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DESCRIPTION AND GENERATION OF PERMUTATIONS CONTAINING CYCLES

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Abstract. *The paper proposes a general approach to generating permutations that contain cycles, based on constructive tools introduced to describe combinatorial sets. Different generation problems for permutations of definite class are formulated and solved. A combinatorial set is introduced to define permutations represented as the multiplication of a definite number of cycles. For this set, combinatorial species and associated generating series are constructed.*

Keywords: *combinatorial generation, permutation, cycle, multiplication of cycles, cyclic permutation, combinatorial species.*

INTRODUCTION

Problems of generating various combinatorial objects are of current importance in mathematical studies and applications. Many monographs and individual papers [1–6] are devoted to combinatorial generation. By generation is understood the construction of all combinatorial structures of certain type [3]. Generation of rather simple objects such as permutations, combinations, partitions, trees, binary sequences, are mainly considered in the literature. The results of the solution of generation problems are used in modeling, combinatorial optimization, and other fields [7–10]. Generating more complex combinatorial objects is difficult because there are no constructive means, and much computational efforts are required since the results of application of well-known generation means are redundant.

Rather complex combinatorial configurations can be generated using constructive means of the description of composition k-mages of combinatorial sets proposed in [11].

Many problems of enumerative combinatorics involve determining whether there are combinatorial objects of certain type and estimating their number [12–14]. Problems of generating combinatorial objects with prescribed properties are inverse problems in a sense.

One of such direct problems is representing a given permutation as a product of cycles and calculating the number of permutations of certain type [4, 12]. The inverse problem is generating permutations based on given cycles.

The purpose of the present paper is to formulate and solve some problems of generating permutations that contain cycles, with the use of constructive description means based on the composition of k-images of combinatorial sets.

REPRESENTATION OF PERMUTATIONS

Various equivalent combinatorial representations of the permutation of elements of the set $S=\{a_1,a_2,\dots,a_n\}$ are known (see, for example, [4, 12]). One of them is to use a relation with the "natural" order of elements of the set S in the first row and the new order in the second one:

$$f=\begin{pmatrix} a_1, & a_2,\dots,a_n \\ a_{j_1}, & a_{j_2},\dots,a_{j_n} \end{pmatrix}. \tag{1}$$

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Other representation is an ordered sequence of elements of the set of the S-word:

$$\pi = (a_{j_1}, a_{j_2}, \dots, a_{j_n}), \tag{2}$$

where it is meant that the permutation transfers a_1 into a_{j_1}, a_2 into a_{j_2},…, and a_n into a_{j_n}. In this case, the record of the permutation coincides with the second row of (1).

There is one more way of representing a permutation: a product of cycles $\pi = \sigma^{n_1} \cdot \sigma^{n_2} \cdot \dots \cdot \sigma^{n_m}$. By a cycle of length m of a permutation π over the set S is meant a subset $\{a_{i_1}, a_{i_2}, \dots, a_{i_m}\} \subset S$, on which the permutation carries out the following:

$$\pi(a_{i_j}) = a_{i_{j+1}}, \quad i_j \in J_m, \quad j \in J_{m-1}, \quad \pi(a_{i_m}) = a_{i_1}, \quad J_n = \{1, 2, \dots, n\}.$$

A cycle of length m is written as $\sigma^m = (a_{i_1}, a_{i_2}, \dots, a_{i_m})$. Following [12], denote by c_i the number of cycles of length i in the given permutation π, the sequence $(c_1, c_2, \dots, c_n)$ being called the type of the permutation π. Denote the total number of cycles of the permutation π by $c(\pi) = c_1 + c_2 + \dots + c_n$. A permutation reresentable by a unique cycle of length n is called cyclic permutation. The number of permutations of type $(c_1, c_2, \dots, c_n)$ is $n!/ 1^{c_1} \cdot c_1! \cdot 2^{c_2} \cdot c_2! \cdot \dots \cdot n^{c_n} \cdot c_n!$ [12]. Note that some enumerative problems for permutations of type $(c_1, c_2, \dots, c_n)$ were analyzed in [13].

