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A NONLINEAR COMPROMISE SCHEME IN MULTICRITERIA EVALUATION AND OPTIMIZATION PROBLEMS

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The concept of a nonlinear compromise scheme in multicriteria problems of evaluation and optimization is presented. It is shown that the problem is to approximate correctly the utility function and construct a substantial mathematical model (scalar convolution) adequate to the given situation to solve various multicriteria problems. In analysis problems, this convolution is an objective function. Its extremization results in a compromise-optimal vector of arguments. An illustrative example is given.

Keywords: *multicriteria problems, situation of decision making, adaptation, model of utility function, nonlinear compromise scheme.*

PROBLEM DESCRIPTION

The problems related to multicriteria problems involve determining the Pareto domain, normalizing the criteria, making allowance for priorities, and choosing a compromise scheme. However, only the first of them has a strong scientific substantiation and does not depend on subjective preferences. Solving the other problems require some information from a decision-maker (DM). This is especially true for the problem of choosing a compromise scheme since a compromise is inherently a prerogative of human.

The essence of the concept of a compromise is in the answer to the following question: how many prize units in one criterion may, in the DM's opinion, compensate for inevitable loss of a unit in another (other) criterion in a given situation? Based on this information, a specific compromise scheme is formulated for the given multicriteria problem and a desired solution is finally found. Note that there are so-called incomparable criteria in decision-making, for which there is no compromise. For example, improving the quality of TV sound cannot compensate for the image degradation. In what follows, we will not consider such incomparable criteria and restrict ourselves to criteria for which a compromise scheme can be obtained.

Thus, finding a multicriteria solution is inherently a compromise and is based on using a subjective information. Given this information and a compromise scheme selected, it is possible to pass from the general vector expression to the scalar convolution of partial criteria, which provides a basis for a constructive apparatus to solve multicriteria problems. Solving the problem is based on the hypothesis that there exists a utility function [1] appearing in the DM's brain during the solution of a specific multicriteria problem. We may state that virtually all the approaches to determining the scalar convolution of criteria are reduced to constructing one mathematical model or another of the DM's utility function.

The problem is to approximate correctly the utility function and to construct a substantial mathematical model (scalar convolution), adequate to the given situation, to solve different multicriteria problems.

FORMALIZATION OF THE PROBLEM

A DM's utility function can generally be represented as $\Phi[y(x), r]$, where $x = \{x_i\}_{i=1}^{n} \in X$ is the vector of feasible solutions defined in the feasible domain X; $y = \{y_k\}_{k=1}^{s} \in M$ is the vector of partial criteria defined in the feasible domain

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$M = \{y \mid 0 \le y_k \le A_k,\ k \in [1, s]\}$; $A = \{A_k\}_{k=1}^{s}$ is the constraint vector; and $r \in R$ is the vector of external conditions defined on the set of feasible factors R.

The situation of making a multicriteria decision is defined by the factors of external conditions r. For example, in the tropics, special attention is paid to the criterion reflecting the performance quality of the engine cooling system. If an aircraft being designed will be used under field conditions, a criterion related to the landing roll and touchdown distance becomes the most important. In solving multicriteria problems, it is usually assumed that the vector r is fixed and specified: $r = r^\circ$. Then the DM's utility function can be represented by the scalar convolution of criteria

$$\Phi[y(x), r]_{|r=r^\circ} = Y[y(x)]^\circ,$$

where $Y[y(x)]^\circ$ is the scalar convolution constructed from the compromise scheme adequate to the given situation.

If the criterion vector $y(x)$ is normalized by the constraint vector A,

$$y_0(x) = \{y_k(x)/A_k\}_{k=1}^{s} = \{y_{0k}(x)\}_{k=1}^{s},$$

then the scalar convolution $Y[y_0(x)]^\circ$ is applied.

In most cases, solving multicriteria problems is restricted to a linearized model

$$Y[y_0(x)]^\circ = \sum_{k=1}^{s} \alpha_k^\circ y_0(x),$$

were α_k° are the weight coefficients composing a vector $\alpha^\circ = \{\alpha_k^\circ\}_{k=1}^{s} \in \Gamma_\alpha$ and defined on a simplex $\Gamma_\alpha = \left\{\alpha \mid \alpha_k \ge 0,\ \sum_{k=1}^{s} \alpha_k = 1\right\}$. The values of the weight coefficients are established based on individual preferences of the DM and correspond to the situation.

