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ADAPTIVE DIAGNOSTIC SYSTEM BASED ON FUZZY RELATIONS

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Causes (diagnoses) are retrieved and identified using observed effects (symptoms) based on fuzzy relations and Zadeh's compositional rule of inference. An approach to designing adaptive fuzzy diagnostic systems is proposed. It allows solving fuzzy logic equations and designing and adjusting fuzzy relations using expert and experimental information.

Keywords: *adaptive diagnostic system, fuzzy relations, solution of a system of fuzzy logic equations, adjustment of a fuzzy model.*

INTRODUCTION

Diagnostics, i.e., establishing the causes of the state of the object being observed, is an important stage of decision-making in various fields of human activity such as medicine, engineering, economy, military science, etc. A convenient tool to formalize cause-and-effect relationships in diagnostic problems are fuzzy relations and Zade's compositional rule of inference [1]. They reduce a diagnostic problem to fuzzy logic equations [2, 3] that relate membership functions of causes (diagnoses) and effects (symptoms).

The papers [4–6] propose a method to solve a system of fuzzy logic equations by reducing it to an optimization problem. Generally, this problem is classed among *NP*-hard ones [7–9]. To overcome the *NP*-hardness, the papers [4–6] use the ideology of genetic optimization [10], which quickly establishes the domain of global minimum of the residual between the left- and right-hand sides of the system of equations followed by a fine adjustment of the solution by search methods available. The genetic algorithm uses all the available experimental information for the optimization, i.e., operates off-line and becomes toilful and inefficient if new experimental data are obtained, i.e., in the on-line mode.

The present paper develops and supplements the studies [4–6] and proposes an adaptive approach to designing diagnostic systems. The essence of the approach is in constructing and training a special neuro-fuzzy network, isomorphic to the diagnostic equations, which allows on-line correction of decisions.

FUZZY DIAGNOSTIC MODEL

An object of the diagnostics r_{IJ} is considered as a black box with n inputs and m outputs (Fig. 1). The outputs are associated with effects (symptoms) E_J being observed. The inputs correspond to the causes of the effects (diagnoses) C_I. A diagnostic problem is to estimate the causes (inputs) from the observed effects (outputs). The inputs and outputs are considered as linguistic variables specified on the appropriate universal sets. To estimate these linguistic variables, fuzzy terms are used.

Let us introduce the following notation: $\{x_1, x_2, \ldots, x_n\}$ is the set of input parameters, $x_i \in [\underline{x}_i, \overline{x}_i]$, $i = \overline{1, n}$; $\{y_1, y_2, \ldots, y_m\}$ is the set of output parameters, $y_j \in [\underline{y}_j, \overline{y}_j]$, $j = \overline{1, m}$; $\{c_{i1}, c_{i2}, \ldots, c_{ik_i}\}$ is the set of linguistic terms to estimate the parameter x_i, $i = \overline{1, n}$; and $\{e_{j1}, e_{j2}, \ldots, e_{jq_j}\}$ is the set of linguistic terms to estimate the parameter y_j, $j = \overline{1, m}$.

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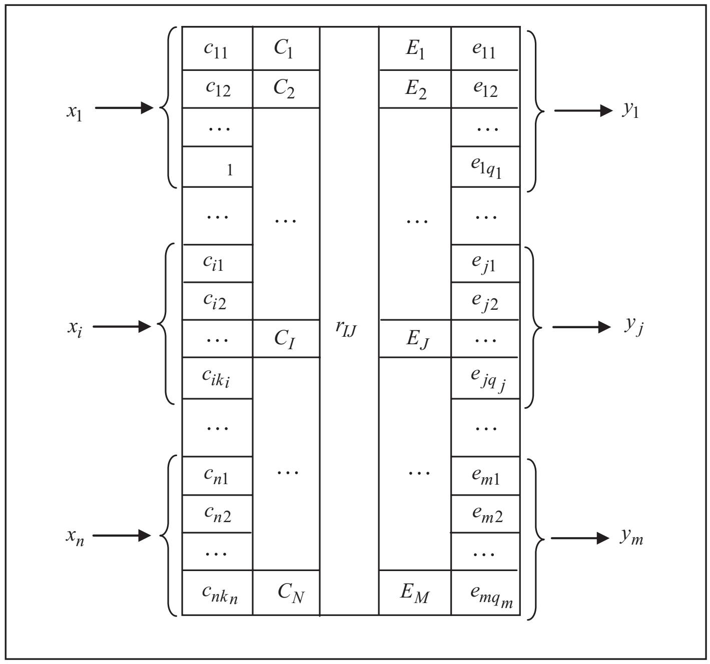


Fig. 1. Structure of the object of diagnostics.

Each term-estimate can be described by a fuzzy set

$$c_{il} = \{(x_i, \mu^{c_{il}}(x_i))\}, \ i = \overline{1,n}, \ l = \overline{1,k_i};$$

$$e_{jp} = \{(y_j, \mu^{e_{jp}}(y_j))\}, \ j = \overline{1,m}, \ p = \overline{1,q_j},$$

where $\mu^{c_{il}}(x_i)$ is the function of membership of the variable x_i in the term estimate c_{il}, $i = \overline{1,n}$, $l = \overline{1,k_i}$; $\mu^{e_{jp}}(y_j)$ is the function of membership of the variable y_j in the term estimate e_{jp}, $j = \overline{1,m}$, $p = \overline{1,q_j}$.

Let us rename the set of input and output estimate terms as follows: $\{C_1, C_2, \dots, C_N\} = \{c_{11}, c_{12}, \dots, c_{1k_1}, \dots, c_{n1}, c_{n2}, \dots, c_{nk_n}\}$ is the set of terms for the estimate of input parameters, where $N = k_1 + k_2 + \dots + k_n$; $\{E_1, E_2, \dots, E_M\} = \{e_{11}, e_{12}, \dots, e_{1q_1}, \dots, e_{m1}, e_{m2}, \dots, e_{mq_m}\}$ is the set of terms for the estimate of output parameters, where $M = q_1 + q_2 + \dots + q_m$.

We will call the set $\{C_I, I = \overline{1,N}\}$ fuzzy causes (diagnoses), and the set $\{E_J, J = \overline{1,M}\}$ fuzzy effects (symptoms).

A diagnostic problem is as follows: using observable output parameters $(y_1^*, y_2^*, \dots, y_m^*)$, estimate input parameters $(x_1^*, x_2^*, \dots, x_n^*)$.

To interrelate the causes and effects, we will use a matrix of fuzzy relations

$$R \subseteq C_I \times E_J = [r_{IJ}, I = \overline{1,N}, J = \overline{1,M}].$$

An element of this matrix represents a number $r_{IJ} \in [0,1]$ that characterizes the degree of influence of the cause C_I on the effect E_J.

Given the matrix *R*, the cause–effect dependence can be described with the use of Zadeh's compositional rule of inference [1]

$$\mu^E = \mu^C \circ R, \tag{1}$$

where $\mu^C = (\mu^{C_1}, \mu^{C_2}, \dots, \mu^{C_N})$ is a fuzzy vector of causes with elements $\mu^{C_I} \in [0,1]$ interpreted as significance measures of causes C_I; $\mu^E = (\mu^{E_1}, \mu^{E_2}, \dots, \mu^{E_M})$ is a fuzzy vector of effects with elements $\mu^{E_J} \in [0,1]$ interpreted as significance measures of effects E_J; and $\circ$ is the max-min operation of the composition [1].

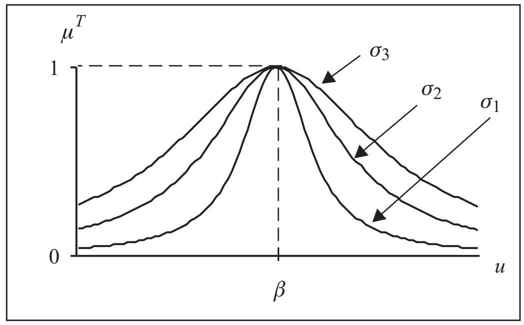


Fig. 2. Model of the membership function.