CONSTRUCTIVE MEANS OF DESCRIPTION AND GENERATION

To solve various problems of the description and generation of permutations containing cycles, we will use the apparatus of composition k-images of combinatorial sets [11]. Let us introduce the following combinatorial set.

Definition. A tuple of cyclic permutations is the composition k-image of combinatorial sets T_m; $P^c_{n_1}, P^c_{n_2}, \dots, P^c_{n_m}$ generated by sets $z^1, z^2, \dots, z^m$; we denote it by $TP^c(N, n_1, n_2, \dots, n_m)$ or TP^c_N, where $N = n_1 + n_2 + \dots + n_m$.

Here, $T_m = \{(t_1, t_2, \dots, t_m) \mid t_j = z^0_j \in \mathbf{Z}^0,\ j \in J_m\}$ is a zero-order set, which is a tuple of m different elements, $z^0 = \{z^0_1, z^0_2, \dots, z^0_m\} \in \mathbf{Z}^0$, $\mathbf{Z}^0$ is a set of all possible tuples of m different elements, $P^c_{n_i}$ are first-order sets of cyclic permutations among n_i elements, $i \in J_m$; $z^i = \{z^i_1, z^i_2, \dots, z^i_{n_i}\}$, $i \in J_m$, $z^i \cap z^j = \varnothing$, $i, j \in J_m$, $i \neq j$, $k = 2$. The set TP^c_N consists of elements of the form $w = (\overline{w}_1, \overline{w}_2, \dots, \overline{w}_m) \in TP^c_N$, where $\overline{w}_i = (z^i_{j_1}, z^i_{j_2}, \dots, z^i_{j_{n_i}}) \in P^c_{n_i}$ is a cyclic permutation of elements of the set z^i.

The set TP^c_N is representable by the composition of mappings of the form [11]:

$$TP^c_N = \Gamma_W \circ \Gamma_{T_m}(z^0), \tag{3}$$

where $\Gamma_{T_m}: \mathbf{Z}_0 \to \mathbf{Y}$, $\mathbf{Y} = \bigcup\limits_{z^0 \in \mathbf{Z}^0} T_m(z^0)$ and $\Gamma_W: \mathbf{Y} \to \mathbf{W}$, $\mathbf{W} = \bigcup\limits_{z^i;\ z^0_i \in P^c_{n_i}(z^i),\ i \in J_m} TP^c_N$.

The mapping Γ_{T_m} describes the construction of the tuple T_m of m different elements, Γ_W is constructed based on operations of n-substitution, n-composition, and mappings $\Gamma_{P^c_{n_i}}$, which specify the sets of cyclic permutations $P^c_{n_i}$ [11]. To solve problems of the description and generation of elements of the set TP^c_N, it is necessary to specify the form of mappings Γ_{T_m}, Γ_W, and $\Gamma_{P^c_{n_i}}$ or to specify the ways of their algorithmic implementation.

A simple formal description and sufficiently investigated properties allow considering tuples and cyclic permutations as base combinatorial sets [11]. The operations of n-substitution and n-composition, along with formal descriptions of base sets, make it possible to obtain both the formal descriptions and generation of the elements of a tuple of cyclic permutations.

Applying the results of the theory of combinatorial species [13] and the constructed combinatorial species of composition k-images of combinatorial sets [15], let us determine the combinatorial species of the tuple of cyclic permutations TP^c_N denoted by STP^c_N. We will construct it according to the algorithm proposed in [15].