Though such an approach has a doubtless advantage (simplicity), it is characterized by shortcomings inherent in the linearization method. For example, a linear model yields correct results only in small neighborhoods of a working point whose position depends on the situation of making a multicriteria decision. Any variation in the situation necessitates weight coefficients of the model to be redetermined. In practical multicriteria problems, it is expedient to construct a nonlinear model of the DM's utility function (the concept of a nonlinear compromise scheme [2]).

The compromise scheme determines the advantage of the resultant multicriteria solution over other Pareto-optimal solutions. Nowadays, a compromise scheme is chosen not theoretically but heuristically, based on individual preferences, professional experience of the developer, and information on the situation in which the multicriteria decision is made.

The main complexity in passing from a vector performance criterion to a scalar convolution is that the convolution should be a conglomerate of partial criteria, the value (importance) of each them in the general estimate varying depending on the situation. In different situations, different partial criteria may become the most important ones. In other words, the scalar convolution of partial criteria should determine the compromise scheme being adapted to the situation. Let this thesis underlie the analysis of the possibilities of formalizing the choice of a compromise scheme.

It is assumed that there are some invariants, rules usually common for all the DMs irrespective of their individualities, to which they equally adhere in one situation or another. According to [3], the DM's subjectivity is limited. Making business decisions requires a person to be rational to give reasons for the choice of a subjective model. Therefore, any DM's preferences should be within the framework of a certain rational system. This is what makes the formalization possible.

In our case, the subject of study is a fine substance such as an imaginary utility function appearing in the DM's brain during the solution of a specific multicriteria problem. Each DM has his utility function. Nevertheless, revealing and analyzing some general laws observed during making multicriteria decisions by different DMs in different situations can provide the information to specify the form of the substantial model of the criterion function.

CONCEPTUAL ANALYSIS OF THE DM'S UTILITY FUNCTION

Depending on the form of the multicriteria problem, the scalar convolution $Y[y_0(x)]$ has different physical meanings. In the analysis problem, this convolution is an evaluation function, its value quantitatively expresses the measure of quality of a multicriteria object for specified values of arguments x. In the optimization problem, the scalar convolution $Y[y_0(x)]$ has the meaning of the objective function. Its extremization results in a compromise-optimal vector of arguments x^*. In what follows, we will consider an optimization problem and assume for definiteness that all the criteria $y_0(x)$ are to be minimized. Then mathematically, the vector optimization problem can be represented as

$$x^* = \arg\min_{x\in X} Y[y_0(x)].$$

Let us introduce the concept of situation intensity as a measure of how relative partial criteria are close to the limit value (unity):

$$\rho_k = 1 - y_{0k}, \quad \rho_k \in [0;1], \quad k \in [1,s].$$

If a multicriteria decision is made in an intense situation, then under the conditions specified, one or several partial criteria may appear dangerously close to the limit values ($\rho_k \approx 0$). And if one of the criteria attains the limit (or is outside it), this event will not be compensated by a possible small level of other criteria (violating any of the constraints is usually prohibited).

In such a situation, it is necessary to interfere (in every possible way) the dangerous increase of the most adverse (i.e., the closest to the limit) partial criterion irrespective of the behavior of other criteria. Therefore, in rather intense situations (for small ρ_k), the DM permits the deterioration of the maximum (most important under those conditions) partial criterion by unity only if this is compensated by a great number of unities that improve the other criteria. And in the first polar case ($\rho_k = 0$), the DM leaves only this unique, most unfavorable partial criterion for consideration and neglects the others. Hence, a minimax Chebyshev model (egalitarian principle)

$$x^* = \arg\min_{x\in X} Y[y_0(x)]^{(1)} = \arg\min_{x\in X}\max_{k\in[1,s]} y_{0k}(x)$$

adequately expresses the compromise scheme in an intense situation.