Finding the vector μ^C reduces to solving the system of fuzzy logic equations

$$\mu^{E_J} = (\mu^{C_1} \wedge r_{1J}) \vee (\mu^{C_2} \wedge r_{2J}) \ldots \vee (\mu^{C_N} \wedge r_{NJ}),\ J = \overline{1, M}, \tag{2}$$

which follows from (1). Since the operations $\vee$ and $\wedge$ are associated with max и min in the theory of fuzzy sets, system (2) can be rearranged as

$$\mu^{E_J} = \max_{I=\overline{1, N}} (\min (\mu^{C_I}, r_{IJ})),\ J = \overline{1, M}. \tag{3}$$

To pass from specific values of input and output variables to the significance measures of causes and effects, it is necessary to determine the membership functions of fuzzy terms C_I and E_J, $I = \overline{1, N}$, $J = \overline{1, M}$. We will use the membership function of a fuzzy term T (Fig. 2)

$$\mu^T(u) = \frac{1}{1 + \left(\dfrac{u - \beta}{\sigma}\right)^2}, \tag{4}$$

where β is the coordinate of the maximum of the function, $\mu^T(\beta) = 1$, and σ is a concentration-tension parameter. This function is introduced in [11] and employed in the identification of nonlinear dependences using IF-THEN fuzzy rules [11–17].

Relations (3) and (4) determine the general form of the fuzzy diagnostic model as follows:

$$\mu^E(Y, B_E, \Omega_E) = F_R(X, R, B_C, \Omega_C), \tag{5}$$

where $B_C = (\beta^{C_1}, \beta^{C_2}, \ldots, \beta^{C_N})$ and $\Omega_C = (\sigma^{C_1}, \sigma^{C_2}, \ldots, \sigma^{C_N})$ are the vectors of β- and σ-parameters of the membership functions of the fuzzy terms $C_1, C_2, \ldots, C_N$; $B_E = (\beta^{E_1}, \beta^{E_2}, \ldots, \beta^{E_M})$ and $\Omega_E = (\sigma^{E_1}, \sigma^{E_2}, \ldots, \sigma^{E_M})$ are the vectors of β- and σ-parameters of the membership functions of the fuzzy terms $E_1, E_2, \ldots, E_M$; and F_R is the operator of the input–outputs relation, corresponding to formulas (3) and (4).

SOLVING THE SYSTEM OF FUZZY LOGIC EQUATIONS

Optimization Problem. Following the approach proposed in [4–6], let us formulate the problem of solving the system of fuzzy logic equations (3). Find a fuzzy vector of causes $\mu^C = (\mu^{C_1}, \mu^{C_2}, \ldots, \mu^{C_N})$ that satisfies the constraints $\mu^{C_I} \in [0, 1]$, $I = \overline{1, N}$, and provides the least distance F between the observed and model significance measures of effects, i.e., between the left- and right-hand sides of each equation in system (3):

$$F = \sum_{J=1}^{M} [\mu^{E_J} - \max_{I=\overline{1, N}} (\min(\mu^{C_I}, r_{IJ}))]^2 = \min_{\mu^C}. \tag{6}$$

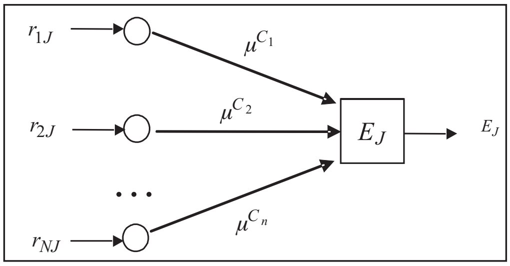
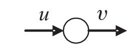
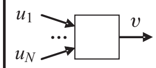
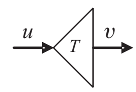
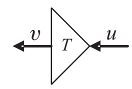


Fig. 3. Neuro-fuzzy model of diagnostic equations.

TABLE 1

Node	**Node name**	**Function**
	Fuzzy input	$v = u$
	Fuzzy output	$v = \max\limits_{i=\overline{1,\, N}} (u_i)$
	Fuzzy term of the cause	$v = \mu^T(u)$
	Fuzzy term of the effect	

According to [3], in the general case, system (3) has a set of solutions $S(R, \mu^E)$, which is determined by the unique maximum solution $\overline{\mu}^C$ and by the set of minimum solutions $S^*(R, \mu^E) = \{\underline{\mu}_l^C, l = \overline{1, T}\}$:

$$S(R, \mu^E) = \bigcup_{\underline{\mu}_l^C \in S^*} [\underline{\mu}_l^C, \overline{\mu}^C], \tag{7}$$

where $\overline{\mu}^C = (\overline{\mu}^{C_1}, \overline{\mu}^{C_2}, \ldots, \overline{\mu}^{C_N})$ and $\underline{\mu}_l^C = (\underline{\mu}_l^{C_1}, \underline{\mu}_l^{C_2}, \ldots, \underline{\mu}_l^{C_N})$ are the vectors of upper and lower bounds of the significance measures of causes C_I, and the operation of union is performed over all the vectors $\underline{\mu}_l^C \in S^*(R, \mu^E)$.

According to [4, 5], intervals (7) are formed by the multiple solution of optimization problem (6) beginning with searching for its zero solution $\mu_0^C = (\mu_0^{C_1}, \mu_0^{C_2}, \ldots, \mu_0^{C_N})$, where $\mu_0^{C_I} \le \overline{\mu}^{C_I}$, $I = \overline{1, N}$. The upper bound $\overline{\mu}^{C_I}$ falls within the interval $[\mu_0^{C_I}, 1]$, the lower bound $\underline{\mu}_l^{C_I}$ is within $[0, \mu_0^{C_I}]$ for $l = 1$ and is within $[0, \overline{\mu}^{C_I}]$ for $l > 1$, the minimum solutions $\underline{\mu}_k^C$, $k < l$, being excluded from the search domain.

Let $\mu^C(t) = (\mu^{C_1}(t), \mu^{C_2}(t), \ldots, \mu^{C_N}(t))$ be a solution of the optimization problem (6) at the tth step of the formation of intervals (7), i.e., $F(\mu^C(t)) = F(\mu_0^C)$ since the values of criterion (6) are identical for all $\mu^C \in S(R, \mu^E)$. In searching for the upper bounds $\overline{\mu}^{C_I}$ we assume that $\mu^{C_I}(t) \ge \mu^{C_I}(t-1)$, and in searching for the lower bounds $\underline{\mu}_l^{C_I}$ that $\mu_l^{C_I}(t) \le \mu_l^{C_I}(t-1)$.

The upper (lower) bounds are established by the following rule: if $\mu^C(t) \ne \mu^C(t-1)$, then $\overline{\mu}^{C_I}(\underline{\mu}_l^{C_I}) = \mu^{C_I}(t)$, $I = \overline{1, N}$. If $\mu^C(t) = \mu^C(t-1)$, then the formation of the interval solution $[\underline{\mu}_l^C, \overline{\mu}^C]$ stops.

Searching for intervals (7) proceeds while $\underline{\mu}_l^C \ne \underline{\mu}_k^C$, $k < l$.

Neuro-Fuzzy Approach to Solving Equations. Figure 3 represents a neuro-fuzzy network isomorphic to the system of fuzzy logic equations (3), and Table 1 shows elements of the neuro-fuzzy network. The network is designed so that the adjusted weights of arcs are the unknown significance measures of causes $C_1, C_2, \ldots, C_N$.

Network inputs are elements of the matrix of fuzzy relations. As follows from the system of fuzzy logic equations (3), the fuzzy relation r_{IJ} is the significance measure of the effect μ^{E_J} provided that the significance measure of the cause μ^{C_I} is equal to unity, and the significance measures of other causes are equal to zero, i.e., $r_{IJ} = \mu^{E_J}(\mu^{C_I} = 1, \mu^{C_K} = 0)$, $K = \overline{1, N}$, $K \ne I$. At the network outputs, actual significance measures of the effects $\max\limits_{I=\overline{1,\, N}} [\min(\mu^{C_I}, r_{IJ})]$ obtained with allowance for the actual weights of arcs μ^{C_I} are united.