Let us consider a combinatorial species of cyclic permutations $C_{n_i}(U_i)$ among n_i elements of the *m*-set $U=(U_1,U_2,\ldots,U_m)$, $U_i=\{z_1^i,z_2^i,\ldots,z_{n_i}^i\}$, $i\in J_m$ [13]. Let us introduce a multisort species $\overline{C}_n^i(U)=C_{n_i}(U_i)$, $i\in J_m$, with the base set $U=(U_1,U_2,\ldots,U_m)$. Let $ST_m(U)$ be an *m*-sort species of tuples with the base set U [15]. Then the combinatorial species $STP_N^c(U)$ is representable by

$$STP_N^c(U)=ST_m \square p(\overline{C}_{n_1}^1,\overline{C}_{n_2}^2,\ldots,\overline{C}_{n_m}^m)[U_1,U_2,\ldots,U_m]=ST_m(U)\square(\overline{C}_{n_1}^1(U),\overline{C}_{n_2}^2(U),\ldots,\overline{C}_{n_m}^m(U)),$$

where $\square$ denotes functorial composition on combinatorial species [13].

To solve enumerative problems on the set TP_N^c, let us construct a generating series associated to the species $STP_N^c(U)$ [13] according to the approach outlined in [15]. Consider that $(n_i-1)!$ cyclic permutations can be constructed among n_i various elements [4, 12]. Moreover, let us assume that a unique tuple $T_{n_i}=(z_1^i,z_2^i,\ldots,z_{n_i}^i)$, $i\in J_m$, can be constructed from elements of the set $U_i=\{z_1^i,z_2^i,\ldots,z_{n_i}^i\}$. Then

$$\begin{aligned} F_{STP_N^c}(x_1,x_2,\ldots,x_n) &= \sum_{n_1,n_2,\ldots,n_m\ge 0}(n_1-1)!(n_2-1)!\ldots(n_m-1)!\frac{x_1^{n_1}}{n_1!}\frac{x_2^{n_2}}{n_2!}\cdots\frac{x_m^{n_m}}{n_m!} \\ &= \sum_{n_1,n_2,\ldots,n_m\ge 0}\frac{x_1^{n_1}}{n_1}\frac{x_2^{n_2}}{n_2}\cdots\frac{x_n^{n_m}}{n_m}. \end{aligned} \tag{4}$$

The introduced combinatorial set TP_N^c allows solving various problems of the description and generation of permutations that contain cycles. Let us formulate some of such problems:

(i) generating a unique permutation based on the cycles specified by chosen cyclic permutations;

(ii) generating all the permutations based on the cycles specified by all possible cyclic permutations of elements of the sets $U_i=\{z_1^i,z_2^i,\ldots,z_{n_i}^i\}$, $i\in J_m$;

(iii) generating all the cycle-containing permutations generated by the set consisting of *N* different elements.

Let us analyze these problems.

GENERATING A PERMUTATION BASED ON GIVEN CYCLES

Problem 1. The number *m* and explicit form of the cycles $\sigma^{n_1},\sigma^{n_2},\ldots,\sigma^{n_m}$ generated by disjoint sets of different elements $z^i=\{z_1^i,z_2^i,\ldots,z_{n_i}^i\}$, $i\in J_m$, are specified. Let $n_1,n_2,\ldots,n_m$ be cycle lengths and $n_1+n_2+\ldots+n_m=N$. It is required to generate a permutation $\pi\in P_N$ representable by the product of cycles $\sigma^{n_1},\sigma^{n_2},\ldots,\sigma^{n_m}$. Here, P_N is the set of permutations among *N* different elements, and an explicitly specified cycle σ^{n_i} means

$$z_{j_1}^i \mapsto z_{j_2}^i \mapsto \ldots \mapsto z_{j_{n_i}}^i \mapsto z_{j_1}^i,\quad j_k\in J_{n_i},\quad k\in J_{n_i},\quad i\in J_m. \tag{5}$$

Note that the type of the permutation π thus specified is completely defined by cycle lengths, and the cycles can be considered as elements of the sets of cyclic permutations $P_{n_i}^c$ generated by the sets $z^i=\{z_1^i,z_2^i,\ldots,z_{n_i}^i\}$, $i\in J_m$. To obtain the unknown permutation π, let us employ the tuple of cyclic permutations TP_N^c constructed above. As follows from the construction of the set TP_N^c, permutation π is one of its elements. It can be obtained as follows.

