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TRANSFORMATIONS OF FUZZY GRAPHS SPECIFIED BY *FD*-GRAMMARS

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The categorical approach is proposed to formalize transformations of FD-graphs that consist of networks of distributed components whose nodes are specified by fuzzy graphs. Necessary and sufficient conditions are formally defined for FD-graph transformations that do not violate structure integrity and can be constructed componentwise. FD-grammars that generalize fuzzy graph grammars are proposed to describe the admissible transformations of FD-graphs.

Keywords: *category theory, distributed systems, fuzzy graphs, graph grammars, model-driven architecture.*

INTRODUCTION

Nowadays, considerable attention is focused on creating formal models of fuzzy information technologies that can be a component of high-performance parallel cluster systems [1, 2]. Such systems logically develop the ideas underlying the architecture of massively parallel systems. To develop formal models of distributed systems, some methods employing graphs are proposed.

The method of *L*-systems, developed by Lindenmayer [3], is one of the most simple methods to create models of distributed systems based on graphs. In this method, a formal grammar based on the rules of generation and transformation of character strings is specified. Parts of a string correspond to states of distributed components, and parallelism occurs as simultaneous application of rules to different parts of the string. The well-known graph method for the specification and analysis of distributed systems is Perti net and its modifications [4].

Graphs can also be applied to describe the topological structure of a distributed system. The graph structure shows of which components the system consists and how they interact with each other. It is convenient to introduce graph transformations to simulate dynamic changes in system structures, for example, redistribution of several components of the system, creating or eliminating communication channels.

The techniques available to specify local states of distributed components do not account for the possibility of representing fuzzy information in separate components of a distributed system. To develop such techniques, fuzzy graphs can be used that model complex relationships of objects inside local components of a software system, which occur, for example, in simulating software architecture [5]. Transformations of fuzzy graphs are defined at this level to specify changes in relations among objects [6–9].

However, formal models (introduced in [10, 11]) of transformation of distributed systems represented by graphs with the nodes being, in turn, fuzzy graphs have not been studied.

In this paper, we attempt to construct and analyze such models. More exactly, the objective of the paper is to develop conceptual fundamentals for the theory of fuzzy graph transformation of a distributed medium using methods of the theory of categories. (See [12] for other applications of category methods in computer science.)

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CATEGORICAL APPROACH TO FUZZY GRAPH TRANSFORMATION

In this section, we consider concepts that continue the ideas of the original categorical approach to fuzzy graph transformation introduced in [6, 7]. In contrast to [6], we define productions over fuzzy graphs by two morphisms that correspond to eliminating and adding elements of a graph, which makes it possible to characterize more precisely the applicability conditions for such productions. See [13, 14] for the other concepts of the theory of categories, including definitions of pushout and colimit.

Definition 1. A fuzzy graph $G=(G^E, G^N, s_G, t_G, \sigma_{GE}, \sigma_{GN})$ is a collection of sets of edges and nodes G^E and G^N, mappings s_G, $t_G: G^E \to G^N$, called the source and target mapping, respectively, and truth mappings for arcs and nodes $\sigma_{GE}: G^E \to [0,1]$ and $\sigma_{GN}: G^N \to [0,1]$, where [0,1] is the interval of truth values for nodes and edges.

The authors showed in [6] that this interval of truth values for edges and nodes of the fuzzy graph G can be generalized to a complete lattice of objects of category **Fuzz**(Q), where Q is a partially ordered support set. Eliminating truth values of nodes and edges of G, we obtain a nonfuzzy graph called the support graph of the fuzzy graph G.

A morphism of fuzzy graphs $f: G \to H$ is a pair of mappings $(f^E: G^E \to H^E, f^N: G^N \to H^N)$ such that f^E and f^N are compatible with the source and target mappings and retain truth values, i.e., $\forall e \in G^E$, $\forall n \in G^N: f^N(s_G(e)) = s_H(f^E(e))$, $f^N(t_G(e)) = t_H(f^E(e))$, $f^E(\sigma_{GE}(e)) = \sigma_{HE}(e)$ and $f^N(\sigma_{GN}(n)) = \sigma_{HN}(n)$, where G and H are fuzzy graphs. A morphism $f=(f^E, f^N)$ of fuzzy graphs is called injective (surjective, bijective) if f^E and f^N are simultaneously injective (surjective, bijective).