Thus, the problem of solving the system of fuzzy logic equations (3) is reduced to the problem of learning of a neuro-fuzzy network (see Fig. 3) with the use of points

$$(r_{1J}, r_{2J}, \ldots, r_{NJ}, \mu^{E_J}), \quad J = \overline{1, M}.$$

The adjustment of parameters of the neuro-fuzzy network employs the recurrent relations

$$\mu^{C_I}(t+1)=\mu^{C_I}(t)-\eta\frac{\partial\varepsilon_t}{\partial\mu^{C_I}(t)} \tag{8}$$

that minimize the criterion

$$\varepsilon_t=\frac{1}{2}(\mu^E(t)-\hat{\mu}^E(t))^2 \tag{9}$$

applied in the neural network theory, where $\mu^E(t)$ and $\hat{\mu}^E(t)$ are a model and an experimental fuzzy vectors of effects at the tth step of learning; $\mu^{C_I}(t)$ are the significance measures of causes C_I at the tth step of learning; and η is a parameter of learning, which can be selected according to the results from [18].

The partial derivatives appearing in (8) characterize the sensitivity of the error ε_t to variations in parameters of the neuro-fuzzy network and can be calculated as follows:

$$\frac{\partial\varepsilon_t}{\partial\mu^{C_I}}=\sum_{J=1}^{M}\left[\frac{\partial\varepsilon_t}{\partial\mu^{E_J}}\cdot\frac{\partial\mu^{E_J}}{\partial\mu^{C_I}}\right].$$

Since determining the element "fuzzy output" from Table 1 involves fuzzy-logic operations min and max, relations (8) for learning are obtained with the use of finite differences.

ADJUSTING THE FUZZY MODEL

Assume that a known training sample is specified as M pairs of experimental data

$$\langle\hat{X}_p,\hat{Y}_p\rangle,\ \ p=\overline{1,M},$$

where $\hat{X}_p=(\hat{x}_1^p,\hat{x}_2^p,\ldots,\hat{x}_n^p)$ and $\hat{Y}_p=(\hat{y}_1^p,\hat{y}_2^p,\ldots,\hat{y}_m^p)$ are the vectors of values of input and output variables in the experiment with the number p.

The essence of the adjustment of the fuzzy diagnostic model (5) consists in selecting a matrix of fuzzy relations R and the vectors of parameters of the membership functions B_C, Ω_C, B_E, and Ω_E that minimize the distance between the model and experimental fuzzy vectors of effects

$$\sum_{p=1}^{M}[F_R(\hat{X}_p,R,B_C,\Omega_C)-\hat{\mu}^E(\hat{Y}_p,B_E,\Omega_E)]^2=\min_{R,B_C,\Omega_C,B_E,\Omega_E}.$$

The neuro-fuzzy model of the object of diagnostics is represented in Fig. 4, and the nodes are in Table 1. The neuro-fuzzy model is obtained by embedding the matrix of fuzzy relations into the neural network so that the weights of arcs subject to learning are fuzzy relations and the membership functions of fuzzy terms of causes and effects.

To adjust the parameters of the neuro-fuzzy network, the recurrent relations

$$\begin{gathered} r_{IJ}(t+1)=r_{IJ}(t)-\eta\frac{\partial\varepsilon_t}{\partial r_{IJ}(t)},\\ \beta^{c_{il}}(t+1)=\beta^{c_{il}}(t)-\eta\frac{\partial\varepsilon_t}{\partial\beta^{c_{il}}(t)},\ \ \sigma^{c_{il}}(t+1)=\sigma^{c_{il}}(t)-\eta\frac{\partial\varepsilon_t}{\partial\sigma^{c_{il}}(t)},\\ \beta^{e_{jp}}(t+1)=\beta^{e_{jp}}(t)-\eta\frac{\partial\varepsilon_t}{\partial\beta^{e_{jp}}(t)},\ \ \sigma^{e_{jp}}(t+1)=\sigma^{e_{jp}}(t)-\eta\frac{\partial\varepsilon_t}{\partial\sigma^{e_{jp}}(t)} \end{gathered} \tag{10}$$

minimizing criterion (9) are used, where $r_{IJ}(t)$ are fuzzy relations at the tth step of learning; $\beta^{c_{il}}(t)$, $\sigma^{c_{il}}(t)$, $\beta^{e_{jp}}(t)$, and $\sigma^{e_{jp}}(t)$ are the parameters of the membership functions of fuzzy terms of causes and effects at the tth step of learning.

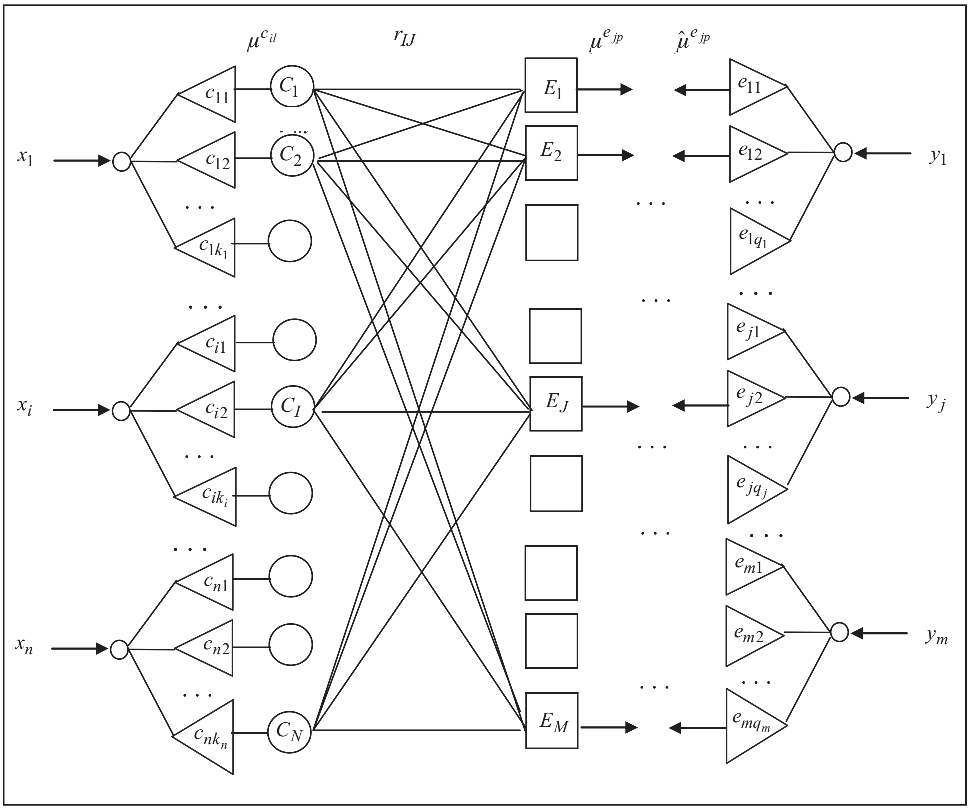


Fig. 4. Neuro-fuzzy model of the object of diagnostics.

The partial derivatives appearing in (10) characterize the sensitivity of the error ε_t to variations in parameters of the neuro-fuzzy network and can be calculated as follows:

$$\frac{\partial \varepsilon_t}{\partial r_{IJ}} = \frac{\partial \varepsilon_t}{\partial \mu^{E_J}(X)} \cdot \frac{\partial \mu^{E_J}(X)}{\partial r_{IJ}};$$

$$\frac{\partial \varepsilon_t}{\partial \beta^{c_{il}}} = \sum_{j=1}^{m} \sum_{p=1}^{q_j} \left[\frac{\partial \varepsilon_t}{\partial \mu^{e_{jp}}(x_i)} \cdot \frac{\partial \mu^{e_{jp}}(x_i)}{\partial \mu^{c_{il}}(x_i)} \cdot \frac{\partial \mu^{c_{il}}(x_i)}{\partial \beta^{c_{il}}} \right];$$

$$\frac{\partial \varepsilon_t}{\partial \sigma^{c_{il}}} = \sum_{j=1}^{m} \sum_{p=1}^{q_j} \left[\frac{\partial \varepsilon_t}{\partial \mu^{e_{jp}}(x_i)} \cdot \frac{\partial \mu^{e_{jp}}(x_i)}{\partial \mu^{c_{il}}(x_i)} \cdot \frac{\partial \mu^{c_{il}}(x_i)}{\partial \sigma^{c_{il}}} \right];$$

$$\frac{\partial \varepsilon_t}{\partial \beta^{e_{jp}}} = \frac{\partial \varepsilon_t}{\partial \mu^{e_{jp}}(y_j)} \cdot \frac{\partial \mu^{e_{jp}}(y_j)}{\partial \beta^{e_{jp}}}; \quad \frac{\partial \varepsilon_t}{\partial \sigma^{e_{jp}}} = \frac{\partial \varepsilon_t}{\partial \mu^{e_{jp}}(y_j)} \cdot \frac{\partial \mu^{e_{jp}}(y_j)}{\partial \sigma^{e_{jp}}}.$$

Since determining the element "fuzzy output" (see Table 1) involves the min and max fuzzy-logic operations, the relations for learning are obtained using finite differences.

