CYCLIC-WAITING SYSTEMS[1]

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Abstract. *The paper analyzes a queuing system where customers are accepted for service either at the time of arrival (if the server if idle) or at the times that differ from it by intervals multiple of cycle time T. Formulas are derived to find the number of customers in the system, waiting time, and the existence condition for ergodic distribution.*

Keywords: *retrial queue, cyclic waiting.*

Introduction. Service discipline plays a key role in the analysis of queuing systems; its nature preconditions the possibility of using one mathematical tool or another. The characteristics of the system depend on the rules governing the choice of a customer to be served. These rules can be rather simple (first arrived first served or, vice versa, last arrived first served). These rules can also be more complex if constraints are imposed on waiting time and on the time individual customers stay in the system, if customers are grouped according to service priority, etc. The analysis of a system with rather simple probabilistic characteristics may be much complicated by specific features of service discipline.

It is proposed to consider a single-server queuing system, where a newly arrived customer is accepted for service either at the time of its arrival at the system (if there is an idle server and no other customers) or at the time that differs from the arrival time by a time multiple of some specified cycle interval (if there is a queue).

For more illustrative presentation, let us give two examples of such systems.

1. *An aircraft approaches to land following a designated route.* If the runway is free, the aircraft lands immediately. If separation norms may not be observed or there are already several aircraft in the waiting zone (going round by aviation terminology) that are waiting for landing clearance, then the newly arrived aircraft is queued (sent to go round). Going round, the aircraft waits for landing clearance. An aircraft obtained the landing clearance should reach the appropriate geometrical point of the go-round route whence it can start landing. The period of time from obtaining the landing clearance till reaching the necessary geometrical position is called idling: the system is ready to accept the aircraft but the landing is possible only after achieving the position when going round.

2. *Transmission of optical signals*. Information in the form of an optical signal arrives at some node, whence it should be transmitted on the FCFS basis. A specific feature of this information is that it cannot be stored in memory as it is intended for a different type of information, for example, in computers. The signal is forwarded to a delay line whence it arrives at the node again in a fixed time. If it is the signal's turn, it is transmitted; if not, it is again forwarded to the delay line. Clearly, a signal can be transmitted from the node at the time of its arrival or at the time that differs from it by a time multiple of the time necessary to pass through the delay line.

This model was first analyzed in [1] and then generalized in [2–5]. Koba [6] has found a sufficient existence condition for ergodic distribution in a system $GI/G/1$. Koba and Mikhalevich [7, 8] compared the *FCFS* and classical disciplines for retrial queuing systems. Such systems were also studied in discrete time. Some models were analyzed by Kárász [9, 10] for customers of two types. The original model have become of interest due to the description of aircraft landing [11, 12], it appeared useful in [13–15] to solve optical signal transmission problems; researchers at the Ghent University work in this field.

[1]The study was supported by the Hungarian Scientific Research Fund (OTKA Grant K60698/2005) and the Hungarian-Ukrainian intergovernmental cooperation in science and technology (Grant UA-28/2008).

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1060-0396/10/4603-0477

Queuing systems can be characterized from the standpoints of both system and individual customers, these characteristics partially overlap. From the system's standpoint, the initial information is the number of customers in it, and from the standpoint of individual customers, waiting time is important. The present paper shows how to evaluate and interrelate these characteristics.

Queue Length. Let us consider a queuing system with Poisson flow of incoming customers. If the system is free, servicing starts immediately. Otherwise, a customer is queued and served on FCFS basis and the service can start at a time of that differs from the arrival time by a time multiple of some time cycle *T*. In our case, the system is characterized by N_{t_n} customers at the instants of time before the beginning of service. Let us show that these quantities form a Markov chain.

Let t_n be the time of the beginning of service of the *n*th customer. The number of customers in the system at the time t_{n+1} is

$$N_{t_{n+1}} = \max\{0, N_{t_n} + \Delta_n - 1\},$$

where Δ_n is the number of customers arriving at the system in the time $[t_n, t_n + i)$. Let us show that they are independent random variables.

First, let us consider the intervals of recording the arriving customers. Let ξ_i and η_i $(i = 1, 2, \dots)$ be two sequences of independent random variables, independent of one another. ξ_i is the time interval between arrivals of the *i*th and $(i+1)$th customers, in the system under study it is exponentially distributed with the parameter λ; and η_i is service time for the *i*th customer (in our case, it is exponentially distributed with the parameter μ).