Definition 2. A fuzzy graph join $D=(B+C)_{/\bar{I}_A}$ is a set of all equivalence classes of the union of B and C, where $\bar{I}_A \subseteq B \times C$ is the equivalence relation produced by the relation $I_A = \{(f(x), g(x) | x \in A\}$, A, B, and C are fuzzy graphs, and $f: A \to B$ and $g: A \to C$ are morhisms of fuzzy graphs.

Morphisms $h: B \to D$ and $k: C \to D$ of fuzzy graphs transform each element B and C into the corresponding equivalence class in D.

The lemma below shows that the concepts of a fuzzy graph, a morphism of fuzzy graphs, and a fuzzy graph join are associated with the well-known concepts of the theory of categories.

LEMMA 1. Fuzzy graphs and morphisms of fuzzy graphs form a category **FGraph**. The disjoint union $\Sigma_1^n G_i$ of the fuzzy graphs G_i, $1 \le i \le n$, with injective morphisms $\mathrm{in}_i: G_i \to \Sigma_1^n G_i$ is a coproduct of all G_i in **FGraph**. The join structure presented in Definition 2 can be specified as a pushout in the category **FGraph**.

Let us describe a sequential transformation of fuzzy graphs performed by applying one production to the source graph. The production describes the nodes and edges to be eliminated, preserved, and recreated. This is expressed by two fuzzy graphs L and R, the left- and right-hand sides. Moreover, there exists an intermediate fuzzy graph I, we will call it a fuzzy graph join. It shows the preserved elements. Their images in the L and R correspond to the preserved elements. Moreover, the preserved elements can be disjoined or joined (united), which is specified by morphisms to the left and to the right, not necessarily injective.

Definition 3. A production $p=(I \xrightarrow{l} L;\ I \xrightarrow{r} R)$ is a set consisting of fuzzy graphs L, I, and R, called the left-hand side, intermediate (or connecting) graph, and the right-hand side, respectively, of p, and of two morphisms, $I \xrightarrow{l} L$ and $I \xrightarrow{r} R$.

The production p is called l-injective (l-surjective) if the morphism l is injective (surjective); the production p is injective if the morphisms l and r are injective. A production $p=(I \xrightarrow{id_I} I;\ I \xrightarrow{id_I} I)$ consisting of two identical morphisms is called identical production.

For each production to be fulfilled, it is necessary to find fuzzy matching of its left-hand side (ε-matching [6]) in the fuzzy graph to which it is applied. Nodes may be eliminated or disjoined if none context edges are pendant after the elimination, i.e., are not connected to the source or target node. If the node is disjoined from a context edge, it is not clear which of the new nodes should be connected to this edge. Matching should not be the isomorphic image of the left-hand side, but it is necessary to ensure that there are no two identified elements, one of which is eliminated and another is preserved. Moreover, it is necessary to create isomorphic images of the eliminated and disjoined elements. These constraints are formulated in the following join condition.

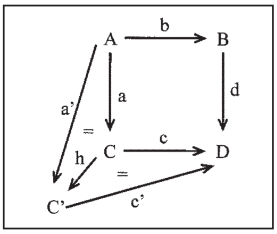

Fig. 1. Diagram of the initial complement of pushout.

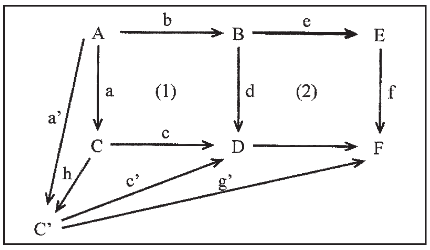

Fig. 2. Pushout composition.

Definition 4. Let a production $p=(I \xrightarrow{l} L;\ I \xrightarrow{r} R)$ be given. A morphism of fuzzy graphs $m{:}L \to G$ satisfies the join condition with respect to p if the following conditions are satisfied:

(i) the identification condition: $\forall x_1, x_2 \in L$ such that $m(x_1) = m(x_2)$, there exist unique $y_1 \in I$ and $y_2 \in I$ such that $l(y_1) = x_1$ and $l(y_2) = x_2$;

(ii) the condition that there are no pendant edges (the no-pendant condition): $\forall e \in G^E - m^E(L^E)$ $L_e^N = \{x \in L^N \mid s_G(e) = m^N(x) \text{ or } t_G(e) = m^N(x)\}$ is defined and $\forall x \in L_e^N$ there exists a unique $y \in I^N$ such that $l^N(y) = x$.

The morphism m is called a matching if it satisfies the join condition with respect to p.