COMPUTER EXPERIMENT

The purpose of the experiment is to check the diagnostic models and algorithms proposed above using a reference input–output model. The reference model is specified by an analytic function $y = f(x)$ approximated by the inference rule (1) and is simultaneously a generator of training and testing samples. The inputs x estimated for each output y are compared with reference values.

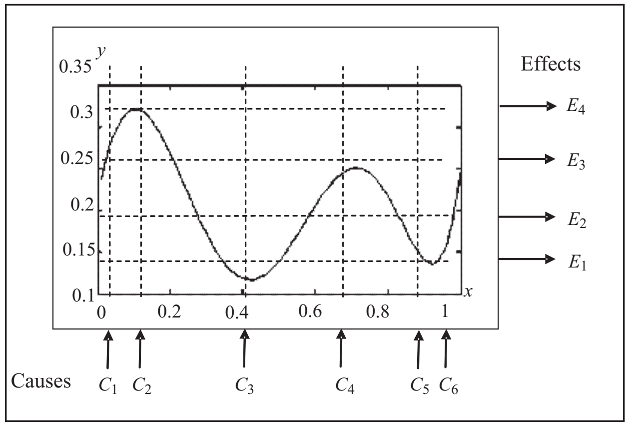


Fig. 5. Curve of the input–output generator model.

TABLE 2

Parameters of the membership functions	Values of the parameters of the membership functions of the fuzzy terms of causes before (after) the adjustment					
	C_1	C_2	C_3	C_4	C_5	C_6
β	0 (0)	0.09 (0.091)	0.4 (0.446)	0.7 (0.711)	0.9 (0.919)	1.0 (1.0)
σ	0.17 (0.114)	0.17 (0.121)	0.17 (0.115)	0.17 (0.118)	0.17 (0.112)	0.08 (0.041)

TABLE 3

Parameters of the membership functions	Values of the parameters of the membership functions of the fuzzy terms of effects before (after) the adjustment			
	E_1	E_2	E_3	E_4
β	0.15 (0.172)	0.2 (0.209)	0.25 (0.259)	0.3 (0.352)
σ	0.05 (0.037)	0.05 (0.040)	0.05 (0.041)	0.05 (0.040)

As the reference model, the formula

$$y = \frac{(1.8x+0.8)(5x-1.1)(4x-2.9)(3x-2.1)(9.5x-9.5)(3x-0.05)+20}{80}$$

is used.

Figure 5 shows the curve of the input–output generator model along with the fuzzy terms of causes C_1 = low, C_2 = below the average, C_3 = average, C_4 = above the average, C_5 = below the high, C_6 = high and with the fuzzy terms of effects E_1 = below the average, E_2 = average, E_3 = above the average, E_4 = high.

Let us represent the expert fuzzy relations R obtained by the method of pairwise comparisons [5, 6].

	E_1	E_2	E_3	E_4
C_1	0.33	1.00	0.67	0.21
C_2	0.12	0.10	0.33	1.00
C_3	1.00	0.23	0.11	0.11
C_4	0.33	0.33	1.00	0.21
C_5	0.91	0.77	0.22	0.34
C_6	0.33	0.90	0.67	0.21

The results of the adjustment of the fuzzy model are shown in Tables 2 and 3.

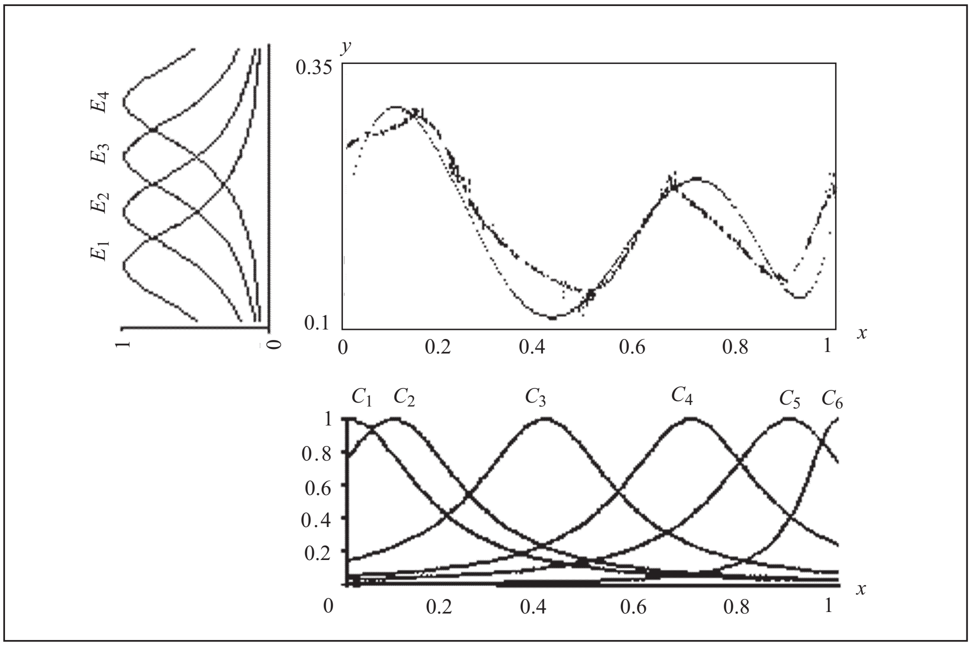


Fig. 6. Estimated reference model and the membership functions of the fuzzy terms of causes and effects before the adjustment.

After adjustment, fuzzy logic equations become

$$\mu^{E_1} = (\mu^{C_1} \wedge 0.27) \vee (\mu^{C_2} \wedge 0.13) \vee (\mu^{C_3} \wedge 0.97) \vee (\mu^{C_4} \wedge 0.20) \vee (\mu^{C_5} \wedge 0.86) \vee (\mu^{C_6} \wedge 0.21),$$

$$\mu^{E_2} = (\mu^{C_1} \wedge 0.93) \vee (\mu^{C_2} \wedge 0.09) \vee (\mu^{C_3} \wedge 0.28) \vee (\mu^{C_4} \wedge 0.44) \vee (\mu^{C_5} \wedge 0.75) \vee (\mu^{C_6} \wedge 0.82),$$

$$\mu^{E_3} = (\mu^{C_1} \wedge 0.63) \vee (\mu^{C_2} \wedge 0.41) \vee (\mu^{C_3} \wedge 0.15) \vee (\mu^{C_4} \wedge 0.95) \vee (\mu^{C_5} \wedge 0.26) \vee (\mu^{C_6} \wedge 0.67),$$

$$\mu^{E_4} = (\mu^{C_1} \wedge 0.12) \vee (\mu^{C_2} \wedge 0.88) \vee (\mu^{C_3} \wedge 0.07) \vee (\mu^{C_4} \wedge 0.08) \vee (\mu^{C_5} \wedge 0.32) \vee (\mu^{C_6} \wedge 0.12). \quad (11)$$

The estimated reference model and the membership functions of the fuzzy terms of causes and effects before and after adjustment are presented in Figs. 6 and 7, respectively.