In less intense situations, other criteria should be satisfied simultaneously, with allowance made for inconsistent interests and objectives of the system. Depending on the situation, the DM varies his estimate of a payoff in one criteria and loss in another ones. In intermediate cases, compromise schemes partially aligning partial criteria in different degrees are selected. As the situation becomes less intense, preferences in partial criteria level off.

Finally, in the second polar case ($\rho_k \approx 1$), the situation is quiet, partial criteria are small, and there is no threat of violating the constraints. The DM considers that a unit deterioration of any partial criterion is compensated quite well by an equivalent unit improvement of any other criterion. This case can be associated with an economic compromise scheme providing the minimum (for the conditions specified) total loss on partial normalized criteria. Such a scheme can be expressed by the model of integral optimality (utilitarian principle)

$$x^* = \arg\min_{x\in X} Y[y_0(x)]^{(2)} = \arg\min_{x\in X}\sum_{k=1}^{s} y_{0k}(x).$$

The analysis reveals a pattern according to which the DM varies the choice from the integral optimality model in quiet situations to the minimax model in intense situations. In intermediate cases, the DM chooses compromise schemes that satisfy partial criteria in different degrees, according to the individual preferences but in view of the situation. If we take the conclusions from this analysis as a logic basis for formalizing the choice of a compromise scheme, we can present various constructive concepts such as the concept of a nonlinear compromise scheme.

NONLINEAR COMPROMISE SCHEME

From the formalization standpoint, it is expedient to replace the problem of choosing a compromise scheme with the equivalent problem of synthesis of a unified scalar convolution of partial criteria which would express different principles of optimality in different situations. Let us state the requirements to the function $Y(y_0)$ being synthesized:

- it should be smooth and differentiable;
- should express the minimax principle in intense situations;
- should express the integral optimality principle under quiet conditions;
- in intermediate cases, should lead to Pareto-optimal solutions giving various measures of partial satisfaction of the criteria.

Thus, a universal convolution should express a compromise scheme adaptable to a situation. We may say that adaptation and adaptability are the main substantial essence of studying multicriteria systems. The scalar convolution should include the explicit characteristics of the situation intensity ρ.

Among the possible functions meeting the above requirements, let us consider an elementary one

$$Y(\alpha, y_0) = \sum_{k=1}^{s} \alpha_k [1 - y_{0k}(x)]^{-1}, \ \alpha \in \Gamma_\alpha, \tag{1}$$

where $\alpha_k = \mathrm{const}$ are formal parameters defined on a simplex and having double physical meaning. On the one hand, these are weight coefficients that express the DM's preferences in partial criteria, and on the other hand, these are coefficients of a substantial regression model of the DM's utility function based on the concept of a nonlinear compromise scheme.

Thus, a nonlinear compromise scheme is associated with a vector optimization model, which explicitly depends on the characteristics of situation intensity ρ:

$$x^* = \arg \min_{x \in X} \sum_{k=1}^{s} \alpha_k [1 - y_{0k}(x)]^{-1}. \tag{2}$$

In contrast to the linear model, defined in a small neighborhood of a working point, the nonlinear model of the DM's utility function is defined on the whole feasible region X and does not require coefficients α_k to be recalculated if the situation varies.

As is seen from (2), if any relative partial criterion, for example, $y_{0i}(x)$, approaches the limit (unity), i.e., the situation becomes intense, the corresponding term $Y_i = 1/\alpha_i[1 - y_{0i}(x)]$ in the sum being minimized increases so that the minimization of the whole sum reduces to the minimization of only this worst term, i.e., of the criterion $y_{0i}(x)$. This is equivalent to a minimax model. To avoid the possible division by zero in intense situations, the following condition is used for the optimization by formula (2): if $y_{0k}(x) \geq 0.95$, it is assumed that $y_{0k}(x) = 0.95$.

If relative partial criteria are far from unity, i.e., the situation is quiet, model (2) operates equivalent to the integral optimality model. In intermediate situations, different degrees of partial alignment of criteria are obtained. Therefore, the nonlinear compromise scheme proposed has the property of continuous adaptation to the situation of making a multicriteria decision. The adaptation is continuous, while the traditional choice of a compromise scheme is discrete and thus the errors due to the quantization of the compromise scheme are added to subjective errors.