If the join condition holds, the production is applicable with respect to the corresponding fuzzy matching. At the first step, all the elements of the current fuzzy graph G that have a pre-image in the left-hand side but not in the interface graph are eliminated. Moreover, according to the production, parts of the fuzzy graph are disjointed, which results in a fuzzy context graph. It shows a form of a transient transformation state where elimination and disjoint are carried out but creation and join still should be done. Then the resultant fuzzy graph is constructed.

Definition 5. A fuzzy context graph C is defined as $C = G - m(L - l(I)) - m(l(\downarrow(p))) + \downarrow(p)$, where $p=(I \xrightarrow{l} L;\ I \xrightarrow{r} R)$ is a production, $m{:}L \to G$ is a matching satisfying the join condition, and disjoint is given by $\downarrow(p) = \{x \in I \mid \exists x' \neq x \in I : l(x) = l(x')\}$.

The step of elimination and disjoint can be described as the inverse to a fuzzy graph join. Join of the left-hand side L and the context fuzzy graph C, which are superimposed in the fuzzy interface graph I, generates the initial fuzzy graph G. If an l-injective production is applied to a fuzzy graph, a context graph can be described by using complement of pushout. Moreover, application of an l-surjective production to a fuzzy graph can be specified by a so-called initial complement of pushout. Since any morphism of fuzzy graphs can be decomposed into a surjective one followed by injective one, such an operation can be performed for any left-hand side morphism of the production. Constructions over both parts are described by pushouts; therefore, both the elimination as a whole and the step of disjoining can be modeled by a pushout.

LEMMA 2. Let $p=(I \xrightarrow{l} L;\ I \xrightarrow{r} R)$ be an l-injective production and $m{:}L \to G$ be a matching of fuzzy graphs. There exist a fuzzy context graph C and two morphisms of fuzzy graphs, $g{:}C \to G$ and $c{:}I \to C$, such that (G, m, g) is a pushout over l and m in the category **FGraph**. The set (C, c, g) is called a complement of the pushout (G, m, l).

Definition 6. A complement of the pushout $(C, a{:}A \to C,\ c{:}C \to D)$ is initial if for any complements $(C', a'{:}A \to C', c'{:}\ C' \to D)$ over the morphisms $b{:}A \to B$ and $d{:}B \to D$, there exists a unique morphism $h{:}C \to C'$ such that $h \circ a = a'$ and $c' \circ h = c$ (Fig. 1).

LEMMA 3. The initial complement of the pushout $(C, a{:}A \to C,\ c{:}C \to D)$ over the morphisms $b{:}A \to B$ and $d{:}B \to D$ is unique up to isomorphism.

Proof. Assume that there exist two initial complements, $(C, a{:}A \to C,\ c{:}C \to D)$ and $(C', a'{:}\ A \to C',\ c'{:}C' \to D)$. Then there exist morphisms $h{:}\ C \to C'$, where $h \circ a = a'$, $c \circ h = c'$, and $h'{:}C' \to C$, where $h' \circ a' = a$, $c' \circ h' = c$. Moreover, there exists a morphism $h' \circ h$, where $c \circ h' \circ h = c' \circ h = c$ and $h' \circ h \circ a = h' \circ a' = a$. Since there also exists $id_C{:}\ C \to C$, where $c \circ id_C = c$, $id_C \circ a = a$, we obtain $h' \circ h = id_C$. Similarly, we obtain $h \circ h' = id_{C'}$. Thus, C and C' are isomorphic. ■

LEMMA 4. Let $p=(I \xrightarrow{l} L;\ I \xrightarrow{r} R)$ be an l-surjective production and $m{:}L \to G$ be a matching of fuzzy graphs. There exist a fuzzy context graph C and two morphisms of fuzzy graphs, $g{:}C \to G$ and $c{:}I \to C$, such that (C, c, g) is the initial complement of pushout over l and m in the category **FGraph**.

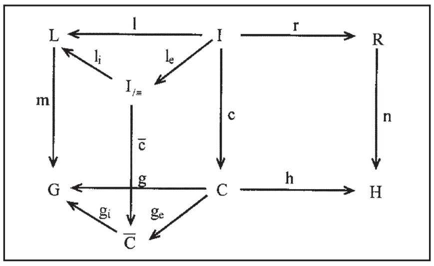

Fig. 3. Diagram of a transformation of fuzzy graphs based on double pushout with disjoint.

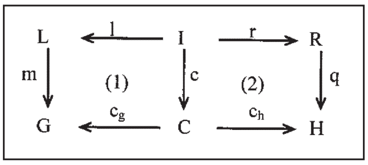

Fig. 4. Direct transformation.