For definiteness, elements of the set z^i are assumed to be ordere
d: $z_1^i\le z_2^i\le\ldots\le z_{n_i}^i$, $i\in J_m$. Let us write the cyclic permutation of elements of the set z^i defined by the cycle σ^{n_i} as

$$f^i=\begin{pmatrix} z_1^i & z_2^i \ldots z_{n_i}^i \\ z_{l_1}^i & z_{l_2}^i \ldots z_{l_{n_i}}^i \end{pmatrix} \tag{6}$$

or as an ordered sequence, but as the second row

$$(z^i_{l_1},\ z^i_{l_2},\ldots,z^i_{l_{n_i}}), \tag{7}$$

meaning that the permutation transfers z^i_1 into $z^i_{l_1}$, z^i_2 into $z^i_{l_2}$, …, and $z^i_{n_i}$ into $z^i_{l_{n_i}}$. To obtain this representation of the cyclic permutation defined by the cycle σ^{n_i}, we will use Algorithm 1, which passes along the chain (5) and forms a cyclic permutation.

Algorithm 1. Forming a cyclic permutation as an ordered sequence based on a cycle of length n_i.

Given are: a cycle σ^{n_i} of length n_i in the form (5). Result: a cyclic permutation $b^i=(b^i_1,b^i_2,\ldots,b^i_{n_i})$ in the form (6) or (7).

1. Put $k=1,\ s=1$.
2. Find $j_l=k$ in the sequence $\{j_1,j_2,\ldots,j_{n_i}\}$.
3. Assign $b^i_k=z^i_{j_{l+1}}$ if $l<n_i$, otherwise go to item 6.
4. Specify $l=l+1,\ k=j_l,\ s=s+1$.
5. Go to item 3 if $s\le n_i$ otherwise stop.
6. Assign $b^i_k=z^i_{j_1},\ l=1,\ s=s+1,\ k=j_1$, go to item 3.

The algorithm produces a cyclic permutation $b^i=(b^i_1,b^i_2,\ldots,b^i_{n_i})$ or

$$f^i=\begin{pmatrix} z^i_1 & z^i_2 \ldots z^i_{n_i} \\ b^i_{l_1} & b^i_{l_2} \ldots b^i_{n_i}\end{pmatrix},$$

corresponding to the cycle σ^{n_i}.

To obtain the permutation π, let us combine into a tuple all the cyclic permutations obtained by Algorithm 1 for all the cycles σ^{n_i}, $i\in J_m$. It is necessary to form the set of elements Z that generate the permutation π and to specify a “natural” order of elements on it, i.e., to form the first row in (1). To this end, we will use Algorithm 2.

Algorithm 2. Forming a permutation π that is the product of cycles $\sigma^{n_1},\sigma^{n_2},\ldots,\sigma^{n_m}$.

Given are: cyclic permutations $b^1,b^2,\ldots,b^m$ obtained from cycles $\sigma^{n_1},\sigma^{n_2},\ldots,\sigma^{n_m}$, respectively, by Algorithm 1. Result: the permutation $\pi=(p_1,p_2,\ldots,p_N)$ in the form (1) and (2) is the product of cycles $\sigma^{n_1},\sigma^{n_2},\ldots,\sigma^{n_m}$ (where p^1_k and p^2_k are elements of the first and second rows, respectively, of the permutation π in (1), $k\in J_N$.

1. Set $i=1$, which is the counter of loops σ^{n_i}, $s=0$.
2. Set $k=1$, which is the counter of elements inside the loop.
3. Assign $p^1_{k+s}=z^i_k,\ p^2_{k+s}=f^i(z^i_k)=b^i_k$.
4. Specify $k=k+1$. Go to item 3 if $k\le n_i$, otherwise go to item 5.
5. Assign $s=s+n_i,\ i=i+1$.
6. Go to item 2 if $i\le m$, otherwise stop.

Thus, sequentially applying Algorithm 1 for each cycle σ^{n_i}, $i\in J_m$, and Algorithm 2 for the obtained cyclic permutations $b^1,b^2,\ldots,b^m$ allows solving Problem 1. Let us consider examples.

