Cybernetics and Systems Analysis, Vol. 47, No. 5, September, 2011

MULTIVALUED DYNAMICS OF SOLUTIONS OF AN AUTONOMOUS DIFFERENTIAL-OPERATOR INCLUSION WITH PSEUDOMONOTONE NONLINEARITY

P. O. Kasyanov

UDC 517.9

Abstract. *This article considers a nonlinear autonomous differential-operator inclusion with a pseudomonotone dependence between determinative problem parameters. The dynamics of all weak solutions defined on the positive semi-axis of time is studied. The existence of trajectory and global attractors is proved and their structure is investigated. A class of high-order nonlinear parabolic equations is considered to be a possible application.*

Keywords: *differential-operator inclusion, global attractor, trajectory attractor, pseudomonotone mapping.*

INTRODUCTION

Qualitative investigations of nonlinear mathematical models of evolutionary processes and fields of different nature, in particular, problems of dynamics of solving nonstationary problems, are performed by many collectives of mathematicians, mechanicians, geophysicists (mainly theorists), and engineers. A list of relevant results that is far from complete is presented in [1–17]. The latest data on the study of multivalued (in the general case) dynamics of solutions of mathematical models with nonlinear nonsmooth discontinuous multivalued nonmonotone interaction functions are based on the theory of global and trajectory attractors for m-semiflows of solutions [1, 5–7]. In this case, to solve the evolutionary problem being considered, the properties connected with system dissipativity and closeness (in a sense) of the resolving operator [1, 5–8, 11, 13, 14] must be fulfilled. Note that such properties of solutions are individually checked for each inclusion on the basis of the linearity or monotonicity of the leading part of the differential operator appearing in the problem [1, 6, 11, 13, 14]. In most cases, quasilinear equations are considered.

At the same time, energy extensions and Nemytskii operators for differential operators occurring in generalized statements of various problems of mathematical physics, problems on a manifold with boundary and without boundary, problems with delay, stochastic partial differential equations, and problems with degeneration, as a rule, possess (if the phase space is properly chosen) common properties connected by growth conditions (the growth often is no more than polynomial), sign conditions, and pseudo-monotonicity [2–4, 12, 15, 16]. Under such constraints imposed on key problem parameters, it is possible to prove in the general case only the existence of weak solutions of a differential-operator inclusion, but the proof is not always cons tructive [2–4, 12, 15, 16]. Thus, the problem of existence and investigation of the structure of trajectory and global attractors for weak solutions of evolutionary inclusions in infinite-dimensional spaces with multivalued interaction multifunctions of pseudomonotone type is an urgent problem.

National Technical University “Kyiv Polytechnic Institute,” National Academy of Sciences of Ukraine and Ministry of Education and Science of Ukraine, Kyiv, Ukraine, *kasyanov@i.ua*. Translated from Kibernetika i Sistemnyi Analiz, No. 5, pp. 150–163, September–October 2011. Original article submitted August 12, 2010.

1060-0396/11/4705-0800

1. PROBLEM STATEMENT

For an evolutionary triple $(V; H; V^*)$ of a multivalued (in the general case) mapping $A: V \rightrightarrows V^*$ and an external force $f \in H$, the problem of investigation of dynamics is considered as $t \to +\infty$ in the phase space H of all weak solutions of a nonlinear autonomous differential-operator inclusion

$$y'(t) + A(y(t)) \ni f \tag{1}$$

that are given for $t \geq 0$, where problem parameters satisfy the following conditions:

(1) $p \geq 2$ and $f \in H$;

(2) the embedding of V into H is compact;

(3) $\exists c > 0: \forall u \in V, \forall d \in A(u), \|d\|_{V^*} \leq c(1 + \|u\|_V^{p-1})$;

(4) $\exists \alpha, \beta > 0: \forall u \in V, \forall d \in A(u), \langle d, u \rangle_V \geq \alpha \|u\|_V^p - \beta$;

(5) $A: V \rightrightarrows V^*$ is (generalized) pseudomonotone [16], i.e.,

- for any $u \in V$, the set $A(u)$ is nonempty, convex, and weakly compact in V^*;
- since $u_n \to u$ weakly in V, $d_n \in A(u_n)\ \forall n \geq 1$, and $\overline{\lim_{n \to +\infty}} \langle d_n, u_n - u \rangle_V \leq 0$, we obtain that $\forall \omega \in V\ \exists d(\omega) \in A(u)$

such that we have

$$\underline{\lim_{n \to +\infty}} \langle d_n, u_n - \omega \rangle_V \geq \langle d(\omega), u - \omega \rangle_V .$$

Here, $\langle \cdot, \cdot \rangle_V : V^* \times V \to \mathbb{R}$ is a pairing in $V^* \times V$; it coincides with the scalar product $(\cdot, \cdot)$ on $H \times V$ in the Hilbert space H.

Comment 1. Conditions (3)–(5) imply that the mapping A is upper semicontinuous since it maps an arbitrary finite-dimensional subspace V into V^* supplied with a weak topology.

A weak solution to evolutionary inclusion (1) on an interval $[\tau, T]$ is understood to be an element u that belongs to a space $L_p(\tau, T; V)$ and is such that, for some $d \in L_q(\tau, T; V^*)$, we have

$$d(t) \in A(y(t)) \text{ for almost all (a.a.) } t \in (\tau, T), \tag{2}$$

$$-\int_\tau^T (\xi'(t), u(t)) dt + \int_\tau^T \langle d(t), \xi(t) \rangle_V dt = \int_\tau^T (f, \xi(t)) dt \quad \forall \xi \in C_0^\infty([\tau, T]; V), \tag{3}$$

where $q > 1$: $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

2. PRELIMINARY RESULTS

For fixed $\tau < T$, we consider

$$X_{\tau,T} = L_p(\tau, T; V), \ X^*_{\tau,T} = L_q(\tau, T; V^*), \ W_{\tau,T} = \{u \in X_{\tau,T} \mid u' \in X^*_{\tau,T}\},$$

$$\mathcal{A}_{\tau,T} : X_{\tau,T} \rightrightarrows X^*_{\tau,T}, \ \mathcal{A}_{\tau,T}(y) = \{d \in X^*_{\tau,T} \mid d(t) \in A(y(t)) \text{ for a.a. } t \in (\tau, T)\},$$

$$f_{\tau,T} \in X^*_{\tau,T}, \quad f_{\tau,T}(t) = f \text{ for a.a. } t \in (\tau, T),$$

where u' is the derivative of an element $u \in X_{\tau,T}$ in the sense of the distribution space $\mathcal{D}([\tau, T]; V^*)$ [2; Definition IV.1.10]. We note that the space $W_{\tau,T}$ is a reflexive Banach space with the following derivative graph norm

[15; statement 4.2.1]:

$$||u||_{W_{\tau,T}} = ||u||_{X_{\tau,T}} + ||u'||_{X^*_{\tau,T}}, \quad u \in W_{\tau,T}. \tag{4}$$

It follows from [3; Lemma 7] and conditions (1)–(5) that $\mathcal{A}_{\tau,T}: X_{\tau,T} \rightrightarrows X^*_{\tau,T}$ satisfies the following conditions:

(a) $\exists C_1 > 0: \ ||d||_{X^*_{\tau,T}} \le C_1(1 + ||y||^{p-1}_{X_{\tau,T}}) \ \forall y \in X_{\tau,T}, \ \forall d \in \mathcal{A}_{\tau,T}(y)$;