LEMMA 5. Let the diagram in Fig. 2 be given, where (1) and (2) are pushouts, the morphism e is injective, and (C, a, c) is the initial complement of b and d. The set $(C, a, g \circ c)$ is the initial complement of pushout in the category **FGraph**.

Proof. A joined diagram (1), (2) is a pushout according to the general properties of pushouts. Let (C', a', g') be a complement presented in Fig. 2. It is necessary to show that there exists a unique morphism $h: C \to C'$ such that $h \circ a = a'$ and $g' \circ h = c \circ g$. According to the assumptions, $g' \circ a' = f \circ e \circ b = g \circ c \circ a$. It is possible to determine that $c' = g^{-1} \circ g'$; g^{-1} is givent for all $g'(x)$, where $x \in C'$, since g and f are surjective according to pushout (2); c' is well defined since for pushouts in **FGraph**, g is injective if e is injective. According to the initial complement (C, a, c), we obtain a unique morphism h, where $h \circ a = a'$ and $c' \circ h = c$. From $g \circ c' = g'$ we get $g' \circ h = g \circ c' \circ h = g \circ c$; therefore $(C, a, g \circ c)$ is initial. ■

LEMMA 6. Let $p = (I \xrightarrow{l} L;\ I \xrightarrow{r} R)$ be a production and $m: L \to G$ be a matching of fuzzy graphs. Then there exist a fuzzy context graph C and complete morphisms of fuzzy graphs $g: C \to G$ and $c: I \to C$ that form the initial complement of pushout m and l.

Proof. It is possible to decompose the morphism l into a surjective morphism $l_e: I \to I_{/\equiv}$ and an injective morphism $l_i: I_{/\equiv} \to L$ (Fig. 3). Accounting for the morphisms m and l_i, we use Lemma 2 to construct the complement $(\overline{C}, g_i: \overline{C} \to G,\ \bar{c}: I_{/\equiv} \to \overline{C})$. Since m satisfies the join condition and $\bar{c}$ is some constraint of m, $\bar{c}$ also satisfies the join condition. Moreover, there exists the initial complement $(C, c: I \to C,\ g_e: C \to \overline{C})$ according to Lemma 4. Let $g = g_i \circ g_e$. According to Lemma 5, there exists the initial complement (C, c, g). ■

It can be shown that $\overline{C}$ is a unique fuzzy complement graph of pushout. But C is not a unique possible fuzzy complement graph. All the remaining fuzzy complement graphs are in a sense decompositions into elements of C. Therefore, we will determine a unique morphism from C into another fuzzy complement graph that is described by the additional property of pushout complement, namely, of being initial. Now it is possible to prove that the construction of C is unique up to isomorphism.

LEMMA 7. If $(C, g: C \to G,\ c: I \to C)$ is the initial complement of $l: I \to L$ and $m: L \to G$, then m satisfies the join condition with respect to the production $p = (I \xrightarrow{l} L;\ I \xrightarrow{r} R)$.

Definition 7. Let a production $p = (I \xrightarrow{l} L;\ I \xrightarrow{r} R)$ and a matching $m: L \to G$ be given. Direct transformation $G \Rightarrow H$ based on p and m (or $G \overset{p,m}{\Rightarrow} H$) from the fuzzy graph G into the fuzzy graph H is given by pushout diagrams (1) and (2) in the category **FGraph**, shown in Fig. 4, (C, c, c_g) being the initial complement.

LEMMA 8. Let $p = (I \xrightarrow{l} L;\ I \xrightarrow{r} R)$ be a production and $m: L \to G$ be a matching of fuzzy graphs. A direct transformation $G \overset{p,m}{\Rightarrow} H$ is unique up to isomorphism.

Proof follows from Lemma 6 based on the fact that pushouts are unique up to isomorphism.

CATEGORY OF FUZZY DISTRIBUTED GRAPHS

States of a distributed system can be described by fuzzy distributed graphs. To describe transformation of fuzzy distributed graphs (*FD*-graphs), we introduce a morphism of *FD*-graphs, which is a special form of morphism. Moreover, it is necessary to construct a pushout of morphisms of *FD*-graphs to analyze possibilities of joining *FD*-graphs. In this section, we specify the sufficient and necessary existence conditions for componentwise construction of such a pushout in view of the distribution of components of the graph.

$$\begin{tikzcd} i \arrow[d, "e"'] & \tilde{G}(i) \arrow[r, "\tilde{f}_i"] \arrow[d, "\tilde{G}(e)"'] & \tilde{H}(f(i)) \arrow[d, "\tilde{H}(f(e))"] \\ j & \tilde{G}(j) \arrow[r, "\tilde{f}_j"'] & \tilde{H}(f(j)) \end{tikzcd}$$