Example 1. Given $m=2$ cycles generated by the sets $z^1=\{a,b,c,d\}$ and $z^2=\{e,f,g\}$, σ^4: $b\mapsto d\mapsto a\mapsto c\mapsto b$, $\sigma^3: f\mapsto e\mapsto g\mapsto f$, generate a permutation $\pi\in P_7$ that is the product of these cycles.

Let us use Algorithm 1 to construct cyclic permutations b^1 and b^2 in the form (6) and (7) from the cycles σ^4 and σ^3:

$$f^1=\begin{pmatrix} a & b & c & d\\ c & d & b & a\end{pmatrix},\ b^1=(c\ d\ b\ a),\ f^2=\begin{pmatrix} e & f & g\\ g & e & f\end{pmatrix},\ b^2=(g\ e\ f).$$

Using Algorithm 2, we obtain the unknown permutation $\pi \in P_7$ generated by the set $Z = \{a, b, c, d, e, f, g\}$:

$$f^{\pi} = \begin{pmatrix} a & b & c & d & e & f & g \\ c & d & b & a & g & e & f \end{pmatrix}, \ \pi = (c\ d\ b\ a\ g\ e\ f).$$

Example 2. Given $m = 2$ cycles generated by the sets $z^1 = \{3,5,7\}$ and $z^2 = \{1,2,4,6\}$, $\sigma^3 : 7 \mapsto 3 \mapsto 5 \mapsto 7$, $\sigma^4 : 2 \mapsto 4 \mapsto 1 \mapsto 6 \mapsto 2$, generate a permutation $\pi \in P_7$, which is the product of these cycles.

Using Algorithm, we construct the cyclic permutations b^1 and b^2 in the form (6) and (7) from the cycles σ^3 and σ^4:

$$f^1 = \begin{pmatrix} 3 & 5 & 7 \\ 5 & 7 & 3 \end{pmatrix}, \ b^1 = (5\ 7\ 3), \ f^2 = \begin{pmatrix} 1 & 2 & 4 & 6 \\ 6 & 4 & 1 & 2 \end{pmatrix}, \ b^2 = (6\ 4\ 1\ 2).$$

Using Algorithm 2, we obtain the permutation $\pi \in P_7$ generated by the set $Z = \{1, 2, 3, 4, 5, 6, 7\}$:

$$f^{\pi} = \begin{pmatrix} 3 & 5 & 7 & 1 & 2 & 4 & 6 \\ 5 & 7 & 3 & 6 & 4 & 1 & 2 \end{pmatrix}, \quad \pi = (5\ 7\ 3\ 6\ 4\ 1\ 2). \tag{8}$$

Note that rearranging elements of the first row of f^{π} together with the corresponding elements of the second row yields a permutation, which is the product of the same cycles as π is:

$$f^{\pi_1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 5 & 1 & 7 & 2 & 3 \end{pmatrix}, \quad \pi_1 = (6\ 4\ 5\ 1\ 7\ 2\ 3). \tag{9}$$

GENERATING ALL PERMUTATIONS FROM THE CYCLES GENERATED BY GIVEN SETS

Problem 2. Given are m disjoint sets of different elements $z^i = \{z_1^i, z_2^i, \ldots, z_{n_i}^i\}$, $i \in J_m$. The sets z^i generate sets of cyclic permutations $P_{n_i}^c(z^i)$, $i \in J_m$. Each cyclic permutation $p^j \in P_{n_i}^c(z^i)$, $j \in J_{M_i}$, $M_i = (n_i - 1)!$, determines a unique cycle $\sigma^{n_i}(p^j)$. It is required to generate all possible different permutations $\pi \in P_N$, $n_1 + n_2 + \ldots + n_m = N$, representable as the product of cycles $\sigma^{n_i}(p^j)$, $j \in J_{M_i}$, $M_i = (n_i - 1)!$, $i \in J_m$.