Let a specific value of the output variable be $y^* = 0.23$. For this value, the membership functions in Fig. 7a can be used to determine the fuzzy vector of effects

$$\mu^E(y^*) = (\mu^{E_1} = 0.29;\ \mu^{E_2} = 0.78;\ \mu^{E_3} = 0.67;\ \mu^{E_4} = 0.10).$$

A genetic algorithm yields a zero solution

$$\mu_0^C = (\mu_0^{C_1} = 0.78,\ \mu_0^{C_2} = 0.10,\ \mu_0^{C_3} = 0.29,\ \mu_0^{C_4} = 0.67,\ \mu_0^{C_5} = 0.07,\ \mu_0^{C_6} = 0.45), \quad (12)$$

for which the value of the optimization criterion (6) is $F = 0.0004$.

The resultant zero solution allows organizing a genetic search for the set of solutions $S(R, \mu^E)$, which is defined by the maximum solution

$$\overline{\mu}^C = (\overline{\mu}^{C_1} = 0.78,\ \overline{\mu}^{C_2} = 0.12,\ \overline{\mu}^{C_3} = 0.29,\ \overline{\mu}^{C_4} = 0.67,\ \overline{\mu}^{C_5} = 0.12,\ \overline{\mu}^{C_6} = 0.78)$$

and by three minimum solutions $S^* = \{\underline{\mu}_1^C, \underline{\mu}_2^C, \underline{\mu}_3^C\}$:

$$\underline{\mu}_1^C = (\underline{\mu}_1^{C_1} = 0.78,\ \underline{\mu}_1^{C_2} = 0,\ \underline{\mu}_1^{C_3} = 0.29,\ \underline{\mu}_1^{C_4} = 0,\ \underline{\mu}_1^{C_5} = 0,\ \underline{\mu}_1^{C_6} = 0.67);$$

$$\underline{\mu}_2^C = (\underline{\mu}_2^{C_1} = 0.78,\ \underline{\mu}_2^{C_2} = 0,\ \underline{\mu}_2^{C_3} = 0.29,\ \underline{\mu}_2^{C_4} = 0.67,\ \underline{\mu}_2^{C_5} = 0,\ \underline{\mu}_2^{C_6} = 0);$$

$$\underline{\mu}_3^C = (\underline{\mu}_3^{C_1} = 0,\ \underline{\mu}_3^{C_2} = 0,\ \underline{\mu}_3^{C_3} = 0.29,\ \underline{\mu}_3^{C_4} = 0,\ \underline{\mu}_3^{C_5} = 0,\ \underline{\mu}_3^{C_6} = 0.78).$$

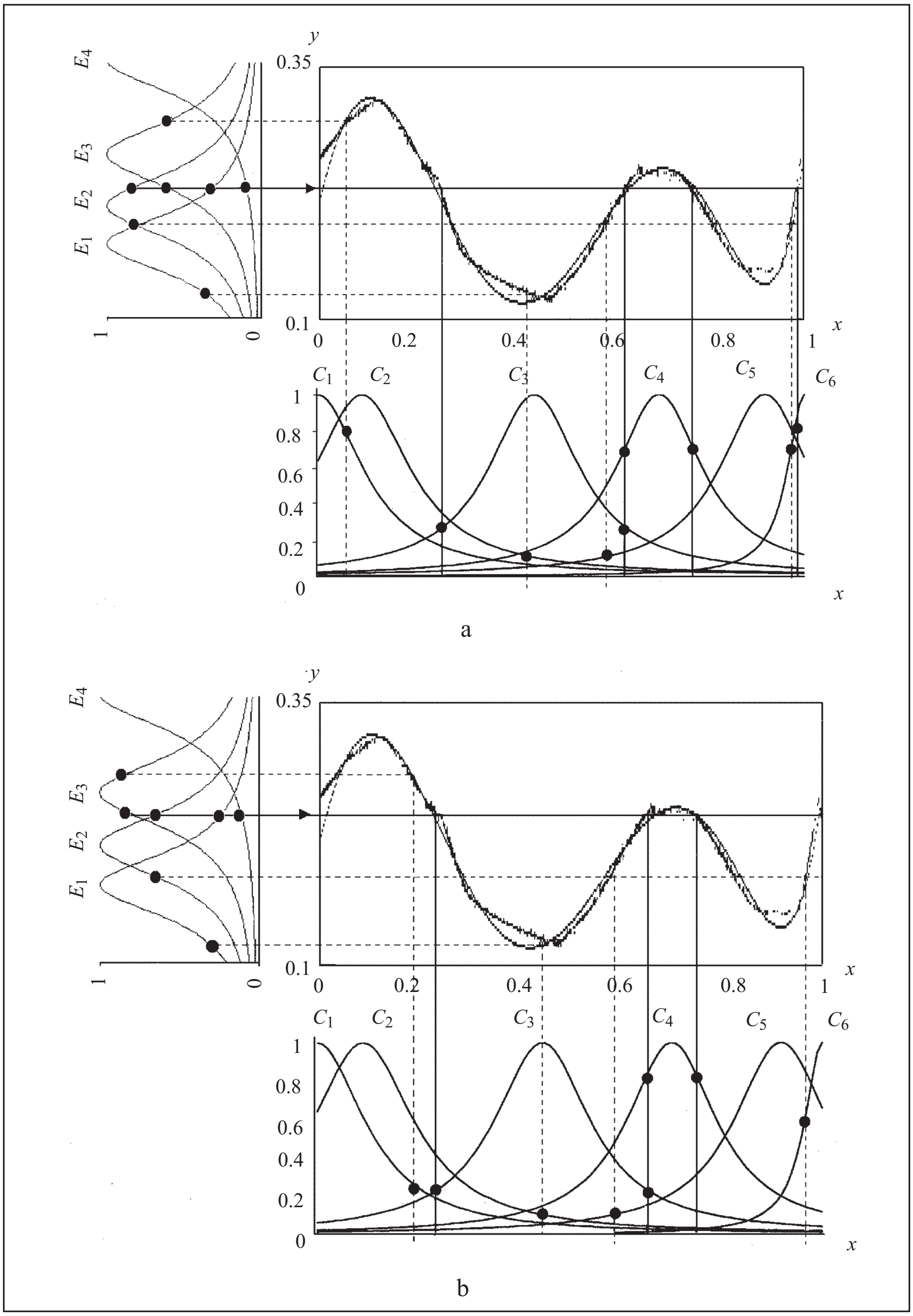


Fig. 7. Estimated reference model and the membership functions of the fuzzy terms of causes and effects after the adjustment: (a) $y^* = 0.23$; (b) $y^* = 0.24$.

Thus, a solution of the system of fuzzy logic equations (11) can be represented as intervals

$$S(R,\mu^E)=\{\mu^{C_1}=0.78;\ \mu^{C_2}\in[0,0.12];\ \mu^{C_3}=0.29;\ \mu^{C_4}\in[0,0.67];\ \mu^{C_5}\in[0,0.12];\ \mu^{C_6}\in[0.67,0.78]\}\cup\{\mu^{C_1}=0.78;\ \mu^{C_2}\in[0,0.12];\ \mu^{C_3}=0.29;\ \mu^{C_4}=0.67;\ \mu^{C_5}\in[0,0.12];\ \mu^{C_6}\in[0,0.78]\}\cup\{\mu^{C_1}\in[0,0.78];\ \mu^{C_2}\in[0,0.12];\ \mu^{C_3}=0.29;\ \mu^{C_4}\in[0,0.67];\ \mu^{C_5}\in[0,0.12];\ \mu^{C_6}=0.78\}. \quad (13)$$

Using membership functions, it is possible to determine the ranges of the input variable (Fig. 7a) for each interval in (13):

— $x^* = 0.060$ or $x^* \in [0.060, 1.0]$ for C_1;

— $x^* \in [0.418, 1.0]$ for C_2;

— $x^* = 0.264$ or $x^* = 0.628$ for C_3;

— $x^* = 0.628$, $x^* \in [0, 0.628]$, $x^* = 0.794$ or $x^* \in [0.794, 1.0]$ for C_4;

— $x^* \in [0, 0.610]$ for C_5;

— $x^* \in [0.971, 0.978]$, $x^* \in [0, 0.978]$ or $x^* = 0.978$ for C_6.