(b) $\exists C_2, C_3 > 0: \ \langle d, y\rangle_{X_{\tau,T}} \ge C_2 ||y||^p_{X_{\tau,T}} - C_3 \ \ \forall y \in X_{\tau,T}, \ \forall d \in \mathcal{A}_{\tau,T}(y)$;

(c) $\mathcal{A}_{\tau,T}: X_{\tau,T} \rightrightarrows X^*_{\tau,T}$ is (generalized) pseudomonotone on $W_{\tau,T}$, i.e.,

- for any $y \in X_{\tau,T}$, the set $\mathcal{A}_{\tau,T}(y)$ is nonempty, convex, and weakly compact in $X^*_{\tau,T}$;
- $\mathcal{A}_{\tau,T}$ is upper semicontinuous as such that maps from an arbitrary finite-dimensional subspace $X_{\tau,T}$ into $X^*_{\tau,T}$ supplied with a weak topology;
- since $y_n \to y$ weakly in $W_{\tau,T}$, $d_n \in \mathcal{A}_{\tau,T}(y_n) \ \forall n \ge 1$, $d_n \to d$ weakly in $X^*_{\tau,T}$, and

$$\overline{\lim_{n\to+\infty}} \langle d_n, y_n - y\rangle_{X_{\tau,T}} \le 0,$$

we obtain that $d \in \mathcal{A}_{\tau,T}(y)$ and $\lim_{n\to+\infty} \langle d_n, y_n\rangle_{X_{\tau,T}} = \langle d, y\rangle_{X_{\tau,T}}$. We note that the measurability condition is not imposed on A.

Here, $\langle\cdot,\cdot\rangle_{X_{\tau,T}}: X^*_{\tau,T} \times X_{\tau,T} \to \mathbb{R}$ is a pairing in $X^*_{\tau,T} \times X_{\tau,T}$; it coincides with the scalar product in $L_2(\tau,T;H)$ on $L_2(\tau,T;H) \times X_{\tau,T}$, i.e.,

$$\forall u \in L_2(\tau,T;H), \ \ \forall v \in X_{\tau,T} \quad \langle u, v\rangle_{X_{\tau,T}} = \int_\tau^T (u(t), v(t))dt.$$

Note also [2; Theorem IV.1.17] that the embedding $W_{\tau,T} \subset C([\tau,T];H)$ is continuous and dense and we have

$$\forall u, v \in W_{\tau,T} \quad (u(T), v(T)) - (u(\tau), v(\tau)) = \int_\tau^T [\langle u'(t), v(t)\rangle_V + \langle v'(t), u(t)\rangle_V] dt. \tag{5}$$

The statement formulated below directly follows from the definition of the derivative in the sense of $\mathcal{D}([\tau,T];V^*)$ and equality (3).

LEMMA 1. Each weak solution $u \in X_{\tau,T}$ of differential-operator inclusion (1) on an interval $[\tau,T]$ belongs to the space $W_{\tau,T}$ and, moreover, we have

$$u' + \mathcal{A}_{\tau,T}(u) \ni f_{\tau,T}. \tag{6}$$

On the contrary, if $u \in W_{\tau,T}$ satisfies inclusion (6), then u is a weak solution of inclusion (1) on $[\tau,T]$.

The existence of a weak solution to Cauchy problem (1) with the initial condition

$$y(\tau) = y_\tau \tag{7}$$

on the interval $[\tau,T]$ for an arbitrary $y_\tau \in H$ is guaranteed by condition (1), conditions (a)–(c), and also the results of [15, Ch. 5]. Thus, the following result takes place.

LEMMA 2. For any $\tau < T$, $y_\tau \in H$, Cauchy problem (1), (7) has a weak solution on the interval $[\tau,T]$. Moreover, each weak solution $u \in X_{\tau,T}$ of Cauchy problem (1), (7) on the interval $[\tau,T]$ belongs to $W_{\tau,T} \subset C([\tau,T];H)$ and satisfies inclusion (6).

Comment 2. Since $W_{\tau,T} \subset C([\tau,T];H)$, initial condition (7) makes sense by virtue of Lemma 1 for each weak solution of problem (1).

For fixed $\tau < T$, we introduce the following denotation: $\mathcal{D}_{\tau,T}(u_\tau) = \{u(\cdot) \mid u$ is a weak solution to inclusion (1) on $[\tau,T], u(\tau) = u_\tau\}$, $u_\tau \in H$.

It follows from Lemma 2 that $\mathcal{D}_{\tau,T}(u_\tau) \ne \varnothing$ and $\mathcal{D}_{\tau,T}(u_\tau) \subset W_{\tau,T} \ \ \forall \tau < T, \ u_\tau \in H$.

Let us prove that a translation and a concatenation of weak solutions are also weak solutions.

LEMMA 3. If $\tau < T$, $u_\tau \in H$, and $u(\cdot) \in \mathcal{D}_{\tau,T}(u_\tau)$, then $v(\cdot) = u(\cdot + s) \in \mathcal{D}_{\tau-s,T-s}(u_\tau)$ $\forall s$. If $\tau < t < T$, $u_\tau \in H$, $u(\cdot) \in \mathcal{D}_{\tau,t}(u_\tau)$, and $v(\cdot) \in \mathcal{D}_{t,T}(u(t))$, then

$$z(s) = \begin{cases} u(s), & s \in [\tau, t], \\ v(s), & s \in [t, T], \end{cases}$$

belongs to $\mathcal{D}_{\tau,T}(u_\tau)$.

Proof follows from the definition of a solution to equality (3), Lemma 1, and the fact that $z \in W_{\tau,T}$ as soon as $v \in W_{\tau,t}$, $u \in W_{t,T}$, and $v(t) = u(t)$. In proving the latter fact, one can use the definition of the derivative in the sense of $\mathcal{D}([\tau,T];V^*)$, formula (5), and Lemma IV.1.12 from [2] on the density of $C^1([t_1,t_2];V)$ in W_{t_1,t_2} when $t_1 < t_2$.

3. ADDITIONAL PROPERTIES OF SOLUTIONS

The proof of the existence of compact global and trajectory attractors of evolutionary inclusions and, in particular, inclusions of type (1) is based on properties of a collection of weak solutions to problem (1) that are connected with the absorbableness of a generated m-semiflow of solutions and its asymptotic compactness (see [5–8] and their references). The following lemmas on a priori estimates of solutions and theorem on the dependence of solutions on initial data play the key role in investigating the dynamics of all weak solutions to problem (1) as $t \to +\infty$.

LEMMA 4. There are $c_4, c_5, c_6, c_7 > 0$ such that, for any finite time interval $[\tau, T]$, each weak solution $u(\cdot)$ to problem (1) on $[\tau, T]$ satisfies the following estimates: $\forall t \ge s$, $t, s \in [\tau, T]$,

$$||u(t)||_H^2 + c_4 \int_s^t ||u(\xi)||_V^p \, d\xi \le ||u(s)||_H^2 + c_5 (1 + ||f||_H^2)(t - s), \tag{8}$$

$$||u(t)||_H^2 \le ||u(s)||_H^2 \, e^{-c_6(t-s)} + c_7 (1 + ||f||_H^2). \tag{9}$$

Proof. The proof standardly follows from the conditions imposed on the parameters of problem (1) and the Gronwall–Bellman lemma.