Assume that there is only one customer in the system at the time of the beginning of service (this is possible if one or none of the customers was in the system at the previous instant of time). If $\xi_i > \eta_i$, then the time before servicing the next customer is ξ_i (the service of the current customer will be completed, and the next customer will appear in the system after the idle period). If $\xi_i < \eta_i$, then the next customer will appear during the service, from this instant of time we reckon time intervals of length *T* until the completion of the service of the first customer is left behind (from the standpoint of arriving customers, we are interested in the interval from the time of arrival of the second customer to the beginning of its service), the length of the interval is some function of ξ_i and η_i, i.e., $f(\xi_i, \eta_i)$.

If at the time of the beginning of service of some customer the next customer is already in the system, the time interval prior to the beginning of service of the second customer is determined as follows. The period of servicing the first customer is divided into intervals of length *T* (the last interval is most likely incomplete). Since the times of the beginning of service of both customers differ from the times of their arrival by multiples of *T*, each interval of length *T* will have one point where the service of the second customer can theoretically be started. Actually, it begins at the first possible time upon the completion of service of the first customer; thus, the necessary time is determined by the ratio between the service time of the first customer and the interarrival time of the next two customers, i.e., is a function of random variables ξ_i and η_i, $f(\xi_i, \eta_i)$.

It is seen that the time intervals on which arriving customers are observed are functions of only random variables ξ_i and η_i, thus, they are independent of one another. Since the inflow is Poisson, the number of customers arriving in these intervals Δ_i are also independent random variables; therefore, N_{t_i} form a Markov chain.

Let us evaluate the transition probabilities. First, let us consider the case where there is only one customer in the system when service starts. Let its service time be *u*, and a new customer arrive at the system in time υ. The probability of the event $\{u - \upsilon < t\}$ (i.e., the time left from the service of the first customer) is

$$P(t) = \{u - \upsilon < t\}$$

$$= \int_0^t \int_0^u \lambda e^{-\lambda \upsilon} \mu e^{-\mu u} d\upsilon\, du + \int_0^\infty \int_{u-t}^u \lambda e^{-\lambda \upsilon} \mu e^{-\mu u} d\upsilon\, du = \frac{\lambda}{\lambda + \mu}(1 - e^{-\mu t}). \quad (1)$$

The time from the arrival of the second customer before the beginning of its service is

$$[I(u - \upsilon) + 1]T,$$

where $I(x)$ is the integer part of x/T. This formula is true almost everywhere, except for points multiple of the time cycle *T*. To determine the transition probabilities, we need the number of customers arrived during this period. By (1), the time from customer's arrival to the beginning of its service is iT with probability

$$\frac{\lambda}{\lambda+\mu}(e^{-\mu(i-1)T}-e^{-\mu T}),$$

the generating function of the number of customers

$$\frac{\lambda}{\lambda+\mu}\sum_{k=0}^{\infty}\sum_{i=1}^{\infty}\{e^{-\mu(i-1)T}-e^{-\mu iT}\}\frac{(\lambda iTz)^k}{k!}e^{-\lambda iT}$$

$$=\frac{\lambda}{\lambda+\mu}\sum_{i=1}^{\infty}\{e^{-\mu(i-1)T}-e^{-\mu iT}\}e^{-\lambda iT(1-z)}=\frac{\lambda}{\lambda+\mu}\frac{e^{-\lambda(1-z)T}(1-e^{-\mu T})}{1-e^{-[\lambda(1-z)+\mu]T}}.$$

This formula is true, if at least one customer appears in the system, the unknown generating function is equal to

$$A(z)=\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}z\frac{(1-e^{-\mu T})e^{-\lambda T(1-z)}}{1-e^{-[\lambda(1-z)+\mu]T}}, \tag{2}$$

where $\frac{\mu}{\lambda+\mu}=\int_0^{\infty}e^{-\lambda x}\mu e^{-\mu x}dx$ is nonarrival probability, and the factor z in the second term on the right-hand side of (2) is due to the obligatory one customer.