Estimating the set of inputs for $y^* = 0.23$, i.e., points (0.264,0.230), (0.628, 0.230), (0.794, 0.230), and (0.978, 0.230), is shown in Fig. 7a, where the solid line represents the significance measures of causes and effects. The other found values of inputs correspond to other values of the output variable with the same significance measures of effects. Estimating these points is shown in Fig. 7a with a dotted line.

Let the value of the output variable has changed with $y^* = 0.23$ to $y^* = 0.24$ (Fig. 7b). For the new value, the fuzzy vector of effects is

$$\mu^E(y^*) = (\mu^{E_1} = 0.23; \mu^{E_2} = 0.62; \mu^{E_3} = 0.82; \mu^{E_4} = 0.11).$$

The neural adjustment of zero solution (12) yields the fuzzy vector of causes

$$\mu_0^C = (\mu_0^{C_1} = 0.17,\ \mu_0^{C_2} = 0.04,\ \mu_0^{C_3} = 0.23,\ \mu_0^{C_4} = 0.82,\ \mu_0^{C_5} = 0.09,\ \mu_0^{C_6} = 0.62),$$

for which the value of the optimization criterion (6) is $F = 0.0001$.

The resultant zero solution has allowed adjusting the interval boundaries in (13) and generating a set of solutions $S(R, \mu^E)$ defined by the maximum solution

$$\bar{\mu}^C = (\bar{\mu}^{C_1} = 0.23,\ \bar{\mu}^{C_2} = 0.12,\ \bar{\mu}^{C_3} = 0.23,\ \bar{\mu}^{C_4} = 0.82,\ \bar{\mu}^{C_5} = 0.12,\ \bar{\mu}^{C_6} = 0.62)$$

and by two minimum solutions $S^* = \{\underline{\mu}_1^C, \underline{\mu}_2^C\}$:

$$\underline{\mu}_1^C = (\underline{\mu}_1^{C_1} = 0.23,\ \underline{\mu}_1^{C_2} = 0,\ \underline{\mu}_1^{C_3} = 0,\ \underline{\mu}_1^{C_4} = 0.82,\ \underline{\mu}_1^{C_5} = 0,\ \underline{\mu}_1^{C_6} = 0.62);$$

$$\underline{\mu}_2^C = (\underline{\mu}_2^{C_1} = 0, \underline{\mu}_2^{C_2} = 0, \underline{\mu}_2^{C_3} = 0.23, \underline{\mu}_2^{C_4} = 0.82, \underline{\mu}_2^{C_5} = 0, \underline{\mu}_2^{C_6} = 0.62).$$

Thus, the solution of the system of fuzzy logic equations (11) for the new value can be represented as the intervals

$$\begin{aligned} S(R, \mu^E) = \{&\mu^{C_1} = 0.23;\ \mu^{C_2} \in [0,\ 0.12];\ \mu^{C_3} \in [0,\ 0.23]; \\ &\mu^{C_4} = 0.82;\ \mu^{C_5} \in [0,\ 0.12];\ \mu^{C_6} = 0.62\} \\ \cup \{&\mu^{C_1} \in [0,\ 0.23];\ \mu^{C_2} \in [0,\ 0.12];\ \mu^{C_3} = 0.23; \\ &\mu^{C_4} = 0.82;\ \mu^{C_5} \in [0, 0.12];\ \mu^{C_6} = 0.62\}. \end{aligned} \tag{14}$$

Solution (14) differs from (13) in the significance measures of causes C_1, C_3, C_4, and C_6, for which the ranges of the input variable were determined (Fig. 7b) using membership functions:

— $x^* = 0.208$ or $x^* \in [0.208, 1.0]$ for C_1;

— $x^* = 0.236$, $x^* \in [0, 0.236]$, $x^* = 0.656$ or $x^* \in [0.656, 1.0]$ for C_3;

— $x^* = 0.656$ or $x^* = 0.766$ for C_4;

— $x^* = 0.968$ for C_6.

The estimated set of inputs for $y^* = 0.24$, i.e., points (0.236, 0.240), (0.656, 0.240), and (0.766, 0.240), is shown in Fig. 7b.

EXAMPLE OF ENGINEERING DIAGNOSTICS

Let us consider a diagnostics of a hydraulic pump.

— The input parameters of the pump (their ranges specified in brackets): x_1 is the engine speed (2600–3200 rpm); x_2 is the filter flow area (30–45 cm^2/kW); x_3 is the impeller clearance (0.1–0.3 mm); x_4 is the suction pipe leakage (0.5–2.0 cm^3/h); and x_5 is the resistance of the pumping main (1.2–3.4 atm (kgf/cm^2)).

— Failure causes: c_{11} is a decreased speed x_1; c_{21} is a reduced flow area x_2, i.e., filter clogging; c_{31} (c_{32}) is a decrease (increase) in the clearance x_3, i.e., assembly defect (wear of the impeller); c_{41} is increased leakage x_4, i.e., seal failure; and c_{51} is a high resistance of the pumping main x_5.

— The output parameters of the pump: y_1 is the pump delivery (20–45 m^3/h); y_2 is the pressure in the pumping main (3.7–5.5 atm (kgf/cm^2)); y_3 is the power input (15–30 kW); and y_4 is the pressure in the suction main (0.5–1.0 atm (kgf/cm^2)).

— Effects observed: e_{11} is the reduction of pump delivery y_1; e_{21} (e_{22}) is the decrease (increase) in the pressure y_2; $e_{31}(e_{32})$ is the decrease (increase) in the power input y_3; and e_{41} is the increase in the suction pressure y_4.

Let us rewrite the set of causes and effects as follows:

$$\{C_1, C_2, C_3, C_4, C_5, C_6\} = \{c_{11}, c_{21}, c_{31}, c_{32}, c_{41}, c_{51}\};$$

$$\{E_1, E_2, E_3, E_4, E_5, E_6\} = \{e_{11}, e_{21}, e_{22}, e_{31}, e_{32}, e_{41}\}.$$

Let us represent the fuzzy expert relations R obtained by the pairwise-comparison method [5, 6].

	E_1	E_2	E_3	E_4	E_5	E_6
C_1	0.23	1.00	0.46	1.00	0.11	0.67
C_2	0.90	0.22	0.56	0.22	0.22	0.89
C_3	0.10	0.44	1.00	0.67	1.00	0.22
C_4	1.00	0.22	0.67	0.22	0.11	0.77
C_5	0.77	0.11	0.33	0.11	0.11	1.00
C_6	0.23	0.56	0.11	0.78	0.44	0.11

To adjust the fuzzy model, the diagnostic results for 340 pumps were used. The results of the adjustment are shown in Table 4 and Fig. 8.

After the adjustment, the diagnostic equations become as follows :

$$\mu^{E_1} = (\mu^{C_1} \wedge 0.21) \vee (\mu^{C_2} \wedge 0.78) \vee (\mu^{C_3} \wedge 0.15) \vee (\mu^{C_4} \wedge 0.84) \vee (\mu^{C_5} \wedge 0.73) \vee (\mu^{C_6} \wedge 0.18),$$

$$\mu^{E_2} = (\mu^{C_1} \wedge 0.97) \vee (\mu^{C_2} \wedge 0.20) \vee (\mu^{C_3} \wedge 0.43) \vee (\mu^{C_4} \wedge 0.18) \vee (\mu^{C_5} \wedge 0.14) \vee (\mu^{C_6} \wedge 0.58),$$

$$\mu^{E_3} = (\mu^{C_1} \wedge 0.48) \vee (\mu^{C_2} \wedge 0.59) \vee (\mu^{C_3} \wedge 0.85) \vee (\mu^{C_4} \wedge 0.63) \vee (\mu^{C_5} \wedge 0.34) \vee (\mu^{C_6} \wedge 0.12),$$

$$\mu^{E_4} = (\mu^{C_1} \wedge 0.94) \vee (\mu^{C_2} \wedge 0.21) \vee (\mu^{C_3} \wedge 0.64) \vee (\mu^{C_4} \wedge 0.18) \vee (\mu^{C_5} \wedge 0.16) \vee (\mu^{C_6} \wedge 0.74),$$

$$\mu^{E_5} = (\mu^{C_1} \wedge 0.16) \vee (\mu^{C_2} \wedge 0.14) \vee (\mu^{C_3} \wedge 0.92) \vee (\mu^{C_4} \wedge 0.08) \vee (\mu^{C_5} \wedge 0.10) \vee (\mu^{C_6} \wedge 0.41),$$

$$\mu^{E_6} = (\mu^{C_1} \wedge 0.64) \vee (\mu^{C_2} \wedge 0.82) \vee (\mu^{C_3} \wedge 0.21) \vee (\mu^{C_4} \wedge 0.72) \vee (\mu^{C_5} \wedge 0.99) \vee (\mu^{C_6} \wedge 0.09). \tag{15}$$