THEOREM 1. Let $\tau < T$, an let $\{u_n\}_{n\ge1}$ be an arbitrary sequence of solutions to problem (1) on $[\tau, T]$ that are weak and such that $u_n(\tau) \to \eta$ weakly in H. Then there are $\{u_{n_k}\}_{k\ge1} \subset \{u_n\}_{n\ge1}$ and $u(\cdot) \in \mathcal{D}_{\tau,T}(\eta)$ such that

$$\forall \varepsilon \in (0, T - \tau) \quad \max_{t \in [\tau+\varepsilon, T]} ||u_{n_k}(t) - u(t)||_H \to 0, \quad k \to +\infty. \tag{10}$$

Proof. We assume that the conditions of Theorem 1 are satisfied. Then, by virtue of Lemma 1, for any $n \ge 1$, we have $u_n(\cdot) \in W_{\tau,T} \subset C([\tau,T];H)$. Moreover, from Lemma 4, condition (4), and relationship (6), we obtain

$$\forall n \ge 1 \;\; \exists d_n \in \mathcal{A}_{\tau,T}(u_n) : u_n'(t) + d_n(t) = f \text{ for a.a. } t \in (\tau, T), \tag{11}$$

$$\exists C > 0 \colon \forall n \ge 1 \;\; ||u_n||_{X_{\tau,T}} + ||u_n'||_{X_{\tau,T}^*} + ||u_n||_{C([\tau,T];H)} + ||d_n||_{X_{\tau,T}^*} \le C. \tag{12}$$

As a result, from the continuity of the embedding $W_{\tau,T} \subset C([\tau,T];H)$ [2; Theorem IV.1.17], conditions (2) and (a), compactness of the embedding $W_{\tau,T} \subset L_2(\tau,T;H)$ [4; Theorem 1.5.1], and also the reflexivity of the space $W_{\tau,T}$ with derivative chart norm (4), we obtain that, up to a subsequence $\{u_{n_k}, d_{n_k}\}_{k\ge1} \subset \{u_n, d_n\}_{n\ge1}$, the following convergences take place for some $u \in W_{\tau,T}$ and $d \in X_{\tau,T}^*$:

$$u_{n_k} \to u \text{ weakly in } X_{\tau,T}, \; u_{n_k}' \to u' \text{ weakly in } X_{\tau,T}^*, \; d_{n_k} \to d \text{ weakly in } X_{\tau,T}^*,$$

$$u_{n_k} \to u \text{ weakly in } C([\tau,T];H), \; u_{n_k} \to u \text{ in } L_2(\tau,T;H),$$

$$u_{n_k}(t) \to u(t) \text{ in } H \text{ for a.a. } t \in (\tau,T), \quad k \to +\infty. \tag{13}$$

We will complete the proof of the theorem in several steps.

Step 1. Let us prove that

$$\forall\, t \in (\tau,T] \quad u_{n_k}(t) \to u(t) \text{ in } H, k \to +\infty. \tag{14}$$

Lemma 4 implies that $\forall\, k \geq 1$, $\forall\, t \geq s$, $t,s \in [\tau,T]$,

$$\|u_{n_k}(t)\|_H^2 - c_5(1+\|f\|_H^2)t \leq \|u_{n_k}(s)\|_H^2 - c_5(1+\|f\|_H^2)s. \tag{15}$$

From convergences (13) we obtain that, for a.a. $s \in (\tau,T)$ and for a.a. $t \in (s,T)$, we have

$$\|u(t)\|_H^2 - c_5(1+\|f\|_H^2)t \leq \|u(s)\|_H^2 - c_5(1+\|f\|_H^2)s.$$

Since $u \in W_{\tau,T} \subset C([\tau,T];H)$, $\forall\, t \geq s$, $t,s \in [\tau,T]$, we have

$$\|u(t)\|_H^2 - c_5(1+\|f\|_H^2)t \leq \|u(s)\|_H^2 - c_5(1+\|f\|_H^2)s. \tag{16}$$

Therefore, the functions

$$J_k(t) = \|u_{n_k}(t)\|_H^2 - c_5(1+\|f\|_H^2)t, \tag{17}$$

$$J(t) = \|u(t)\|_H^2 - c_5(1+\|f\|_H^2)t \tag{18}$$

are continuous and monotonically nonincreasing on $[\tau,T]$.

Since $u_{n_k}(t) \to u(t)$ in H for a.a. $t \in (\tau,T)$, we have

$$J_k(t) \to J(t), \quad k \to +\infty \text{ for a.a. } t \in (\tau,T). \tag{19}$$

Let us show that

$$\overline{\lim_{k\to+\infty}}\, J_k(t) \leq J(t) \quad \forall\, t \in (\tau,T]. \tag{20}$$

It follows from convergence (19) that $\forall\, t \in (\tau,T]$, $\forall\, \varepsilon > 0\, \exists\, \bar{t} \in (\tau,t)$: $|J(\bar{t}) - J(t)| < \varepsilon$ and that $\lim_{k\to+\infty} J_k(\bar{t}) = J(\bar{t})$. Therefore, $\forall\, k \geq 1$, we have

$$J_k(t) - J(t) \leq J_k(\bar{t}) - J(t) \leq |J_k(\bar{t}) - J(\bar{t})| + |J(\bar{t}) - J(t)| < \varepsilon + |J_k(\bar{t}) - J(\bar{t})|.$$

Thus, we obtain

$$\forall\, t \in (\tau,T],\ \forall\, \varepsilon > 0 \quad \overline{\lim_{k\to+\infty}}\, J_k(t) \leq J(t) + \varepsilon,$$

which implies inequalities (20) and, in particular, the inequality

$$\overline{\lim_{k\to+\infty}}\, \|u_{n_k}(t)\|_H^2 \leq \|u(t)\|_H^2 \quad \forall\, t \in (\tau,T].$$

The weak convergence of $u_{n_k}(t)$ to $u(t)$ in H as $k \to +\infty$ $\forall\, t \in [\tau,T]$, inequality (20), and the result of [2; Theorem I.5.12] implies convergence (14).

Step 2. Let us show that

$$u' = f_{\tau,T} - d. \tag{21}$$

By virtue of Lemma 1, for any $k \geq 1$ and $\xi \in C_0^\infty([\tau,T];V)$, the following relationship is true:

$$-\langle \xi', u_{n_k} \rangle_{X_{\tau,T}} + \langle d_{n_k}, \xi \rangle_{X_{\tau,T}} = \langle f_{\tau,T}, \xi \rangle. \tag{22}$$

Passing to limit in relationship (22) as $k \to +\infty$, we obtain

$$\forall\, \xi \in C_0^\infty([\tau,T];V) - \langle \xi', u \rangle_{X_{\tau,T}} + \langle d, \xi \rangle_{X_{\tau,T}} = \langle f_{\tau,T}, \xi \rangle.$$

Thus, using properties of the Bochner integral, $\forall \varphi \in C_0^\infty([\tau,T])$, $\forall h \in V$, we have

$$-\left(\int_\tau^T u(s)\varphi'(s)ds, h\right) = -\int_\tau^T (h, u(s))_H \varphi'(s)ds = \int_\tau^T \langle f - d(s), h\rangle_V \varphi(s)ds = \left\langle \int_\tau^T [f_{\tau,T}(s) - d(s)]\varphi(s)ds, h\right\rangle_V .$$

Relationship (21) directly follows from the definition of the derivative of an element $u \in X_{\tau,T}$ in the sense of $\mathcal{D}^*([\tau,T];\ V^*)$.