UNIFICATION

As indicated above, choosing a compromise scheme is a prerogative of the human, it reflects his subjective utility function in solving a specific multicriteria problem. Nevertheless, it has been possible to reveal some patterns and use this objective basis to construct the scalar convolution of criteria whose form depends on the substantial representations of the essence of the phenomenon under study. The phenomenon of DM's individual preferences is formally represented by the presence of the vector α in the structure of the substantial models (1) and (2).

The role of subjective factors in the solution of multicriteria problems can be evaluated in different ways. The subjectivity is admissible and is even expedient if such problem is solved based on the interests of a specific human or small collectives of people with similar preferences. Therefore, the mechanism of individual preferences is applied quite intensively to solve multicriteria problems. For example, a tailor-made suit from a fashion atelier is usually better but more expensive than that bought in a ready-made shop.

If the resultant solution is for general use, it should be quite objective and unified. In these cases, the mechanism of individual preferences is excluded from the solution of multicriteria problems to prevent subjectivity and ambiguity of the solution results.

If the result of the solution of a multicriteria problem is intended for a wide use, it is unified and individual preferences are leveled statistically. Then the Bernoulli–Laplaces principle of insufficient basis becomes applicable:

TABLE 1

Quality category	Ranges of normalized inverse fundamental scale for estimates y_0 and Y_0
Unacceptable	1.0 – 0.7
Low	0.7 – 0.5
Satisfactory	0.5 – 0.4
Good	0.4 – 0.2
High	0.2 – 0.0

if a priori probabilities of possible hypotheses are unknown, they should be assumed equal, i.e., equiprobable. As applied to a multicriteria problem, this means that all the weight coefficients α_k, $k \in [1,s]$, in (1) should be taken equal if there are no preliminary data that the criteria are of different values [4]. Unification assumes that all the criteria are of identical importance, and all the weight coefficients in expression (1) should be assumed equal: $\alpha_k \equiv 1/s \ \forall k \in [1,s]$. Then

$$Y(\alpha, y_0) = \frac{1}{s}\sum_{k=1}^{s}[1 - y_{0k}(x)]^{-1}.$$

Since multiplying by $1/s$ is a monotonic transformation, which (according to the Germeier theorem) does not change the results of comparison, we pass to the unified expression for the scalar convolution of criteria

$$Y(y_0) = \sum_{k=1}^{s}[1 - y_{0k}(x)]^{-1}. \tag{3}$$

It is expedient to apply formula (3) in all the cases where the multicriteria problem is solved not in the interests of one specific DM but for a wide use.

Doroshenko [4] proposes to start a multicriteria solution with formula (3) in all the cases. The result and the corresponding values of partial criteria are shown to the DM to evaluate them. If the solution obtained does not satisfy the DM and needs a correction according to his individual preferences, the procedure of determining weight coefficients α_k, $k \in [1,s]$, is organized, and formulas (1) and (2) are used for the optimization. Turning back to the analogy, note that a ready-made suit needs usually just insignificant adjustment. The solution of multicriteria optimization problems is described in [4, 5, 7].

MULTICRITERIA EVALUATION PROBLEMS

In contrast to optimization problems, multicriteria evaluation is classed among analysis problems. Convolution (1) or (3) is not objective but an evaluation function, and its value quantitatively expresses the measure of quality of a multicriteria object under specified values of arguments *x*.

Multicriteria evaluation of alternatives often needs not only analytic but also a qualitative estimate. To this end, we should normalize the expression for the scalar convolution of criteria $Y(\alpha, y_0)$ and associate the resultant value Y_0 with an inverted normalized fundamental scale. The general concept of a serial fundamental scale is described in [6]. Table 1 presents an interval normalized inverse scale and relates the qualitative gradations of properties of the objects and the corresponding quantitative estimates y_0 and Y_0.