Fig. 5. Diagram of morphism of *FD*-graphs.

$$\begin{tikzcd} i \arrow[d, "e"'] & \tilde{G}(i) \arrow[r, "\tilde{g}_i"] \arrow[d, "\tilde{G}(e)"'] & \tilde{K}(g(i)) \arrow[r, "\tilde{h}_{g(i)}"] \arrow[d, "\tilde{K}(g(e))"'] & \tilde{H}(h(g(i))) \arrow[d, "\tilde{H}(h(g(e)))"] \\ j & \tilde{G}(j) \arrow[r, "\tilde{g}_j"] & \tilde{K}(g(j)) \arrow[r, "\tilde{h}_{g(j)}"] & \tilde{H}(h(g(j))) \end{tikzcd}$$

Fig. 6. Composition of morphisms of *FD*-graphs.

Definition 8. A diagram $\tilde{G}:G \to$ **FGraph** is called a fuzzy distributed graph (*FD*-graph), where G is a graph of the category **Graph**, called a graph of a network of components, and **FGraph** is a category of fuzzy graphs and their complete morphisms that preserve truth values.

Denote a fuzzy graph by $\tilde{G}(i)$ for any network node $i \in G^N$. For any network edge $e \in G^E$, the fuzzy graph $\tilde{G}(s(e))$ is a graph of interface, or the source graph, $\tilde{G}(t(e))$ is called a target graph, and $\tilde{G}(e)$ specifies the complete morphism of fuzzy graphs.

If it is obvious which mapping to the source or to the target is meant, we write $\tilde{G}(s(e))(\tilde{G}(t(e)))$ rather than $\tilde{G}(s_G(e))\ (\tilde{G}(t_G(e)))$.

Definition 9. A morphism $\tilde{f} = (\eta, f\ ): \tilde{G} \to \tilde{H}$ of *FD*-graphs is a natural transformation $\eta\ = (\tilde{f}_i\ )_{i \in G^N}$ in the category **FGraph**, where $\tilde{G}$ and $\tilde{H}$ are *FD*-graphs, $f: G \to H$ is the existing morphism of graphs in the **Graph**, called a network morphism.

If f is injective (surjective), then $\tilde{f}$ is called *n*-injective (*n*-surjective). If, moreover, all $\tilde{f}_i$, where $i \in G^N$, are injective, then $\tilde{f}$ is also called injective.

The morphism $\tilde{f}$ of *FD*-graphs associates each node $i \in G^N$ with a morphism of fuzzy graphs $\tilde{f}_i : \tilde{G}(i) \to \tilde{H}(f(i))$, called local morphism of graphs such that $\forall e \in G^E$, where $s(e) = i$ and $t(e) = j$, the diagram in Fig. 5 commutes. In particular, if fuzzy graphs $\tilde{G}(i)$ and $\tilde{G}(j)$ connected by an edge $\tilde{G}(e\ )$ are given, and they are mapped by local morphisms of graphs $\tilde{f}_i$ and $\tilde{f}_j$ onto graphs $\tilde{H}(f(i))$ and $\tilde{H}(f(j))$, then the edge $\tilde{G}(e)$ should be mapped onto $\tilde{H}(f(e))$.

Definition 10. A composition $\tilde{f} = (\eta_{f,}\ f\ ): \tilde{G} \to \tilde{H}$ of two morphisms $\tilde{g} = (\eta_{g,}\ g): \tilde{G} \to \tilde{K}$ and $\tilde{h} = (\eta_h, h): \tilde{K} \to \tilde{H}$ of *FD*-graphs is defined as follows: $f = h \circ g$ and $\eta_f\ = (\eta_h \circ g) \circ \eta_g : \tilde{G} \to \tilde{K} \circ g\ \to \tilde{H} \circ h \circ g$.

LEMMA 9. All *FD*-graphs and morphisms of *FD*-graphs form the category **FDGraph**.

Proof. A composition $\tilde{f}: \tilde{G} \to \tilde{H}$ of two morphisms $\tilde{g}: \tilde{G} \to \tilde{K}$ and $h\ : \tilde{K} \to \tilde{H}$ of *FD*-graphs is a morphism of *FD*-graphs. According to Definition 10, $\tilde{f}_i = \tilde{h}_{g(i)} \circ \tilde{g}_i$ for all $i \in G^N$ and for all $e \in G^E$ such that $s_G(e) = i$ and $t_G(e) = i$ diagrams in Fig. 6 commute. Since the composition of morphisms of *FD*-graphs is obtained from morphisms of fuzzy graphs, it is associative.