Step 3. We fix an arbitrary $\varepsilon \in (0, T-\tau)$ and, using the pseudo-monotonicity of $\mathcal{A}_{\tau+\varepsilon,T}$ on $W_{\tau+\varepsilon,T}$, show that

$$d(t) \in A(u(t)) \text{ for a.a. } t \in (\tau+\varepsilon, T). \tag{23}$$

Let us consider restrictions $u_{n_k}(\cdot)$, $d_{n_k}(\cdot)$, $u(\cdot)$, and $d(\cdot)$ to the interval $[\tau+\varepsilon, T]$. For simplicity, we denote them by the same symbols $u_{n_k}(\cdot)$, $d_{n_k}(\cdot)$, $u(\cdot)$, and $d(\cdot)$, respectively. From convergences (13) and (14), we have

$$u_{n_k} \to u \text{ weakly in } W_{\tau+\varepsilon,T}, \quad d_{n_k} \to d \text{ weakly in } X^*_{\tau+\varepsilon,T},$$

$$\forall t \in [\tau+\varepsilon, T] \quad u_{n_k}(t) \to u(t) \text{ in } H,\ k \to +\infty. \tag{24}$$

Let us show that

$$\lim_{k\to+\infty} \langle d_{n_k}, u_{n_k} - u\rangle_{X_{\tau+\varepsilon,T}} = 0. \tag{25}$$

In fact, we have

$$\forall k \ge 1 \int_{\tau+\varepsilon}^T \langle d_{n_k}(s), u_{n_k}(s) - u(s)\rangle_V\, ds = \int_{\tau+\varepsilon}^T (f, u_{n_k}(s) - u(s))ds - \int_{\tau+\varepsilon}^T \langle u'_{n_k}, (s), u_{n_k}(s) - u(s)\rangle_V\, ds. \tag{26}$$

It follows from convergence (24) that

$$\int_{\tau+\varepsilon}^T (f, u_{n_k}(s) - u(s))ds \to 0,\ k \to +\infty. \tag{27}$$

From statements (5) and (24) we obtain

$$\int_{\tau+\varepsilon}^T \langle u'_{n_k}(s), u(s) - u_{n_k}(s)\rangle_V\, ds = \int_{\tau+\varepsilon}^T \langle u'_{n_k}(s), u(s)\rangle_V - \frac{1}{2}(\|u_{n_k}(T)\|_H^2 - \|u_{n_k}(\tau+\varepsilon)\|_H^2)$$

$$\to \int_{\tau+\varepsilon}^T \langle u'(s), u(s)\rangle_V - \frac{1}{2}(\|u(\tau)\|_H^2 - \|u(\tau+\varepsilon)\|_H^2) = 0,\ k \to +\infty. \tag{28}$$

Passing to limit in statement (26) as $k \to +\infty$, we obtain statement (25) from statements (27) and (28).

Thus, we obtain relation (23) from statements (11), (24), and (25) and the pseudo-monotonicity of $\mathcal{A}_{\tau+\varepsilon,T}$ on $W_{\tau+\varepsilon,T}$.

Step 4. The arbitrariness of $\varepsilon \in (0, T-\tau)$, convergences (13), relation (23), and definition of $\mathcal{A}_{\tau,T}$ imply $u(\cdot) \in \mathcal{D}_{\tau,T}(\eta)$.

Step 5. We prove convergence (10) by contradiction. Let us assume that $\exists\, \varepsilon > 0$, $\exists\, L > 0$, and $\exists\, \{u_{k_j}\}_{j\ge1} \subset \{u_{n_k}\}_{k\ge1}$ such that

$$\forall j \ge 1 \quad \max_{t\in[\tau+\varepsilon,T]} \|u_{k_j}(t) - u(t)\|_H = \|u_{k_j}(t_j) - u(t_j)\|_H \ge L.$$

Without loss of generality, we can consider that $t_j \to t_0 \in [\tau+\varepsilon, T]$, $j \to +\infty$. Hence, by virtue of the continuity of $u\colon [\tau,T] \to H$, we have

$$\varliminf_{j\to+\infty} \|u_{k_j}(t_j) - u(t_0)\|_H \ge L. \tag{29}$$

At the same time, let us show that

$$u_{k_j}(t_j) \to u(t_0) \text{ in } H,\ j \to +\infty. \tag{30}$$

Step 5.1. We first prove that

$$u_{k_j}(t_j) \to u(t_0) \text{ weakly in } H,\ j \to +\infty. \tag{31}$$

For a fixed $h \in V$, it follows from convergences (13) that the sequence of real functions $(u_{n_k}(\cdot), h):[\tau, T] \to \mathbb{R}$ is uniformly bounded and equipotentionally continuous. Taking into account inequality (12) and the density of the embedding $V \subset H$, we obtain that $u_{n_k}(t) \to u(t)$ weakly in H and uniformly on $[\tau, T]$, $k \to +\infty$, which implies convergence (31).

Step 5.2. We prove that

$$\overline{\lim_{j \to +\infty}} \|u_{k_j}(t_j)\|_H \le \|u(t_0)\|_H. \tag{32}$$

Let us consider the continuous monotonically nonincreasing functions J_{k_j} and J, $j \ge 1$, defined as functions (17) and (18). We fix an arbitrary $\varepsilon_1 > 0$. It follows from convergence (19) and the continuity of J that

$$\exists\, \bar{t} \in (\tau, t_0): \lim_{j \to +\infty} J_{k_j}(\bar{t}) = J(\bar{t}), \ |J(\bar{t}) - J(t_0)| < \varepsilon_1.$$

Then, for sufficiently large $j \ge 1$, we have

$$J_{k_j}(t_j) - J(t_0) \le |J_{k_j}(\bar{t}) - J(\bar{t})| + |J(\bar{t}) - J(t_0)| \le |J_{k_j}(\bar{t}) - J(\bar{t})| + \varepsilon_1.$$

Hence, we have $\overline{\lim_{j \to +\infty}} J_{k_j}(t_j) \le J(t_0) + \varepsilon_1$. From the arbitrariness of $\varepsilon_1 > 0$ and since $t_j \to t_0$ and $j \to +\infty$, we obtain inequality (32).

Step 5.3. Convergence (31), inequality (32), and [2; Theorem I .5.12] directly imply convergence (30).

Step 5.4. To complete the proof of the theorem, we note that convergence (30) contradicts inequality (29).

The theorem is proved.

COROLLARY 1. Let $\tau < T$, let $\{u_n\}_{n \ge 1}$ be an arbitrary sequence of weak solutions to problem (1) on $[\tau, T]$, and let the sequence be such that $u_n(\tau) \to \eta$ in H, $n \to +\infty$. Then there are $u(\cdot) \in \mathcal{D}_{\tau,T}(\eta)$ and $\{u_{n_k}\}_{k \ge 1} \subset \{u_n\}_{n \ge 1}$ such that $u_{n_k} \to u$ in $C([\tau, T]; H)$, $k \to +\infty$.

Proof. The unique vital difference from the proof of Theorem 1 consists of checking the inequality $\overline{\lim_{j \to +\infty}} J_{k_j}(t_j) \le J(t_0)$ when $t_0 = \tau$, $t_j \to t_0$, $j \to +\infty$, and $\{t_j\}_{j \ge 1} \subset [\tau, T]$ (see step 5.2 of the proof of Theorem 1). In this case, $\forall\, j \ge 1$, $J_{k_j}(t_j) - J(\tau) \le J_{k_j}(\tau) - J(\tau)$. Since $u_n(\tau) \to u(\tau)$ in H, $n \to +\infty$, we have $J_{k_j}(\tau) \to J(\tau)$, $j \to +\infty$. Hence, we obtain $\overline{\lim_{j \to +\infty}} J_{k_j}(t_j) \le J(t_0)$.