Let us find the transition probabilities for other states. In this case, at the beginning of service of a customer, the next customer is already in the system. Let $x=u-I(u)T$, and y be the time between arrivals of two customers mod T. Obviously, y has truncated exponential distribution with distribution function

$$\frac{1-e^{-\lambda y}}{1-e^{-\lambda T}},\ 0\le y\le T.$$

The time between the beginning of service of two neighboring customers is

$$\begin{cases} I(u)+y & \text{if } x\le y,\\ (I(u)+1)T+y & \text{if } x>y.\end{cases}$$

The probability that k customers arrive during these periods is

$$\frac{(\lambda\{I(u)T+y\})^k}{k!}\exp(-\lambda\{I(u)T+y\}), \tag{3}$$

$$\frac{(\lambda\{[I(u)+1]T+y\})^k}{k!}\exp(-\lambda\{[I(u)+1]T+y\}). \tag{4}$$

Let us fix y and divide the service time into intervals of length T consisting of two parts, y and $T-y$; probabilities (3) and (4) correspond to them. The generating function of the number of arrived customers has the following form (provided that the time between the arrivals moduluo T is equal to y):

$$M(z^{\xi}\mid y)=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\left\{\int_{iT}^{iT+y}\frac{[\lambda(iT+y)z]^k}{k!}e^{-\lambda(iT+y)}\mu e^{-\mu u}du\right.$$

$$\left.+\int_{iT+y}^{(i+1)T}\frac{[\lambda((i+1)T+y)z]^k}{k!}e^{-\lambda((i+1)T+y)}\mu e^{-\mu u}du\right\}$$

$$=\frac{1}{1-e^{-[\lambda(1-z)+\mu]T}}\{e^{-\lambda(1-z)y}-e^{-[\lambda(1-z)+\mu]y}$$

$$+e^{-\lambda(1-z)T}e^{-[\lambda(1-z)+\mu]y}-e^{-\lambda(1-z)y}e^{-[\lambda(1-z)+\mu]T}\},$$

where ζ is a random variable that means the number of customers arrived during the time specified. Multiplying this expression by $\dfrac{\lambda e^{-\lambda y}}{1-e^{-\lambda T}}$ and integrating from 0 to T, we obtain the generating function of transition probabilities

$$B(z)=\sum_{i=0}^{\infty} b_i z^i = \frac{1}{(1-e^{-\lambda T})(1-e^{-[\lambda(1-z)+\mu]T})}\left\{\frac{1}{2-z}(1-e^{-\lambda(2-z)T})(1-e^{-[\lambda(1-z)+\mu]T})\right.$$

$$\left. -\frac{\lambda}{\lambda(2-z)+\mu}(1-e^{-[\lambda(2-z)+\mu]T})(1-e^{-\lambda(1-z)T})\right\}. \tag{5}$$

Hence, the theorem below follows.

THEOREM 1. Consider a queuing system with Poisson inflow of customers with the parameter λ; the service time is exponentially distributed with the parameter μ. Service of a customer can start at the time of its arrival (if the system is free) or at a time that differs from it by a time interval multiple of some time T (if the system is busy or there is a queue). Customers are serviced on the FCFS basis. If the server is idle, there is no customer arrived earlier in the system, the current customer is at the corresponding position, the service will necessarily begin. Let us introduce a Markov chain whose states correspond to the number of customers in the system at the instants of time $t_k - 0$ (t_k is the time of the beginning of service of the kth customer). Its matrix of transition probabilities has the form

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & b_0 & b_1 & b_2 & \cdots \\ 0 & 0 & b_0 & b_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix}, \tag{6}$$

and its elements are defined by the generating functions (2) and (5). Let us write the generating function of ergodic distribution of this chain:

$$P(z)=p_0\frac{B(z)(\lambda z+\mu)-zA(z)(\lambda+\mu)}{\mu[B(z)-z]},$$

where

$$p_0 = 1-\frac{\lambda}{\lambda+\mu}\frac{1-e^{-(\lambda+\mu)T}e^{-\lambda T}}{(1-e^{-\mu T})}. \tag{7}$$

Then the existence condition for the ergodic distribution has the form

$$\frac{\lambda}{\mu}<\frac{e^{-\lambda T}(1-e^{-\mu T})}{1-e^{-\lambda T}}. \tag{8}$$