Identical morphisms $i\tilde{d}_{\tilde{G}} = (\eta_{id_G}, id_G\ ): \tilde{G} \to \tilde{G}$ for any *FD*-graph $\tilde{G}$ are defined by the identities $i\tilde{d}_{\tilde{G}(i)}: \tilde{G}(i) \to \tilde{G}(i)$ for all $i \in G^N$. The equalities $i\ \tilde{d}_{\tilde{G}} \circ \tilde{g} = \tilde{g}$ and $\tilde{f} \circ i\tilde{d}_{\tilde{G}} = \tilde{f}$ hold for all morphisms $\tilde{f} = (\eta_f, f\ ): \tilde{G} \to \tilde{H}$ and $\tilde{g} = (\eta_{g,}\ g\ ): \tilde{H} \to \tilde{G}$ of *FD*-graphs since $id_G \circ g = g$ and $\eta_g\ = (\eta_{id_G} \circ g) \circ \eta_g$: $\tilde{H} \to \tilde{G} \circ g \to \tilde{G} \circ g$. Moreover, $f \circ id_G\ = f$ and $\eta_f\ = \eta_f \circ \eta_{id_G}$: $\tilde{G} \to \tilde{G} \to \tilde{H}\ \circ f$. ■

LEMMA 10. Let $\tilde{G}$ be an *FD*-graph, $\Sigma\tilde{G}(n)$ be a coproduct of all $\tilde{G}(n)$, $n \in G^N$, and $i_n: \tilde{G}(n) \to \Sigma\tilde{G}(n)$ be the corresponding injections. Let $\equiv_G$ be the equivalence closure $\{(i_{s(e)}(x),\ i_{t(e)}\ (\tilde{G}(e)(x))) | x \in \tilde{G}(s(e)),\ e \in G^E\}$. Then $\tilde{G}' = \Sigma\tilde{G}(n)_{/\equiv_G}$; $\tilde{g}': \Sigma\tilde{G}(n) \to \tilde{G}'$ is a function of partition into equivalence classes, and the morphisms $\tilde{g}' \circ i_n: \tilde{G}(n) \to \tilde{G}$ $\forall n \in G^N$ specify the colimit of $\tilde{G}$.

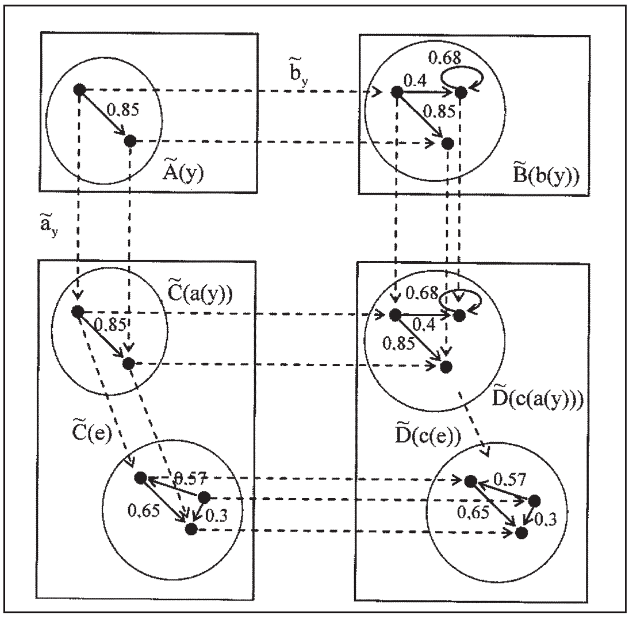

Fig. 7. Morphisms of *FD*-graphs with no join possible.

Proof. Let us construct a colimit of an arbitrary diagram in a category **Set** of sets and functions. This construction may be valid for the **FGraph** category, which is described as a comma category over **Set**. ■

Definition 11. The kernel of a morphism a is a set of pairs $\approx_a = \{(x, y) \in A \times A | a(x) = a(y)\}$.

Denote by $equiv(\approx_a)$ the minimum equivalence relation generated by $\approx_a$. For the equivalence relation $\approx$, let $\widetilde{A}_{[x]}$ be an *FD*-graph consisting of all fuzzy graphs $\widetilde{A}(y)$ for $y \in [x]$ and all morphisms of fuzzy graphs $\widetilde{A}(e)$ for $s(e), t(e) \in [x]$ of each class $[x] \in A^N_{/\approx}$. The graph $\widetilde{A}_{[x]}$ is called a choice of join $[x]$ of the *FD*-graph $\widetilde{A}$.