As follows from [4, 12], since the sets z^i, $i \in J_m$, and hence the cycles defined by them are disjoint, the order of cycles representing a permutation as a product of cycles does not matter. Therefore, in the problem under study we do not distinguish permutations represented as a product of identical cycles, for example, permutations (8) and (9). Moreover, any cyclic shifts of elements in cycle specifications do not change the result of the generation of permutation π. Therefore, dealing with the generation of all possible different permutations in the problem, we mean only those represented by products of cycle elements that differ in the structure and (or) in the order. Each such cycle $\sigma^{n_i}(p^j)$ is uniquely determined by the cyclic permutation $p^j \in P_{n_i}^c(z^i)$, $j \in J_{M_i}$, $M_i = (n_i - 1)!$ [4, 12]. The number of cycles generated by the set z^i is equal to the number of different cyclic permutations that can be constructed from elements of the set z^i. And the number of different permutations representable as the product of cycles from elements of the given sets is equal to the number of elements of the tuple of cyclic permutations TP_N^c introduced above. This number of elements can be found by using the generating series (4). Thus, Problem 2 can be replaced with the equivalent problem of generating all elements of a tuple of cyclic permutations.

Since a tuple of cyclic permutations is a composition k-image of combinatorial sets, its elements can be described and generated using mappings based on base combinatorial sets of tuples and cyclic permutations [11]. In solving problems of the generation of permutations containing cycles, we will be oriented to the algorithmic implementation of the mappings Γ_{T_m}, Γ_W, and $\Gamma_{P_{n_i}^c}$, which describe elements of the set TP_N^c in (3).

In implementing the mapping Γ_W, each element of the zero-order set (the tuple T_m) is replaced with an element of the sets of cyclic permutations $P_{n_1}^c(z^1)$, $P_{n_2}^c(z^2)$,..., $P_{n_m}^c(z^m)$, respectively. As a result, one element of the set TP_N^c is formed: permutation π possessing the required properties.

To obtain all elements of the set TP_N^c in the described way, we will implement the mapping Γ_{T_m} algorithmically. The mapping $\Gamma_{P_{n_i}^c}$ will be obtained based on the method of the generation of elements of a set of cyclic permutations. One of the algorithms to generate cyclic permutations is described in [1]. Applying this algorithm allows generating all elements of the base combinatorial sets of cyclic permutations. Generating one element for each set $P_{n_1}^c(z^1)$, $P_{n_2}^c(z^2)$, …, $P_{n_m}^c(z^m)$ yields m cycles. Using Algorithm 2, which implements mapping Γ_W, they can be rearranged to the permutation π representeble as the product of these cycles.

Thus, the solution of Problem 2 reduces to the repeated solution of Problem 1 based on the generation of elements of sets of cyclic permutations according to the chosen algorithm. Let us consider an example.

Example 3. Given $m = 2$ disjoint sets $z^1 = \{3, 5, 7\}$ and $z^2 = \{1, 2, 4, 6\}$ generating the sets of cyclic permutations $P_{n_1}^c(z^1)$ and $P_{n_2}^c(z^2)$, generate all possible different permutations $\pi \in P_7$ representable as the product of cycles $\sigma^{n_i}(p^j)$, $j \in J_{M_i}$, $M_i = (n_i - 1)!$, $i \in J_2$.

As indicated above, the number of cyclic permutations (hence cycles as well) composed of n elements is $(n-1)!$. Two different cycles σ^{n_1} can be constructed from the set z^1: $3 \mapsto 5 \mapsto 7 \mapsto 3$, $3 \mapsto 7 \mapsto 5 \mapsto 3$ and six cycles σ^{n_2} from the set z^2: $1 \mapsto 2 \mapsto 4 \mapsto 6 \mapsto 1$, $1 \mapsto 4 \mapsto 2 \mapsto 6 \mapsto 1$, $1 \mapsto 2 \mapsto 6 \mapsto 4 \mapsto 1$, $1 \mapsto 4 \mapsto 6 \mapsto 2 \mapsto 1$, $1 \mapsto 6 \mapsto 2 \mapsto 4 \mapsto 1$, and $1 \mapsto 6 \mapsto 4 \mapsto 2 \mapsto 1$. It is possible to construct 12 different permutations representable as the product of cycles σ^{n_1} and σ^{n_2}. Combining different pairs of cycles σ^{n_1} and σ^{n_2} yields 12 variants of the initial data for Problem 1. Following Algorithms 1 and 2 in each case, we obtain 12 permutations of the form (6):