4. GLOBAL ATTRACTORS

Let us consider constructions introduced in [7]. We denote by $P(H)$ ($\mathcal{B}(H)$) the collection of all nonempty (nonempty bounded) subsets of the space H. We recall that an m-semiflow is understood to be a multivalued mapping $G: \mathbb{R} \times H \to P(H)$ for which

- $G(0, \cdot) = Id$ (identity mapping);
- $G(t+s, x) \subset G(t, G(s, x))$ $\forall\, x \in H$, $t, s \in \mathbb{R}_+$;

an *m*-semiflow is strict if $G(t+s, x) = G(t, G(s, x))$ $\forall\, x \in H$, $t, s \in \mathbb{R}_+$.

It follows from Lemmas 3 and 4 that any weak solution can be extended to a global solution defined on $[0, +\infty)$. Let, for an arbitrary $y_0 \in H$, $\mathcal{D}(y_0)$ be a collection of all weak solutions (defined on $[0, +\infty)$) to problem (1) with the initial data $y(0) = y_0$.

We define the m-semiflow G as follows: $G(t, y_0) = \{y(t) \mid y(\cdot) \in \mathcal{D}(y_0)\}$.

LEMMA 5. The m-semiflow G is strict.

Proof. Let $y \in G(t+s, y_0)$. Then $y = u(t+s)$, where $u(\cdot) \in \mathcal{D}(y_0)$. Lemma 3 implies that $v(\cdot) = u(s + \cdot) \in \mathcal{D}(u(s))$. Hence, we have $y = v(t) \in G(t, u(s)) \subset G(t, G(s, y_0))$.

Conversely, if $y \in G(t, G(s, y_0))$, then $\exists\, u(\cdot) \in \mathcal{D}(y_0)$, $v(\cdot) \in \mathcal{D}(u(s))$: $y = v(t)$. Let us define the following mapping:

$$z(\xi) = \begin{cases} u(\xi), & \xi \in [0, s], \\ v(\xi - s), & \xi \in [s, t+s]. \end{cases}$$

It follows from Lemma 3 that $z(\cdot) \in \mathcal{D}(y_0)$. Hence, $y = z(t+s) \in G(t+s, y_0)$.

We recall that a set $\mathcal{A}$ is called a global attractor of G if

- $\mathcal{A}$ is negatively semiinvariant (i.e., $\mathcal{A} \subset G(t,\mathcal{A})\ \forall\, t \geq 0$);
- $\mathcal{A}$ is an attracting set, i.e.,

$$\text{dist}\,(G(t,B),\mathcal{A}) \to 0,\ t \to +\infty\ \ \forall\, B \in \mathcal{B}(H), \tag{33}$$

where $\text{dist}\,(C,D) = \sup\limits_{c\in C} \inf\limits_{d\in D} ||c-d||_H$ is the Hausdorff hemimetric;

- for any closed set $Y \subset H$ satisfying property (33), we have $\mathcal{A} \subset Y$ (minimality).

A global attractor is called invariant if $\mathcal{A} = G(t,\mathcal{A})\ \forall\, t \geq 0$.

Let us prove the existence of an invariant compact global attractor.

THEOREM 2. An m-semiflow G possesses an invariant global attractor $\mathcal{A}$ compact in the phase space H.

Proof. It follows from Lemma 4 that

$$\exists\, R,\, \widetilde{\alpha} > 0:\ \forall\, y_0 \in H, y(\cdot) \in \mathcal{D}(y_0),\ t \geq 0\ \ ||y(t)||_H^2 \leq ||y_0||_H^2\, e^{-\widetilde{\alpha} t} + R. \tag{34}$$

Thus, a sphere $B_0 = \{u \in H\,|\,||u||_H \leq \sqrt{R+1}\}$ is an absorbing set, i.e., $\forall\, B \in \mathcal{B}(H)\ \exists\, T(B) > 0:\ \forall\, t \geq T(B)\ G(t,B) \subset B_0$. In particular, inequality (34) implies that the set $\cup_{t\geq 0} G(t,B)$ is bounded in $H\ \forall\, B \in \mathcal{B}(H)$.

We also note that, by Theorem 1, a mapping $G(t,\cdot): H \to \mathcal{B}(H)$ assumes compact values and is compact when $t > 0$ in the sense that it translates bounded sets into precompact sets.

Let us prove that the mapping $u_0 \to G(t,u_0)$ is upper semicontinuous [9; Definition 1.4.1]. To this end, it suffices to show [10; p. 48] that $\forall\, u_0 \in H,\ \forall\, \varepsilon > 0\ \exists\, \delta(u_0,\varepsilon) > 0:\ \forall\, u \in B_\delta(u_0)\ G(t,u) \subset B_\varepsilon(G(t,u_0)) = \{z \in H\,|\text{dist}\,(z, G(t,u_0)) < \varepsilon\}$. If this is not the case, then there are $u_0 \in H$, $\varepsilon > 0$, $\{\delta_n\}_{n\geq 1} \subset (0,+\infty)$, and $\{u_n\}_{n\geq 1} \subset H$ such that $\forall\, n \geq 1\ u_n \in B_{\delta_n}(u_0)$, $G(t,u_n) \not\subset B_\varepsilon(G(t,u_0))$, and $\delta_n \to 0$, $n \to +\infty$. Then we have $\forall\, n \geq 1\ \exists\, v_n(\cdot) \in \mathcal{D}(u_n):\ v_n(t) \notin B_\varepsilon(G(t,u_0))$. Since $u_n \to u_0$ in H, $n \to +\infty$, Theorem 1 implies that $v_n(t) \to v(t) \in G(t,u_0)$ in H, $n \to +\infty$, for some $v(\cdot) \in \mathcal{D}(u_0)$. This contradicts the fact that $\forall\, n \geq 1\ \ ||v_n(t) - v(t)||_H \geq \varepsilon$.

Thus, the existence of a global attractor with required properties directly follows from [7; Proposition 2, Theorem 3, and Remark 8].

The theorem is proved.

5. TRAJECTORY ATTRACTORS

Let us consider the family $\mathcal{K}_+ = \cup_{y_0\in H} \mathcal{D}(y_0)$ of all weak solutions to inclusion (1) that are defined on $[0,+\infty)$. Note that $\mathcal{K}_+$ is translationally invariant, i.e., $\forall\, u(\cdot) \in \mathcal{K}_+$, $\forall\, h \geq 0\ u_h(\cdot) \in \mathcal{K}_+$, where $u_h(s) = u(h+s)$, $s \geq 0$. We specify a semigroup of translations $\{T(h)\}_{h\geq 0}$, $T(h)u(\cdot) = u_h(\cdot)$, $h \geq 0$, $u \in \mathcal{K}_+$ on $\mathcal{K}_+$. By virtue of the translational invariance of $\mathcal{K}_+$, we conclude that $T(h)\mathcal{K}_+ \subset \mathcal{K}_+$ when $h \geq 0$.

We construct an attractor of the translational semigroup $\{T(h)\}_{h\geq 0}$ acting on $\mathcal{K}_+$. On $\mathcal{K}_+$, we consider the topology induced from the Frechet space $C^{\text{loc}}(\mathbb{R}_+;H)$. Note that

$$f_n(\cdot) \to f(\cdot)\ \text{in}\ C^{\text{loc}}(\mathbb{R}_+;H) \Leftrightarrow\ \forall\, M > 0\ \Pi_M f_n(\cdot) \to \Pi_M f(\cdot)\ \text{in}\ C([0,M];H),$$

where Π_M is the restriction operator on an interval $[0,M]$ [6; p. 18]. We denote by Π_+ the restriction operator on $[0,+\infty)$.

