koba and I. N. Kovalenko, "Ergodicity condition for a system with queuing and servicing of objects with a complicated structure," Cybern. Syst. Analysis, **43**, No. 5, 625–628 (2007).
3. L. Lakatos, "A discrete cycle-waiting queueing problem," Theoriya Veroyatn. i yeyo Primenen., **42**, No. 2, 405–406 (1997).
4. L. Lakatos, "On a simple discrete cyclic-waiting queueing problem," J. Math. Sci., **92**, No. 4, 4031–4034 (1999).
5. L. Lakatos, "On a cyclic-waiting queueing system," Theory of Stochastic Processes, **18**, No. 2, 177–181 (1996).
6. L. Lakatos, "A probability model connected with landing of airplanes," in: Safety and Reliability, Brookfield, Rotterdam (1999), pp. 151–154.
7. L. Lakatos, "Limit distribution for some cyclic-waiting systems," in: Proc. Ukrainian Math. Congress, 2001, Kyiv (2002), pp. 102–106.
8. L. Lakatos, "A special cycling-waiting queueing system with refusals," J. Math. Sci., **111**, No. 3, 3541–3544 (2002).
9. E. V. Koba, "On a GI/G/1 retrial queuing system with FIFO service discipline," Dop. NAN Ukrainy, No. 6, 101–103 (2000).
10. E. V. Koba, "On a GI/G/1 retrial queueing system with a FIFO queueing discipline," Theory of Stochastic Processes, **24**, No. 8, 201–207 (2002).
11. O. V. Koba, "Stationary characteristics of a GI/G/1 *T*-retrial queuing system with FIFO service discipline," Visnyk NAU, No. 1, 122–125 (2003).
12. V. S. Korolyuk and Yu. V. Borovskikh, Analytic Asymptotics of Probability Distributions [in Russian], Naukova Dumka, Kyiv (1981).
13. E. V. Koba, "Ergodicity condition for a generalized Lakatos queuing model," Dop. NAN Ukrainy, No. 11, 70–74 (2004).
14. A. A. Borovkov, Probabilistic Processes in Queuing Theory [in Russian], Nauka, Moscow (1972).
15. O. V. Koba and K. V. Mikhalevich, "Comparing M/M/1 systems with fast retrials for different service disciplines," Syst. Doslidzh. Inform. Tekhnologii, No. 2, 59–68 (2003).
16. S. V. Pustova, "Investigation of call centers as retrial queuing systems," Cybern. Syst. Analysis, **46**, No. 3, 494–499 (2010).
17. S. V. Pustova, "Dependence of performance indices of a call center on the distribution of call's sojourn time in the orbit," Cybern. Syst. Analysis, **45**, No. 2, 314–325 (2009).
18. S. Pustova, "Modeling call center operation with taking into account repeated attempts of subscribers," Visnyk NAU, No. 3, 21–24 (2006).
19. O. V. Koba and S. V. Pustova, "Analytical model of call center operation," Dop. NANU, No. 2, 17–25 (2007).
20. E. V. Koba and S. V. Pustova, "Call center as retrial queuing system," J. Autom. Inform. Sci., Vol. 39, Issue 5, 37–47 (2007).
21. W. Rogiest, K. Laevens, D. Fiems, and H. Bruneel, "Analysis of a Lakatos-type queueing system with general service times," in: Abstracts of the 20th Conf. on Quantitative Methods for Decision Making, ORBEL 20, Ghent (2006), pp. 95–97.
22. Compact optical buffer with ring resonators, IBM, 2006, http://domino.research.ibm.com/comm/research_projects.nsf /pages/photonics.ringbuffer.html.