We denote the equivalence relation $\approx$ for nodes (edges) by $\approx^N$ $(\approx^E)$. Let $a(\approx) = \{(a(x), a(y) \in B \times B) | (x, y) \in \approx\}$.

LEMMA 11. A morphism of fuzzy graphs $\widetilde{f}: \widetilde{G} \to \widetilde{H}$ is a composition of an *n*-surjective morphism $\widetilde{g}: \widetilde{G} \to \widetilde{K}$ of *FD*-graphs and an injective morphism $\widetilde{h}: \widetilde{K} \to \widetilde{H}$ of *FD*-graphs such that $\widetilde{f} = \widetilde{h} \circ \widetilde{g}$. Moreover, the fuzzy graph $\widetilde{K}$ is unique up to isomorphism, $\widetilde{g}$ is an epimorphism, and $\widetilde{h}$ is a monomorphism.

Proof. It is possible to decompose the morphism of graphs f into a surjective morphism of graphs g and an injective morphism of graphs h, where $f = h \circ g$ is such that K is unique up to isomorphism.

Let the set $(C_{[x]}, \cup_{y \in [x]} c_y : \widetilde{G}(y) \to C_{[x]})$ be a coproduct for all $[x] \in G^N_{/\approx f}$. Moreover, $\forall [x] \in G^N_{/\approx f}$ $u_{[x]}$: $C_{[x]} \to \widetilde{H}(f(x))$ are generated morphisms of the coproduct. Each of them can be decomposed into a surjective morphism $s_{[x]} : C_{[x]} \to \widetilde{K}(g(x))$ of graphs and an injective morphism $h_{[x]} : \widetilde{K}(g(x)) \to \widetilde{H}(f(x))$ of fuzzy graphs. $\widetilde{K}(g(e))$ is defined as $h^{-1}_{[t(e)]} \circ \widetilde{H}(f(e)) \circ h_{[s(e)]}$ for each edge $e \in G^E$ and $\widetilde{g}_x = s_{[x]} \circ c_x$ $\forall x \in G^N$. It remains to show that $\widetilde{g}$ and $\widetilde{h}$ are morphisms of *FD*-graphs and $\widetilde{K}$ is unique up to isomorphism. Moreover, it is necessary to show that $\widetilde{g}$ and $\widetilde{h}$ are an epi- and monomorphism, respectively. ■

A pushout over *n*-injective morphismsof *FD*-graphs $\widetilde{a}: \widetilde{A} \to \widetilde{C}$ and $\widetilde{b}: \widetilde{A} \to \widetilde{B}$ can be constructed componentwise in the following steps. First, a pushout on morphisms of the network is created. For all nodes of the network in A, a pushout on their morphisms of fuzzy graphs in $\widetilde{B}$ and $\widetilde{C}$ is constructed. All the other fuzzy graphs $\widetilde{B}$ and $\widetilde{C}$ are transferred into $\widetilde{D}$ unchanged. A network edge in D, which has a pre-image in A, is supplemented with a generated morphism between its source and target fuzzy graphs of pushout. All the other morphisms $\widetilde{B}$ and $\widetilde{C}$ of fuzzy graphs are defined with respect to their new target graphs. The source graphs should be mapped into structural-equivalent ones according to pushout conditions.

It is possible to verify that a join of *FD*-graphs in which a join of corresponding graphs of the network is computed first is not distributed. Graphs, especially with injective morphisms among them, can be joined elementwise. For example, a node in *B* and a node in *C* that have a common origin in *A* are glued. All the remaining nodes are just copied into *D*. These actions can be performed in parallel.

A componentwise construction of a pushout based on *n*-injective morphisms can always be performed if the following two pushout conditions are satisfied. These conditions require, for each edge of the network having a source node glued to another one, that a fuzzy graph of this node be mapped isomorphically to the pushout graph.

Definition 12. Pushout conditions (i) and (ii) are satisfied for *n*-injective morphisms $\widetilde{a}:\widetilde{A}\to\widetilde{C}$ and $\widetilde{b}:\widetilde{A}\to\widetilde{B}$ of *FD*-graphs if:

(i) $\forall e\in C^E - a(A^E)\ \forall y\in A^N: a(y)=s(e)\Rightarrow \widetilde{b}_y$ is surjective and $\approx_{\widetilde{b}_y}\subseteq\approx_{\widetilde{a}_y}$;

(ii) $\forall e\in B^E - b(A^E)\ \forall y\in A^N: b(y)=s(e)\Rightarrow \widetilde{a}_y$ is surjective and $\approx_{\widetilde{a}_y}\subseteq\approx_{\widetilde{b}_y}$.