We recall that a set $\mathcal{P} \subset C^{\text{loc}}(\mathbb{R}_+;H) \cap L_\infty(\mathbb{R}_+;H)$ is called attracting for the space of trajectories $\mathcal{K}_+$ of inclusion (1) in the topology of $C^{\text{loc}}(\mathbb{R}_+;H)$ if, for any set $\mathcal{B} \subset \mathcal{K}_+$ bounded in $L_\infty(\mathbb{R}_+;H)$ and an arbitrary number $M \geq 0$, the following relationship is satisfied:

$$\text{dist}_{C([0,M];H)}(\Pi_M T(t)\mathcal{B}, \Pi_M \mathcal{P}) \to 0,\quad t \to +\infty. \tag{35}$$

A set $\mathcal{U} \subset \mathcal{K}_+$ is called a trajectory attractor in the space of trajectories $\mathcal{K}_+$ with respect to the topology of $C^{\text{loc}}(\mathbb{R}_+; H)$ [6; Definition 1.2] if

- $\mathcal{U}$ is compact in $C^{\text{loc}}(\mathbb{R}_+; H)$ and is bounded in $L_\infty(\mathbb{R}_+; H)$;
- $\mathcal{U}$ is strictly invariant with respect to $\{T(h)\}_{h\ge 0}$, i.e., $T(h)\mathcal{U} = \mathcal{U}\ \forall h \ge 0$;
- $\mathcal{U}$ is an attracting set for the space of trajectories $\mathcal{K}_+$ in the topology of $C^{\text{loc}}(\mathbb{R}_+; H)$.

Let us consider inclusion (1) on the entire number axis. By analogy with the space $C^{\text{loc}}(\mathbb{R}_+; H)$, the space $C^{\text{loc}}(\mathbb{R}; H)$ is supplied with the topology of local uniform convergence on every interval $[-M, M] \subset \mathbb{R}$ [6; p. 198]. A function $u \in C^{\text{loc}}(\mathbb{R}; H) \cap L_\infty(\mathbb{R}; H)$ is called a complete trajectory of inclusion (1) if $\forall h \in \mathbb{R}\ \Pi_+ u_h(\cdot) \in \mathcal{K}_+$ [6; p. 198]. Let $\mathcal{K}$ be the totality of all complete trajectories of inclusion (1). We note that

$$\forall h \in \mathbb{R},\ \forall u(\cdot) \in \mathcal{K}\ \ u_h(\cdot) \in \mathcal{K}. \tag{36}$$

LEMMA 6. The set $\mathcal{K}$ is nonempty, is compact in $C^{\text{loc}}(\mathbb{R}; H)$, and is bounded in $L_\infty(\mathbb{R}; H)$; moreover, we have

$$\forall y(\cdot) \in \mathcal{K},\ \forall t \in \mathbb{R}\quad y(t) \in \mathcal{A}, \tag{37}$$

where $\mathcal{A}$ is the global attractor from Theorem 2.

Proof. Step 1. Let us show that $\mathcal{K} \ne \varnothing$. It may be noted that [15] and also conditions (1) and (3)–(5) imply that $\exists v \in V$: $A(v) \ni f$. We put $u(t) = v\ \forall t \in \mathbb{R}$. Then $u \in \mathcal{K} \ne \varnothing$.

Step 2. Let us prove statement (37). For any $y \in \mathcal{K}\ \exists d > 0: \|y(t)\|_H \le d\ \forall t \in \mathbb{R}$. We put $B = \cup_{t\in\mathbb{R}} \{y(t)\} \in \mathcal{B}(H)$. Note that, $\forall \tau \in \mathbb{R}$, $\forall t \in \mathbb{R}_+\ \ y(\tau) = y_{\tau-t}(t) \in G(t, y_{\tau-t}(0)) \subset G(t, B)$. It follows from Theorem 2 and convergence (33) that $\forall \varepsilon > 0\ \exists T > 0: \forall \tau \in \mathbb{R}\ \text{dist}(y(\tau), \mathcal{A}) \le \text{dist}(G(T, B), \mathcal{A}) < \varepsilon$. Therefore, taking into account the compactness of $\mathcal{A}$ in H, for any $u(\cdot) \in \mathcal{K}$ and $\tau \in \mathbb{R}$, we have $u(\tau) \in \mathcal{A}$.

Step 3. The boundedness of $\mathcal{K}$ in $L_\infty(\mathbb{R}_+; H)$ follows from statement (37) and the boundedness of $\mathcal{A}$ in H.

Step 4. We check the compactness of $\mathcal{K}$ in $C^{\text{loc}}(\mathbb{R}; H)$. To this end, is suffices to check its precompactness and closedness.

Step 4.1. We check the precompactness of $\mathcal{K}$ in $C^{\text{loc}}(\mathbb{R}; H)$. If this is not the case, then, by virtue of statement (36), $\exists M > 0: \Pi_M \mathcal{K}$ is not a precompact set in $C([0, M]; H)$. Hence, there is a sequence $\{v_n\}_{n\ge1} \subset \Pi_M \mathcal{K}$ that does not have a subsequence converging in $C([0, M]; H)$. At the same time, we have $v_n = \Pi_M u_n$, where $u_n \in \mathcal{K}$, $v_n(0) = u_n(0) \in \mathcal{A}$, $n \ge 1$. Since $\mathcal{A}$ is a compact set in H (see Theorem 2), by virtue of Corollary 1, $\exists \{v_{n_k}\}_{k\ge1} \subset \{v_n\}_{n\ge1}$, $\exists \eta \in H$, $\exists v(\cdot) \in \mathcal{D}_{0,M}(\eta)$: $v_{n_k}(0) \to \eta$ in H and $v_{n_k} \to v$ in $C([0, T]; H)$, $k \to +\infty$. We arrive at a contradiction.

Step 4.2. We check the closedness of $\mathcal{K}$ in $C^{\text{loc}}(\mathbb{R}; H)$. Let $\{v_n\}_{n\ge1} \subset \mathcal{K}$, $v \in C^{\text{loc}}(\mathbb{R}; H)$: $v_n \to v$ in $C^{\text{loc}}(\mathbb{R}; H)$, $n \to +\infty$. The boundedness of $\mathcal{K}$ in $L_\infty(\mathbb{R}; H)$ implies $v \in L_\infty(\mathbb{R}; H)$. From Corollary 1 we obtain that, $\forall M > 0$, the restriction $v(\cdot)$ to an interval $[-M, M]$ belongs to $\mathcal{D}_{-M,M}(v(-T))$. Hence, $v(\cdot)$ is a complete trajectory of inclusion (1). Thus, $v \in \mathcal{K}$.