Proof. The system operation can be described by an embedded Markov chain with the matrix of transition probabilities (6). Denote the ergodic probabilities by p_i ($i=0,1,\dots$) and introduce the generating function $P(z)=\sum_{i=0}^{\infty} p_i z^i$. Then

$$p_j = p_0 a_j + p_1 a_j + \sum_{i=2}^{j+1} p_i b_{j-i+1},\ j=1,2,\dots, \tag{9}$$

$$p_0 = p_0 a_0 + p_1 a_0. \tag{10}$$

From (9) and (10),

$$P(z)=\frac{p_0[zA(z)-B(z)]+p_1 z[A(z)-B(z)]}{z-B(z)}.$$

This expression contains the unknown probabilities p_0 and p_1; using (10), we can write p_1 in terms of p_0; p_0 can be found from the condition $P(1)=1$, i.e.,

$$p_0 = \frac{1-B'(1)}{1+A'(1)-B'(1)+\lambda\mu\,[A'(1)-B'(1)]}.$$

Since the chain is irreducible, $p_0 > 0$. Using

$$A'(1)=\frac{\lambda}{\lambda+\mu}\left\{1+\frac{\lambda T}{1-e^{-\mu T}}\right\},\ B'(1)=1-\frac{\lambda T e^{-\lambda T}}{1-e^{-\lambda T}}+\frac{\lambda}{\lambda+\mu}\lambda T\frac{1-e^{-(\lambda+\mu)T}}{(1-e^{-\lambda T})(1-e^{-\mu T})}$$

yields

$$\left(1+\frac{\lambda}{\mu}\right)A'(1)-\frac{\lambda}{\mu}B'(1)=\frac{\lambda}{\lambda+\mu}\lambda T\frac{1-e^{-(\lambda+\mu)T}}{(1-e^{-\lambda T})(1-e^{-\mu T})}>0$$

since the condition $1-B'(1)>0$ should be satisfied. This leads to the inequality

$$\lambda T e^{-\lambda T}\, 1-e^{-\lambda T}\, -\lambda\lambda+\mu\lambda T 1-e^{-(\lambda+\mu)T}\,(1-e^{-\lambda T})(1-e^{-\mu T})>0,$$

whence

$$\frac{\lambda}{\lambda+\mu}<\frac{e^{-\lambda T}(1-e^{-\mu T})}{1-e^{-(\lambda+\mu)T}},$$

which is equivalent to (8).

The theorem is proved.

By busy period are meant intervals necessary to achieve the position to begin the service; as T decreases, their influence decreases too, in the limiting case service becomes continuous.

COROLLARY. The limit distribution of the system described in the theorem as $T\to 0$

$$P^*(z)=\frac{1-\rho}{1-\rho z}\ \left(\rho=\frac{\lambda}{\mu}\right),$$

i.e., is geometrical distribution with the parameter ρ.

Proof. Let us find p_0, $A(z)$, and $B(z)$ as $T\to 0$, denote the limiting values by p_0^*, $A^*(z)$, and $B^*(z)$. Considering (7), (2), and (5), we obtain

$$p_0^* = \lim_{T\to 0} p_0 = \lim_{T\to 0}\left(1-\frac{\lambda}{\lambda+\mu}\frac{1-e^{-(\lambda+\mu)T}}{e^{-\lambda T}(1-e^{-\mu T})}\right)=1-\frac{\lambda}{\mu}=1-\rho,$$

$$\lim_{T\to 0} A(z)=\lim_{T\to 0} B(z)=\frac{\mu}{\lambda(1-z)+\mu},$$

i.e.,

$$P^*(z)=(1-\rho)\frac{(\lambda z+\mu)\dfrac{\mu}{\lambda(1-z)+\mu}-z(\lambda+\mu)\dfrac{\mu}{\lambda(1-z)+\mu}}{\mu\left[\dfrac{\mu}{\lambda(1-z)+\mu}\right]}=\frac{1-\rho}{1-\rho z}.$$

This formula is the generating function of the ergodic distribution for an $M/M/1$ system, which coincides with classical results.