The structure of the nonlinear compromise scheme allows normalizing the scalar convolution not to the maximum (usually unknown) but to the minimum value. Putting, in the expression for the nonlinear scalar convolution (1), the ideal (zero) values of the minimized criteria $y_{0k}(x) = 0$ and taking into account that the weight coefficients are normalized on the simplex, $\sum_{k=1}^{s}\alpha_k = 1$, yields $Y_{0\min} = 1$. The normalized minimized scalar convolution has the form

$$Y_0 = 1 - \frac{1}{Y(\alpha, y_0)}. \tag{4}$$

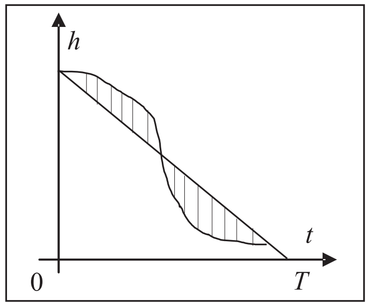

Fig. 1

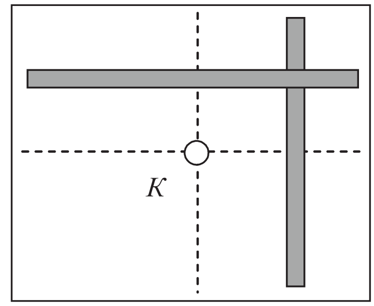

Fig. 2

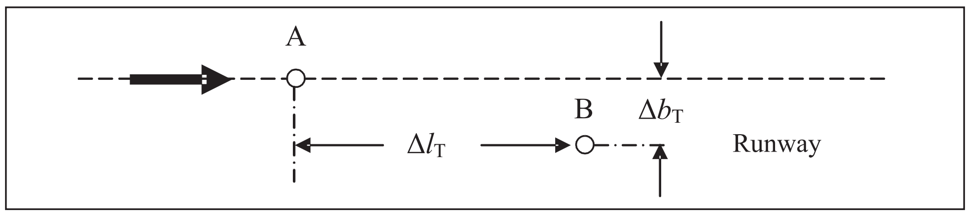

Fig. 3

A qualitative (linguistic) estimate of an alternative can be obtained by comparing the analytic estimate Y_0 with the normalized inverse fundamental scale. Evaluating alternatives using a unified normalized fundamental scale makes it possible to solve multicriteria problems both in traditional formulations and in the case where an alternative should be selected from a set of inhomogeneous alternatives for which a unified set of quantitative criteria of an estimate cannot be formulated, and to estimate the unique alternative.

MODEL EXAMPLE

Let us show the capabilities of a nonlinear compromise scheme in a multicriteria analysis problem such as the quality evaluation, using several criteria, of the glide landing of an aircraft.

Figure 1 shows schematically how the height of an aircraft h varies during the glide landing in the coordinates (h, t). The height h is assumed zero at the instant of time $t = T$. How the position of the aircraft b varies relative to the central line of the runway in the lateral plane during the glide landing can be represented similarly.

During the glide landing, the pilot navigates the aircraft using the director device shown schematically in Fig. 2. The position of the bars shown in the figure means that the aircraft is above the glide path and to the right from the axial line of the runway. The control consists in bringing the bar crosspoint into coincidence with the central point of the device K.

Figure 3 shows that at the instant of time $t = T$ the aircraft has touched the runway at the point B, spaced at Δl_T from the reference point A and at Δb_T from the axial line of the runway.

To evaluate the landing quality, we will use three terminal ($t = T$) performance criteria ($y_1 - y_3$) and two integral criteria (y_4, y_5):

$y_1 = |\Delta l_T| < A_1$ is the absolute value of the deviation from the reference contact point in the longitudinal plane;

$y_2 = |\Delta b_T| < A_2$ is the absolute value of the deviation of the contact point from the longitudinal axis of the runway in the lateral plane;

$y_3 = V_h^{(T)} < A_3$ is the vertical speed at the terminal point;

$y_4 = \frac{1}{T}\int_0^T |\Delta h| dt < A_4$ is the mean deviation from the glide path in the vertical plane;

$y_5 = \frac{1}{T}\int_0^T |\Delta b| dt < A_5$ is the mean deviation from the glide path in the horizontal plane.

Moreover, the following criteria may characterize aircraft landing quality: the deviation y_6 from the calculated landing speed at the terminal point; the aspect angle y_7 at the terminal point; the bank angle y_8 at the terminal point; the deviation y_9 from the calculated pitch angle at the terminal point; the mean amount of controls y_{10} on the glide path (integral criterion), etc. We assume that the last criteria are satisfied in all the landings and do not appear here.