LEMMA 7. Let $\mathcal{A}$ be the global attractor from Theorem 2. Then

$$\forall y_0 \in \mathcal{A}\ \ \exists y(\cdot) \in \mathcal{K}:\ y(0) = y_0. \tag{38}$$

Proof. Let $y_0 \in \mathcal{A}$, and let $u(\cdot) \in \mathcal{D}(y_0)$. From inequality (9) and convergence (33) we obtain $\forall t \in \mathbb{R}_+\ y(t) \in \mathcal{A}$. Theorem 2 implies that $G(1, \mathcal{A}) = \mathcal{A}$. Therefore, we have

$$\forall \eta \in \mathcal{A}\ \ \exists \xi \in \mathcal{A},\ \exists \varphi_\eta(\cdot) \in \mathcal{D}_{0,1}(\xi):\ \varphi_\eta(1) = \eta.$$

For any $t \in \mathbb{R}$, we put

$$y(t) = \begin{cases} u(t), & t \in \mathbb{R}_+, \\ \varphi_{y(-k+1)}(t+k), & t \in [-k, -k+1),\ k \in \mathbb{N}. \end{cases}$$

We note that $y \in C^{\text{loc}}(\mathbb{R}; H)$ and $y(t) \in \mathcal{A}\ \forall t \in \mathbb{R}$ (hence, $y \in L_\infty(\mathbb{R}; H)$) and, by virtue of Lemma 3, $y \in \mathcal{K}$ and, at the same time, $y(0) = y_0$.

THEOREM 3. Let $\mathcal{A}$ be the global attractor from Theorem 2. Then there is a trajectory attractor $\mathcal{P} \subset \mathcal{K}_+$ in the space $\mathcal{K}_+$. In this case, we have

$$\mathcal{P} = \Pi_+ \mathcal{K} = \Pi_+ \{y \in \mathcal{K} \mid y(t) \in \mathcal{A} \ \forall t \in \mathbb{R}\}. \tag{39}$$

Proof. It follows from Lemma 6 and the continuity of the operator $\Pi_+ : C^{\text{loc}}(\mathbb{R}; H) \to C^{\text{loc}}(\mathbb{R}_+; H)$ that the set $\Pi_+ \mathcal{K}$ is nonempty, is compact in $C^{\text{loc}}(\mathbb{R}_+; H)$, and is bounded in $L_\infty(\mathbb{R}_+; H)$. Moreover, the second equality in formula (39) is fulfilled. The strict invariance of $\Pi_+ \mathcal{K}$ follows from the autonomy of inclusion (1).

Let us prove that $\Pi_+ \mathcal{K}$ is an attracting set for the space of trajectories $\mathcal{K}_+$ in the topology of $C^{\text{loc}}(\mathbb{R}_+; H)$. Let $B \subset \mathcal{K}_+$ be a bounded set in $L_\infty(\mathbb{R}_+; H)$, and let $M \ge 0$. Let us check the fulfillment of relationship (35). If it is not fulfilled, then there are sequences $t_n \to +\infty$, $v_n(\cdot) \in B$ such that

$$\forall n \ge 1 \ \ \text{dist}_{\,C([0,T];H)}(\Pi_M v_n(t_n + \cdot), \Pi_M \mathcal{K}) \ge \varepsilon. \tag{40}$$

At the same time, the boundedness of B in $L_\infty(\mathbb{R}_+; H)$ implies that $\exists R > 0 : \forall v(\cdot) \in B$, $\forall t \in \mathbb{R}_+ \ \|v(t)\|_H \le R$. Thus, $\exists N \ge 1$: $\forall n \ge N \ v_n(t_n) \in G(t_n, v_n(0)) \subset G(1, G(t_n - 1, v_n(0))) \subset G(1, \overline{B}_R)$, where $\overline{B}_R = \{u \in H \,|\, \|u\|_H \le R\}$.

Hence, taking into account convergence (33) and the compactness of the mapping $G(1,\cdot): H \to \mathcal{B}(H)$ (see the proof of Theorem 2), we obtain $\exists \{v_{n_k}(t_{n_k})\}_{k \ge 1} \subset \{v_n(t_n)\}_{n \ge 1}$, $\exists z \in \mathcal{A}$: $v_{n_k}(t_{n_k}) \to z$ in H, $k \to +\infty$. Next, $\forall k \ge 1$, we put $\varphi_k(t) = v_{n_k}(t_{n_k} + t)$, $t \in [0, M]$. Note that, $\forall k \ge 1$, $\varphi_k(\cdot) \in \mathcal{D}_{0,M}(v_{n_k}(t_{n_k}))$. Then, from Corollary 1, we obtain a subsequence $\{\varphi_{k_j}\}_{j \ge 1} \subset \{\varphi_k\}_{k \ge 1}$ and an element $\varphi(\cdot) \in \mathcal{D}_{0,M}(z)$,

$$\varphi_{k_j} \to \varphi \ \text{ in } C([0,M]; H), \quad j \to +\infty. \tag{41}$$

In this case, taking into account the invariance of $\mathcal{A}$ (see Theorem 2), $\forall t \in [0, M] \ \varphi(t) \in \mathcal{A}$. By Lemma 7, there are $y(\cdot), v(\cdot) \in \mathcal{K}$ such that $y(0) = z$ and $v(0) = \varphi(M)$. For any $t \in \mathbb{R}$, we put

$$\psi(t) = \begin{cases} y(t), & t \le 0, \\ \varphi(t), & t \in [0, M], \\ v(t - M), & t \ge M. \end{cases}$$

By Lemma 3, $\psi(\cdot) \in \mathcal{K}$. Hence, from statement (40), we obtain

$$\forall k \ge 1 \ \|\Pi_M v_{n_k}(t_{n_k} + \cdot) - \Pi_M \psi(\cdot)\|_{C([0,M];H)} = \|\varphi_k - \varphi\|_{C([0,M];H)} \ge \varepsilon,$$

contrary to convergence (41).

Thus, the set $\mathcal{P}$ in the construction of attractor (39) is a trajectory attractor in the space of trajectories $\mathcal{K}_+$ with respect to the topology of $C^{\text{loc}}(\mathbb{R}_+; H)$.

The theorem is proved.

EXAMPLES

We consider the class of nonlinear boundary problems in which the dynamics of solutions can be investigated as $t \to +\infty$, making no pretense to the generality of the presentation.

We assume that $n \ge 2$, $m \ge 1$, $p \ge 2$, $1 < q \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary $\Gamma = \partial\Omega$. We denote by N_1 (respectively, by N_2) the number of derivations of order $\le m-1$ (respectively, of order $= m$) with respect to x. Let also $A_\alpha(x, \eta; \xi)$ be a family of real functions ($|\alpha| \le m$) defined in $\Omega \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and satisfying the conditions

(a) for a.a. $x \in \Omega$, a function $(\eta, \xi) \to A_\alpha(x, \eta, \xi)$ is continuous in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$;

(b) $\forall (\eta, \xi) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, a function $x \to A_\alpha(x, \eta, \xi)$ is measurable in Ω;

(c) there are $c_1 \geq 0$ and $k_1 \in L_q(\Omega)$ such that, for a.a. $x \in \Omega$, $\forall (\eta, \xi) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, we have

$$|A_\alpha(x, \eta, \xi)| \leq c_1[|\eta|^{p-1} + |\xi|^{p-1} + k_1(x)];$$

(d) there are $c_2 > 0$ and $k_2 \in L_1(\Omega)$ such that, for a.a. $x \in \Omega$, $\forall (\eta, \xi) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, we have

$$\sum_{|\alpha|=m} A_\alpha(x, \eta, \xi)\xi_\alpha \geq c_2 |\xi|^p - k_2(x);$$

(e) for a.a. $x \in \Omega$, $\forall \eta \in \mathbb{R}^{N_1}$, $\forall \xi, \xi^* \in \mathbb{R}^{N_2}$, and $\xi \to \xi^*$, we have

$$\sum_{|\alpha|=m} (A_\alpha(x, \eta, \xi) - A_\alpha(x, \eta, \xi^*))(\xi_\alpha - \xi_\alpha^*) > 0.$$

We introduce the denotations $D^k u = \{D^\beta u, |\beta| = k\}$ and $\delta u = \{u, Du, \ldots, D^{m-1}u\}$ [4; p.194].