Waiting Time. Let us consider a queuing system described in the previous section and use the results obtained by Koba [6] to find the waiting time distribution. Let t_n be the time of arrival of the nth customer; then its service will begin at the time $t_n + T\cdot X_n$, where X_n is a non-negative integer. Let $\xi_n = t_{n+1}-t_n$, and η_n be the service time of the nth customer. The following relation holds for X_n and X_{n+1}: if $(k-1)T < iT+\eta_n-\xi_n \le kT$ $(k\ge 1)$, then $X_{n+1}=k$; if

$iT+\eta_n-\xi_n\le 0$, then $X_{n+1}=0$. Hence, X_n is a homogeneous Markov chain with the transition probabilities p_{ik}, where

$$p_{ik}=P\{(k-i-1)T<\eta_n-\xi_n\le(k-i)T\}$$

if $k\ge 1$;

$$p_{i0}=P\{\eta_n-\xi_n\le -iT\}.$$

Let us introduce the notation

$$f_j=P\{(j-1)T<\eta_n-\xi_n\le jT\}, \tag{11}$$

$$p_{ik}=f_{k-i},\ k\ge 1,\ p_{i0}=\sum_{j=-\infty}^{-i} f_j=\hat{f}_i. \tag{12}$$

With (12) taken into account, the ergodic distribution of the chain will satisfy the system of equations

$$p_j=\sum_{i=0}^{\infty} p_i p_{ij},\ j\ge 0,\ \sum_{j=0}^{\infty} p_j=1. \tag{13}$$

THEOREM 2. Let us consider the system described in Theorem 1 and introduce a Markov chain whose states correspond to the waiting time at the time of arrival of customers. The matrix of transition probabilities of this chain has the form

$$\begin{bmatrix} \sum_{j=-\infty}^{0} f_j & f_1 & f_2 & f_3 & f_4 & \cdots \\ \sum_{j=-\infty}^{-1} f_j & f_0 & f_1 & f_2 & f_3 & \cdots \\ \sum_{j=-\infty}^{-2} f_j & f_{-1} & f_0 & f_1 & f_2 & \cdots \\ \sum_{j=-\infty}^{-3} f_j & f_{-2} & f_{-1} & f_0 & f_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{14}$$

its elements are defined by (11) and (12). Then the generating function of the ergodic distribution has the form

$$P(z)=\left[1-\frac{\lambda}{\mu}\frac{1-e^{-\lambda T}}{e^{-\lambda T}(1-e^{-\mu T})}\right]\frac{\frac{\mu}{\lambda+\mu}-\frac{\mu(1-e^{-\lambda T})}{\lambda+\mu}\frac{z}{z-e^{-\lambda T}}}{1-\frac{\lambda(1-e^{-\mu T})}{\lambda+\mu}\frac{z}{1-ze^{-\mu T}}-\frac{\mu(1-e^{-\lambda T})}{\lambda+\mu}\frac{z}{z-e^{-\lambda T}}}, \tag{15}$$

and the existence condition for the ergodic distribution can be expressed by the inequality

$$\frac{\lambda}{\mu}<\frac{e^{-\lambda T}(1-e^{-\mu T})}{1-e^{-\lambda T}}. \tag{16}$$

Proof. We have

$$P\{\xi<x\}=1-e^{-\lambda x},\ P\{\eta<x\}=1-e^{-\mu x}.$$

For the distribution function $\eta-\xi$ we obtain

$$F(x)=\begin{cases} \frac{\mu}{\lambda+\mu}e^{\lambda x} & \text{if } x\le 0, \\ 1-\frac{\lambda}{\lambda+\mu}e^{-\mu x} & \text{if } x>0. \end{cases}$$

The transition probabilities of the Markov chain are as follows: for $j>0$

$$f_j = 1-\frac{\lambda}{\lambda+\mu}e^{-\mu(j-1)T} - 1 + \frac{\lambda}{\lambda+\mu}e^{-\mu jT} = \frac{\lambda}{\lambda+\mu}(1-e^{-\mu T})e^{-\mu(j-1)T},$$

for negative values $(j \geq 0)$

$$f_{-j} = \frac{\mu}{\lambda+\mu}e^{-\lambda jT} - \frac{\mu}{\lambda+\mu}e^{-\lambda(j+1)T} = \frac{\mu}{\lambda+\mu}(1-e^{-\lambda T})e^{-\lambda jT},$$

$$p_{i0} = \hat{f}_i = \sum_{j=-\infty}^{-i} f_j = \sum_{j=i}^{\infty}\frac{\mu}{\lambda+\mu}(1-e^{-\lambda T})e^{-\lambda jT} = \frac{\mu}{\lambda+\mu}e^{-\lambda iT}.$$