To calculate the integral criteria, we will employ the method of approximate integration, which is illustrated using the criterion y_4 as an example. Divide the time interval of the descent along the glide path $[0,T]$ into N subintervals Δt. During each of these subintervals, the quantity $|\Delta h|_i$, $i \in [1,N]$, is measured and assumed constant. Then we can pass from the integral in the formula for the integral criterion to the summation

$$y_4 = \frac{1}{T}\int_0^T |\Delta h| dt \approx \frac{1}{T}\sum_{i=1}^{N} |\Delta h_i| \Delta t_i .$$

If all the subintervals are identical, i.e., $\Delta t_i = \Delta t \ \forall i$, then $T = N\Delta t$ and

$$y_4 \approx \frac{\Delta t}{N\Delta t}\sum_{i=1}^{N} |\Delta h_i| = \frac{1}{N}\sum_{i=1}^{N} |\Delta h_i| .$$

The criterion y_5 can be calculated similarly.

The scalar convolution of criteria (1) or (3) can be used as the estimate function. Below, we will consider convolution (1) as a more general one. The weight coefficients α can be determined in the interactive procedure described in [4]. Let the following values of the weight coefficients be obtained:

$$\alpha_1 = 0.25;\ \alpha_2 = 0.22;\ \alpha_3 = 0.28;\ \alpha_4 = 0.16;\ \alpha_5 = 0.09.$$

The following values of constraints in the criteria are specified (note that the example is model):

$$A_1 = 15\,\mathrm{m};\ A_2 = 10\,\mathrm{m};\ A_3 = 1\ \mathrm{m/sec};\ A_4 = 30\,\mathrm{m};\ A_5 = 20\,\mathrm{m}.$$

Then we will use the nonlinear compromise scheme to evaluate the quality of two aircraft landings for different numerical values of partial criteria.

Landing 1. Let $y_1 = 6\mathrm{m}$; $y_2 = 3\mathrm{m}$; $y_3 = 0.2\,\mathrm{m/sec}$; $y_4 = 10.5\,\mathrm{m}$; and $y_5 = 7.25\,\mathrm{m}$. The normalization $y_{0k} = \dfrac{y_k}{A_k}$, $k \in [1,s]$, yields the relative partial criteria: $y_{01} = 0.4$; $y_{02} = 0.3$; $y_{03} = 0.2$; $y_{04} = 0.35$; and $y_{05} = 0.36$.

Let us calculate the scalar convolution of the criteria using the nonlinear compromise scheme (formula (1)):

$$Y = 0.25\frac{1}{1-0.4} + 0.22\frac{1}{1-0.3} + 0.28\frac{1}{1-0.2} + 0.16\frac{1}{1-0.35} + 0.09\frac{1}{1-0.36} = 1.47.$$

With the normalization (4) we get

$$Y_0 = 1 - \frac{1}{1.47} = 0.32 .$$

Comparing this value with the qualitative grades of the normalized inverse fundamental scale (see Table 1) allows concluding that the landing is good.

Landing 2. Let $y_1 = 3\mathrm{m}$; $y_2 = 4\mathrm{m}$; $y_3 = 0.6\,\mathrm{m/sec}$; $y_4 = 13.25\,\mathrm{m}$; and $y_5 = 10.5\,\mathrm{m}$. The relative partial criteria are as follows: $y_{01} = 0.2$; $y_{02} = 0.4$; $y_{03} = 0.6$; $y_{04} = 0.44$; and $y_{05} = 0.52$.

Based on formula (1) we obtain

$$Y = 0.25\frac{1}{1-0.2} + 0.22\frac{1}{1-0.4} + 0.28\frac{1}{1-0.6} + 0.16\frac{1}{1-0.44} + 0.09\frac{1}{1-0.52} = 1.85.$$

Normalization (4) yields

$$Y_0 = 1 - \frac{1}{1.85} = 0.46 .$$

According to the normalized inverse fundamental scale, landing 2 is satisfactory. This conclusion follows from the criterion $y_3 = 0.6\,\mathrm{m/sec}$ (hard landing).

The multicriteria evaluation procedure is applicable for example to learning and training pilots and in similar cases in other subject domains.

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