We investigate the dynamics of all weak (generalized) solutions defined on $[0, +\infty)$ for an arbitrary fixed external force $f \in L_2(\Omega)$ as $t \to +\infty$ in the following problem:

$$\frac{\partial y(x,t)}{\partial t} + \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (A_\alpha(x, \delta y(x,t), D^m y(x,t))) = f(x) \text{ in } \Omega \times (0, +\infty), \tag{42}$$

$$D^\alpha y(x,t) = 0 \text{ on } \Gamma \times (0, +\infty), \; |\alpha| \leq m-1. \tag{43}$$

We introduce the following denotations [4; p. 195]: $H = L_2(\Omega)$, $V = W_0^{m,p}(\Omega)$ is the Sobolev space of real-valued functions, and

$$a(u, \omega) = \sum_{|\alpha| \leq m} \int_\Omega A_\alpha(x, \delta u(x), D^m u(x)) D^\alpha \omega(x) dx, \quad u, \omega \in V.$$

Condition (2) takes place according to the Sobolev theorem on the compactness of embedding. Taking into account conditions (a)–(e) and the reasoning from [4; pp. 192–199], the operator $A: V \to V^*$ defined by the formula $\langle A(u), \omega \rangle_V = a(u, \omega)\ \forall u, \omega \in V$ satisfies conditions (3)–(5). Hence, it is possible to pass from problem (42), (43) to the corresponding problem in "generalized" statement (1). We note that

$$A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (A_\alpha(x, \delta u, D^m u)) \;\; \forall u \in C_0^\infty(\Omega).$$

Thus, for weak (generalized) solutions of problem (42), (43), all the statements from the previous sections, in particular, Theorems 1–3 and Lemmas 1–7 are fulfilled.

Comment 3. New classes of problems can also be considered as applications such as problems with degeneration, problems on manifolds with boundary and without boundary, problems with delay, stochastic partial differential equations, and other problems with differential operators of pseudomonotone type with the corresponding choice of phase spaces [4, 11–13].

CONCLUSIONS

It follows from the results of Secs. 4 and 5 that an m-semiflow G constructed over all weak solutions to problem (1) has a compact invariant global attractor $\mathcal{A}$. For all weak solutions to problem (1) that are defined on $[0, +\infty)$, there is a trajectory attractor $\mathcal{P}$. At the same time, we have $\mathcal{A} = \mathcal{P}(0) = \{y(0) \mid y \in \mathcal{K}\}$, $\mathcal{P} = \Pi_+ \mathcal{K}$, where $\mathcal{K}$ is the totality of all complete trajectories of differential-operator inclusion (1) in $C^{\text{loc}}(\mathbb{R}; H) \cap L_\infty(\mathbb{R}; H)$. Thus, the equality of global attractors is proved both in the sense of [7; Definition 6] and in the sense of [6; Definition 2.2]. The questions of connectedness and dimensionality of the constructed attractors remain open in the general case. We note that the approaches proposed in [6, 7] are based on properties of solutions of evolutionary objects, in particular, in this work, on properties of the interaction function A from problem (1) and properties of phase spaces.

Analyzing the proofs of the presented results, the following weaker condition imposed on the mapping $A: V \rightrightarrows V^*$ can be considered instead of condition (5): since $u_n \to u$ weakly in V, $d_n \in A(u_n)\ \forall\, n \ge 1$, $d_n \to d$ weakly in V^*, $n \to +\infty$, and $\lim_{n \to +\infty} \langle d_n, u_n - u \rangle_V = 0$, we have $d \in A(u)$.

For the class of autonomous differential-operator inclusions with a pseudomonotone nonlinear dependence between key problem parameters, the dynamics of all global weak solutions defined on $[0, +\infty)$ is investigated as $t \to +\infty$. The existence of a global compact attractor and a compact trajectory attractor is proved, their structures are studied, and the equality of global attractors is checked both in the sense of Definitions 6 from [7] and in the sense of Definition 2.2 from [6]. The obtained results allow one to investigate the dynamics of solutions of new classes of evolutionary inclusions from nonlinear mathematical models of geophysical and sociotechnical processes and fields with a pseudomonotone interaction function satisfying the condition of no more than polynomial growth and standard sign condition.

REFERENCES

1. J. M. Ball, "Continuity properties and global attractors of generalized semiflows and the Navier–Stokes equations," J. Nonlinear Sci., **7**, No. 5, 475–502 (1997).
2. H. Gajewski, K. Grцger, and K. Zacharias, Nonlinear Operator Equations and Operator Differential Equations [Russian translation], Mir, Moscow (1978).
3. S. Migorski, "Boundary hemivariational inequalities of hyperbolic type and applications," J. Global Optim., **31**, No. 3, 505–533 (2005).
4. J.-L. Lions, Some Methods of Solution of Nonlinear Boundary-Value Problems [Russian translation], Mir, Moscow (1972).
5. O. A. Ladyzhenskaya, Mathematical Problems in the Dynamics of a Viscous Incompressible Fluid [in Russian], Fizmatgiz, Moscow (1961).
6. M. I. Vishik and V. V. Chepyzhov, "Trajectory and global attractors of the three-dimensional Navier–Stokes systems," Mat. Zametki, **71**, No. 2, 194–213 (2002).
7. V. S. Melnik and J. Valero, "On attractors of multivalued semi-flows and differential inclusions," Set-Valued Analysis, **6**, No. 1, 83–111 (1998).
8. J. Valero and A. V. Kapustyan, "On the connectedness and asymptotic behaviour of solutions of reaction-diffusion systems," J. Math. Analysis and Appl., **323**, No. 1, 614–633 (2006).
9. J. P. Aubin and H. Frankowska, Set-Valued Analysis, Birkhauser, Boston (1990).
10. J. P. Aubin and A. Sellina, Set-Valued Analysis and Viability Theory, Springer, Berlin (1984).
11. G. R. Sell and Yu You, Dynamics of Evolutionary Equations, Springer, New York (2002).
12. Yu. A. Dubinsky, "Nonlinear higher order parabolic equations," Itogi Nauki i Tekhniki, Sovrem. Probl. Matematiki, Nov. Dostizh., **37**, 89–166 (1990).
13. V. V. Chepyzhov and M. I. Vishik, "Trajectory attractor for reaction-diffusion system with diffusion coefficient vanishing in time," Discr. and Contin. Dynam. Systems, **27**, No. 4, 1498–1509 (2010).
14. I. D. Chueshov, "Global attractors in nonlinear problems of mathematical physics," Usp. Mat. Nauk, **48**, No. 3 (291), 135–162 (1993).
15. M. Z. Zgurovsky, P. O. Kasyanov, and V. S. Melnik, Differential-Operator Inclusions and Variational Inequalities in Infinite Dimensional Spaces [in Russian], Naukova Dumka, Kyiv (2008).
16. P. O. Kasyanov, V. S. Mel'nik, and S. Toscano, "Solutions of Cauchy and periodic problems for evolution inclusions with multi-valued w_{λ_0}-pseudomonotone maps," J. Differ. Equations, **249**, No. 6, 1258–1287 (2010).
17. A. A. Chikrii, Conflict-Controlled Processes, Kluwer, Dordrecht (1997).