Using the matrices of transition probabilities (14), we obtain the system of equations

$$\begin{aligned}
p_0 &= p_0\hat{f}_0 + p_1\hat{f}_1 + p_2\hat{f}_2 + p_3\hat{f}_3 + \dots \\
p_1 &= p_0 f_1 + p_1 f_0 + p_2 f_{-1} + p_3 f_{-2} + \dots \\
p_2 &= p_0 f_2 + p_1 f_1 + p_2 f_0 + p_3 f_{-1} + \dots \\
&\vdots
\end{aligned}$$

Multiplying the jth equation by z^j and summing from zero to infinity, we obtain for the generating function $P(z) = \sum_{j=0}^{\infty} p_j z^j$

$$P(z) = P(z)F_+(z) + \sum_{j=1}^{\infty} p_j z^j \sum_{i=0}^{j-1} f_{-i} z^{-i} + \sum_{j=0}^{\infty} p_j \hat{f}_j. \tag{17}$$

In our case,

$$F_+(z) = \sum_{i=1}^{\infty} f_i z^i = \frac{\lambda z}{\lambda+\mu}(1-e^{-\mu T})\sum_{i=1}^{\infty} e^{-\mu(i-1)T} z^{i-1} = \frac{\lambda(1-e^{-\mu T})}{\lambda+\mu}\frac{z}{1-ze^{-\mu T}},$$

$$\sum_{i=0}^{j-1} f_{-i} z^{-i} = \frac{\mu(1-e^{-\lambda T})}{\lambda+\mu}\sum_{i=0}^{j-1} e^{-\lambda iT} z^{-i} = \frac{\mu(1-e^{-\lambda T})}{\lambda+\mu}\frac{1-(e^{-\lambda T}z)^j}{1-\frac{e^{-\lambda T}}{z}},$$

$$\sum_{i=0}^{\infty} p_i \hat{f}_i = \sum_{i=0}^{\infty} p_i \frac{\mu}{\lambda+\mu}e^{-\lambda iT} = \frac{\mu}{\lambda+\mu}P(e^{-\lambda T}).$$

Using these expressions yields

$$P(z) = P(z)F_+(z) + \sum_{j=1}^{\infty} p_j z^j \frac{\mu(1-e^{-\lambda T})}{\lambda+\mu}\frac{1-\left(\frac{e^{-\lambda T}}{z}\right)^j}{1-\frac{e^{-\lambda T}}{z}} + \frac{\mu}{\lambda+\mu}P(e^{-\lambda T})$$

$$= P(z)F_+(z) + \frac{\mu(1-e^{-\lambda T})}{\lambda+\mu}\frac{z}{z-e^{-\lambda T}}[P(z)-P(e^{-\lambda T})] + \frac{\mu}{\lambda+\mu}P(e^{-\lambda T})$$

or

$$P(z)\left[1-F_+(z)-\frac{\mu(1-e^{-\lambda T})}{\lambda+\mu}\frac{z}{z-e^{-\lambda T}}\right] = P(e^{-\lambda T})\left[\frac{\mu}{\lambda+\mu}-\frac{\mu(1-e^{-\lambda T})}{\lambda+\mu}\frac{z}{z-e^{-\lambda T}}\right].$$

To find $P(e^{-\lambda T})$ we will use the fact that $P(1)=1$, whence

$$P(e^{-\lambda T}) = 1-\frac{\lambda}{\mu}\frac{1-e^{-\lambda T}}{e^{-\lambda T}(1-e^{-\mu T})}.$$

For the generating function, we obtain expression (15), whence the probability of zero waiting time is

$$p_0 = \left[1 - \frac{\lambda}{\mu} \frac{1-e^{-\lambda T}}{e^{-\lambda T}(1-e^{-\mu T})}\right] \frac{\mu}{\lambda+\mu}.$$

For $p_0 > 0$ to hold, the inequality

$$\frac{\lambda}{\mu} \frac{1-e^{-\lambda T}}{e^{-\lambda T}(1-e^{-\mu T})} < 1$$

should be satisfied, which leads to the ergodicity condition (16).

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