Let the vector of observed parameters for a specific pump be

$$Y^* = (y_1^* = 26.12\,\text{m}^3/\text{h};\ y_2^* = 5.08\,\text{atm (kgs/cm}^2);\ y_3^* = 24\ \text{kW};\ y_4^* = 0.781\,\text{atm (kgs/cm}^2).$$

A fuzzy vector of effects can be determined for these values using the membership functions in Fig. 8b:

$$\mu^E(Y^*) = (\mu^{E_1} = 0.71;\ \mu^{E_2} = 0.34;\ \mu^{E_3} = 0.63;\ \mu^{E_4} = 0.18;\ \mu^{E_5} = 0.12;\ \mu^{E_6} = 0.71).$$

TABLE 4

Parameters of the membership functions	Values of the parameters of the membership functions of fuzzy terms of causes and effects after the adjustment					
	C_1 / E_1	C_2 / E_2	C_3 / E_3	C_4 / E_4	C_5 / E_5	C_6 / E_6
β	2700 / 22.98	32.27 / 3.86	0.11 / 5.37	0.28 / 16.45	1.82 / 28.92	3.19 / 0.89
σ	104.57 / 4.93	2.94 / 0.87	0.03 / 0.38	0.06 / 3.54	0.31 / 1.82	0.54 / 0.17

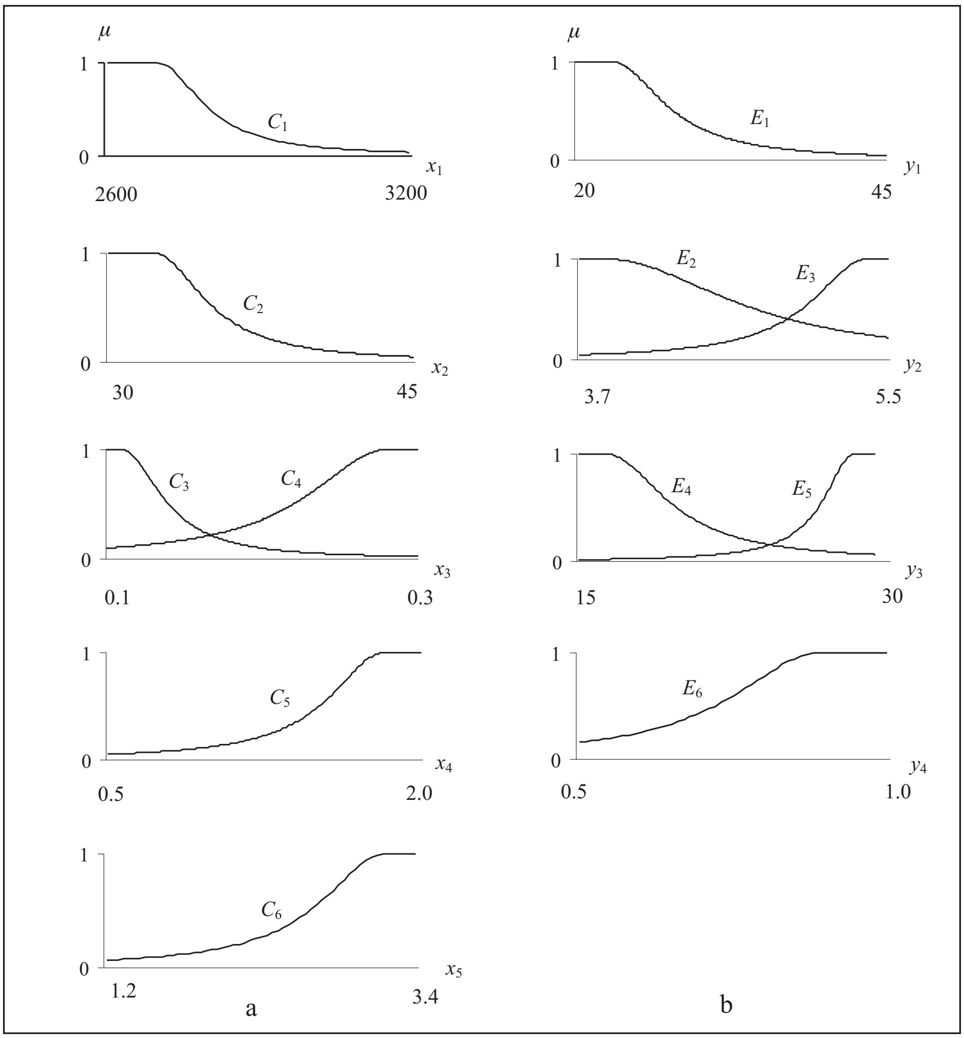


Fig. 8. Membership functions of fuzzy terms of causes (a) and effects (b) after the adjustment.

A genetic algorithm yields the zero solution

$$\mu_0^C = (\mu_0^{C_1} = 0.26,\ \mu_0^{C_2} = 0.54,\ \mu_0^{C_3} = 0.14,\ \mu_0^{C_4} = 0.69,\ \mu_0^{C_5} = 0.71,\ \mu_0^{C_6} = 0.08), \qquad (16)$$

for which the value of the optimization criterion (6) is $F = 0.0144$.

The zero vector found has allowed organizing a genetic search for the set $S(R, \mu^E)$ defined by the maximum solution

$$\overline{\mu}^C = (\overline{\mu}^{C_1} = 0.26, \overline{\mu}^{C_2} = 0.71, \overline{\mu}^{C_3} = 0.16, \overline{\mu}^{C_4} = 0.71, \overline{\mu}^{C_5} = 0.71, \overline{\mu}^{C_6} = 0.16)$$

and by three minimum solutions $S^* = (\underline{\mu}_1^C, \underline{\mu}_2^C, \underline{\mu}_3^C)$:

$$\underline{\mu}_1^C = (\underline{\mu}_1^{C_1} = 0.26,\ \underline{\mu}_1^{C_2} = 0.71,\ \underline{\mu}_1^{C_3} = 0,\ \underline{\mu}_1^{C_4} = 0.63,\ \underline{\mu}_1^{C_5} = 0,\ \underline{\mu}_1^{C_6} = 0);$$
$$\underline{\mu}_2^C = (\underline{\mu}_2^{C_1} = 0.26,\ \underline{\mu}_2^{C_2} = 0,\ \underline{\mu}_2^{C_3} = 0,\ \underline{\mu}_2^{C_4} = 0.71,\ \underline{\mu}_2^{C_5} = 0,\ \underline{\mu}_2^{C_6} = 0);$$
$$\underline{\mu}_3^C = (\underline{\mu}_3^{C_1} = 0.26,\ \underline{\mu}_3^{C_2} = 0,\ \underline{\mu}_3^{C_3} = 0,\ \underline{\mu}_3^{C_4} = 0.63,\ \underline{\mu}_3^{C_5} = 0.71,\ \underline{\mu}_3^{C_6} = 0).$$