$$\begin{pmatrix} 3&5&7&1&2&4&6 \\ 5&7&3&2&4&6&1 \end{pmatrix}, \begin{pmatrix} 3&5&7&1&2&4&6 \\ 5&7&3&4&6&2&1 \end{pmatrix}, \begin{pmatrix} 3&5&7&1&2&4&6 \\ 5&7&3&4&6&1&4 \end{pmatrix}, \begin{pmatrix} 3&5&7&1&2&4&6 \\ 5&7&3&4&1&6&2 \end{pmatrix},$$

$$\begin{pmatrix} 3&5&7&1&2&4&6 \\ 5&7&3&6&4&1&2 \end{pmatrix}, \begin{pmatrix} 3&5&7&1&2&4&6 \\ 5&7&3&6&1&2&4 \end{pmatrix}, \begin{pmatrix} 3&5&7&1&2&4&6 \\ 7&3&5&2&4&6&1 \end{pmatrix}, \begin{pmatrix} 3&5&7&1&2&4&6 \\ 7&3&5&4&6&2&1 \end{pmatrix},$$

$$\begin{pmatrix} 3&5&7&1&2&4&6 \\ 7&3&5&2&6&1&4 \end{pmatrix}, \begin{pmatrix} 3&5&7&1&2&4&6 \\ 7&3&5&4&1&6&2 \end{pmatrix}, \begin{pmatrix} 3&5&7&1&2&4&6 \\ 7&3&5&6&4&1&2 \end{pmatrix}, \begin{pmatrix} 3&5&7&1&2&4&6 \\ 7&3&5&6&1&2&4 \end{pmatrix}.$$

GENERATING ALL PERMUTATIONS GENERATED BY SET PARTITIONS

Problem 3. Given is a set $Z = \{z_1, z_2, \dots, z_N\}$ consisting of different elements. The set Z is assumed to be partitioned into disjoint subsets $z^{ik} = \{z_1^{ik}, z_2^{ik}, \dots, z_{n_i}^{ik}\}$, $i \in J_{m_k}$, $n_1 + n_2 + \ldots + n_{m_k} = N$, $k \in J_K$, where K is the number of variants of partitioning the set Z into disjoint subsets. Each subset z^{ik} generates a set of cyclic permutations $P_{n_i}^c(z^{ik})$ and a set of corresponding cycles $\sigma^{n_i}(p^j)$, $j \in J_{M_i}$, $M_i = (n_i - 1)!$. It is required to generate all possible different permutations of elements of the set Z, $\pi \in P_N$, representable as the product of cycles $\sigma^{n_i}(p^j)$ generated by all possible cyclic permutations: $p^j \in P_{n_i}^c(z^{ik})$, $j \in J_{M_i}$, $M_i = (n_i - 1)!$, $k \in J_K$.

The mechanism of the generation of permutations in this problem can be based on the solution of Problem 2 for each variant of the partition of the set Z into disjoint subsets. Each variant of the partition $z^{ik} = \{z_1^{ik}, z_2^{ik}, \dots, z_{n_i}^{ik}\}$, $i \in J_{m_k}$, $n_1 + n_2 + \ldots + n_{m_k} = N$, $k \in J_K$, generates sets of cyclic permutations $P_{n_i}^c(z^{ik})$, $j \in J_{M_i}$, $M_i = (n_i - 1)!$ and the corresponding cycles $\sigma^{n_i}(p^j)$, $p^j \in P_{n_i}^c(z^{ik})$, which are initial data for Problem 2. Thus, to solve Problem 3, it is necessary to determine the way of generating all possible partitions of the set Z into disjoint subsets.

The problem of partitioning a set into subsets is classical in combinatorics [2,3,5,12]. Different variants of partitioning a set into disjoint nonempty subsets (blocks) are associated with quantitative estimates. The number of partitions of an n-set into k blocks is estimated by the Stirling number of the second kind $S(n,k)$ and the total number of partitions of an n-set by the Bell number $B(n)$, where $B(n)=\sum_{k=1}^{n} S(n,k)$. The algorithms of generating partitions of a set into blocks are described in [2, 5] and are focused on the generation of both all possible partitions (Hutchinson algorithm) and partitions of an n-set exactly into k blocks. Using these algorithms allows forming set-generating cycles for the problem of the generation of permutations representable as a product of cycles.

It is possible to consider a variant of the problem of representing a permutation as a product of exactly k cycles and a product of different number of cycles (according to the number of subsets into which the set Z is partitioned). The number of permutations having exactly k cycles is estimated by the Stirling number of the first kind $s(n,k)$ [12]. It is also possible to solve the problem of generating all permutations of a prescribed type among elements of the set Z. To this end, when generating partitions of the set Z into subsets, it is necessary to choose only those variants of partitions that correspond to the specified type of permutation in the number of subsets of different cardinality. In all these cases, Problems 1 and 2 can be considered as base problems of the generation of permutations containing cycles. All more general problems of the generation of permutations of the specified class reduce to them in one way or another.

ESTIMATING THE METHOD OF THE GENERATION OF PERMUTATIONS CONTAINING CYCLES

Note that the set P_N generated by the union of the sets of generating elements of cyclic permutations $\widetilde{Z}=\bigcup_{i=1}^{m} z^i$ contains all elements of the tuple of cyclic permutations TP_N^c. Therefore, all permutations that satisfy the conditions of Problems 1 and 2 can be carried out by generating the permutations of all elements of the set $\widetilde{Z}$ and checking each of them for admissibility by decomposing into a product of cycles. The number of all such permutations obtained by one of the well-known methods [1] is Card $P_N = N! = (n_1 + n_2 + \ldots + n_m)!$, where $n_1, n_2, \ldots, n_m$ are the lengths of cycles in the decomposition of the permutation into a product of cycles. Let us compare the number of permutations that satisfy the conditions of Problems 1 and 2, i.e., the number of elements of the tuple of cyclic permutations TP_N^c generated by the proposed method, with the number of all possible permutations among N elements. According to (4), the number of all the permutations representable as a product of m cycles of length $n_1, n_2, \ldots, n_m$ is Card $TP_N^c = (n_1 - 1)! \cdot (n_2 - 1)! \cdot \ldots \cdot (n_m - 1)!$, respectively. Let us consider the ratio of cardinalities of these sets:

$$\frac{\text{Card } P_N}{\text{Card } TP_N^c} = \alpha = \frac{N!}{(n_1-1)!\cdot(n_2-1)!\cdot\ldots\cdot(n_m-1)!} = \frac{N!}{n_1!\cdot n_2!\cdot\ldots\cdot n_m!}\cdot n_1\cdot n_2\cdot\ldots\cdot n_m. \tag{10}$$

The first factor on the right-hand side of (10) is the number of permutations with repetitions among m different elements, where the first element repeats n_1 times, the second n_2 times, …, the mth element n_m times, i.e.,

$$\alpha = n_1 \cdot n_2 \cdot \ldots \cdot n_m \cdot \text{Card } P(N, n_1, n_2, \ldots, n_m). \tag{11}$$

Thus, the following statement is true.

Statement. The number of all permutations among N different elements representable as the product of m cycles of lengths $n_1, n_2, \ldots, n_m$, respectively, is α times less than the number of all permutations among N different elements, where α is defined by (11).

CONCLUSIONS

The approach proposed in the paper to the generation of permutations containing cycles is sufficiently universal. It can be used to generate permutations containing cycles in view of a large set of requirements to the number of cycles, their length, etc.

Note that the approaches to the generation of permutations containing cycles considered in the paper have high computational complexity. However, applying the proposed methods is justified if it is required to generate complex combinatorial objects in various problems. The application of these methods in such a situation substantially decreases the redundancy typical of universal methods of generation.

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