Thus, a solution of the system of fuzzy logic equations (15) can be represented as the intervals

$$\begin{aligned} S(R, \mu^E) = \{&\mu^{C_1} = 0.26;\ \mu^{C_2} = 0.71;\ \mu^{C_3} \in [0, 0.16];\\ &\mu^{C_4} \in [0.63, 0.71];\ \mu^{C_5} \in [0, 0.71];\ \mu^{C_6} \in [0, 0.16]\}\\ &\cup \{\mu^{C_1} = 0.26;\ \mu^{C_2} \in [0, 0.71];\ \mu^{C_3} \in [0, 0.16];\\ &\mu^{C_4} = 0.71;\ \mu^{C_5} \in [0, 0.71];\ \mu^{C_6} \in [0, 0.16]\}\\ &\cup \{\mu^{C_1} = 0.26;\ \mu^{C_2} \in [0, 0.71];\ \mu^{C_3} \in [0, 0.16];\\ &\mu^{C_4} \in [0.63, 0.71];\ \mu^{C_5} = 0.71;\ \mu^{C_6} \in [0, 0.16]\}. \end{aligned} \tag{17}$$

The ranges of the input variable can be determined for each interval in the solution (17) using the membership functions in Fig. 8b:

— $x_1^* = 2877$ rpm for C_1;

— $x_2^* = 34.15$ or $x_2^* \in [34.15, 45]$ cm^2/kW for C_2;

— $x_3^* \in [0.178, 0.300]$ mm for C_3;

— $x_3^* = 0.242$ or $x_3^* \in [0.234, 0.242]$ mm for C_4;

— $x_4^* = 1.62$ or $x_4^* \in [0.5, 1.62]$ cm^3/h for C_5;

— $x_5^* \in [1.2, 1.95]$ kgf/cm^2 for C_6.

The solutions allow preliminary conclusions. A pump failure may be because of filter clogging, wear of the impeller or seal failure (flow area decreased to 34.15–45 cm^2/kW, clearance increased up to 0.234–0.242 mm, and leakage increased up to 0.5–1.62 cm^3/h), since the significance measures of the causes C_2, C_4, and C_5 are sufficiently high. An assembly defect of the impeller for the clearance within 0.178–0.300 mm should be excluded since the significance measure of the cause C_3 is small. The engine speed reduced to 2877 rpm may also affect the operation of the pump, which is manifested by the significance measure of the cause C_1. Resistance of the pumping main increased up to 1.2–1.95 kgf/cm^2 slightly affects a pump failure since the significance measure of the cause C_6 is small.

Let a repeated measurement has revealed a decrease in the pump delivery till $y_1^* = 24.97$ m^3/h and an increase in the suction pressure up to $y_4^* = 0.792$ atm, the values of μ^{E_1} increasing up to 0.86, μ^{E_6} up to 0.75, and the values of other parameters remaining unchanged.

A neural adjustment of zero solution (16) has yielded a fuzzy vector of causes

$$\mu_0^C = (\mu_0^{C_1} = 0.26,\ \mu_0^{C_2} = 0.17,\ \mu_0^{C_3} = 0.10,\ \mu_0^{C_4} = 0.93,\ \mu_0^{C_5} = 0.75,\ \mu_0^{C_6} = 0.05),$$

for which the value of the optimization criterion (6) has constituted $F = 0.0148$.

The resultant zero solution has allowed adjusting the ranges in the solution (17) and generating the set of solutions $S(R, \mu^E)$ defined by the maximum solution

$$\overline{\mu}^C = (\overline{\mu}^{C_1} = 0.26,\ \overline{\mu}^{C_2} = 0.75,\ \overline{\mu}^{C_3} = 0.16,\ \overline{\mu}^{C_4} = 1.00,\ \overline{\mu}^{C_5} = 0.75,\ \overline{\mu}^{C_6} = 0.16)$$

and by two minimum solutions $S^* = \{\underline{\mu}_1^C, \underline{\mu}_2^C\}$:

$$\underline{\mu}_1^C = (\underline{\mu}_1^{C_1} = 0.26,\ \underline{\mu}_1^{C_2} = 0.75,\ \underline{\mu}_1^{C_3} = 0,\ \underline{\mu}_1^{C_4} = 0.84,\ \underline{\mu}_1^{C_5} = 0,\ \underline{\mu}_1^{C_6} = 0);$$
$$\underline{\mu}_2^C = (\underline{\mu}_2^{C_1} = 0.26,\ \underline{\mu}_2^{C_2} = 0,\ \underline{\mu}_2^{C_3} = 0,\ \underline{\mu}_2^{C_4} = 0.84,\ \underline{\mu}_2^{C_5} = 0.75,\ \underline{\mu}_2^{C_6} = 0).$$

TABLE 5

Fuzzy term of the cause (diagnosis)	Number of failures in the sample	Probability of a correct diagnosis		
		before adjustment	after adjustment	
			zero solution (genetic algorithm)	refined diagnosis (neural network)
C_1	105	$83/105 = 0.79$	$99/105 = 0.94$	$103/105 = 0.98$
C_2	203	$164/203 = 0.81$	$186/203 = 0.92$	$197/203 = 0.97$
C_3	59	$52/59 = 0.88$	$54/59 = 0.91$	$57/59 = 0.97$
C_4	187	$154/187 = 0.82$	$174/187 = 0.93$	$178/187 = 0.95$
C_5	94	$85/94 = 0.90$	$90/94 = 0.96$	$93/94 = 0.99$
C_6	75	$64/75 = 0.85$	$69/75 = 0.92$	$73/75 = 0.97$

Thus, a solution of the system of fuzzy logic equations (15) can be represented as the intervals

$$\begin{aligned} S(R, \mu^E) = \{&\mu^{C_1} = 0.26;\ \mu^{C_2} = 0.75;\ \mu^{C_3} \in [0, 0.16]; \\ &\mu^{C_4} \in [0.84, 1.00];\ \mu^{C_5} \in [0, 0.75];\ \mu^{C_6} \in [0, 0.16]\} \\ &\cup \{\mu^{C_1} = 0.26;\ \mu^{C_2} \in [0, 0.75];\ \mu^{C_3} \in [0, 0.16]; \\ &\mu^{C_4} \in [0.84, 1.00];\ \mu^{C_5} = 0.75;\ \mu^{C_6} \in [0, 0.16]\}. \end{aligned} \tag{18}$$

Solution (17) differs from (18) in the significance measures of the causes C_2, C_4, and C_5. The ranges of input variable (see Fig. 8a) have been determined for these causes using the membership functions: $x_2^* = 33.97$ or $x_2^* \in [33.97, 45]$ cm^2/kW for C_2; $x_3^* \in [0.254, 0.300]$ mm for C_4; $x_4^* = 1.64$ or $x_4^* \in [0.5, 1.64]$ cm^3/h for C_5.

The solution obtained allows the final conclusions. The state of the pump being observed is due to the wear of the impeller (increase in the clearance to 0.254–0.300 mm) since the significance measure of the cause C_4 is maximum. The causes of the pump failure are still filter clogging and seal failure (the flow area decreased to 33.97–45 cm^2/kW and the leakage increased to 0.5–1.64 cm^3/h) since the significance measures of the causes C_2 and C_5 are reasonably high. The values of other parameters have not changed.

Testing the fuzzy model employed the results of diagnostics of 250 pumps with different failures. Table 5 shows the measures of effectiveness of the adjustment algorithm for a testing sample. Attaining a 95% correctness of the diagnostics required 30 min of the operation of a genetic algorithm and 4 min of the operation of a neural network (Celeron 700). Experience of applying genetic optimization algorithms in problems of diagnostic, prediction, and knowledge acquisition [11–17] suggests that the problem time for about 500 controlled (optimized) variables [16] does not exceed 200 min on a Celeron 700.

CONCLUSIONS

The approach to constructing adaptive diagnostic systems considered in the paper is based on fuzzy relations. A diagnostic problem is formulated as estimating causes from the effects being observed. Using expert information represented as fuzzy relations, cause-and-effect relationships are formalized as a system of fuzzy logic equations. Solving a system of equations is reduced to an optimization problem that minimizes the residual between the left- and right-hand sides of these equations. The adaptive approach proposed here provides a solution of the optimization problem using recurrent relations that correspond to on-line learning of a specially constructed neuro-fuzzy network, isomorphic to the system of diagnostic equations. The efficiency of the approach is confirmed with a computer experiment and a real example of diagnostics.

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