Figure 7 illustrates a situation where the pushout condition (i) (Definition 12) is not satisfied. All the morphisms of fuzzy graphs are depicted by dashed arrows between the corresponding nodes. Mapping of edges can be obtained uniquely. Morphisms of networks are not shown explicitly.

The morphism $\widetilde{b}_y$ is not surjective. A node with a closed loop and its adjacent edges in $\widetilde{B}(b(y))$, which are copied in $\widetilde{D}(c(a(y)))$, cannot be mapped as $\widetilde{D}(c(e\ ))$.

THEOREM 1. Let $\widetilde{a}:\widetilde{A}\to\widetilde{C}$ and $\widetilde{b}:\widetilde{A}\to\widetilde{B}$ be *n*-injective morphisms of *FD*-graphs that satisfy the pushout conditions (i) and (ii). Then the pushout of $\widetilde{a}$ and $\widetilde{b}$ in the **FDGraph** category exists and can be constructed componentwise.

Proof. Let *D*, $c:C\to D$ and $d:B\to D$ form the pushout of *a* and *b* in the **FGraph**. The way it is constructed is specified as a fuzzy graph join according to Definition 2. The pushout graph $\widetilde{D}: D\to$ **FGraph** is constructed as follows.

Let $\mathrm{PUSH}(\widetilde{a}_y, \widetilde{b}_y)$ be a pushout graph over $\widetilde{a}_y$ and $\widetilde{b}_y$. For each node $x\in D^N$ there exists at least one node $y\in A^N$ such that $c\circ a(y)=x$ since *a* and *b* are injective. Therefore, we can uniquely denote $\mathrm{PUSH}(\widetilde{a}_y, \widetilde{b}_y)$ by PUSH_x. Let us define the following mapping:

$$\widetilde{D}(x)=\begin{cases}\mathrm{PUSH}_x & \text{if } \exists y\in A^N, \text{ where } c\circ a(y)=x,\\ \widetilde{C}(z) & \text{if } \exists z\in C^N - a(A^n), \text{ where } c(z)=x,\\ \widetilde{B}(v) & \text{if } \exists v\in B^N - b(A^n), \text{ where } d(v)=x,\\ \mathrm{GEN}(\mathrm{PUSH}_{s(x)}, \mathrm{PUSH}_{t(x)}) & \text{if } \exists e\in A^E, \text{ where } c\circ a(e)=x,\\ \widetilde{c}_{t(x)}\circ\widetilde{C}(e)\circ\widetilde{c}^{-1}_{s(x)} & \text{if } \exists e\in C^E - a(A^E), \text{where } c(e)=x,\\ \widetilde{d}_{t(x)}\circ\widetilde{B}(e)\circ\widetilde{d}^{-1}_{s(x)} & \text{if } \exists e\in B^E - b(A^E), \text{ where } d(e)=x.\end{cases}$$

Here $\mathrm{GEN}(\mathrm{PUSH}_{s(x)}, \mathrm{PUSH}_{t(x)})$ is a generated morphism from the pushout graph $\widetilde{D}(s(x))$ into the pushout graph $\widetilde{D}(t(x))$, where $\widetilde{a}_{t(e)}\circ\widetilde{A}(e)=\widetilde{C}(a(e))\circ\widetilde{a}_{s(e)}$ and $\widetilde{b}_{t(e)}\circ\widetilde{A}(e)=\widetilde{B}(b(e))\circ\widetilde{b}_{s(e)}$. The morphisms of the pushout $\widetilde{c}:\widetilde{C}\to\widetilde{D}$ and $\widetilde{d}:\widetilde{B}\to\widetilde{D}$ are defined as follows:

$$\widetilde{c}_x=\begin{cases} id_{\widetilde{C}(x)} & \text{if } x\in C^N - a(A^N),\\ \text{morphism of } \mathrm{PUSH}_{c(x)} & \text{otherwise,}\end{cases}\qquad \widetilde{d}_x=\begin{cases} id_{\widetilde{B}(x)} & \text{if } x\in B^N - b(A^N),\\ \text{morphism of } \mathrm{PUSH}_{d(x)} & \text{otherwise.}\end{cases}$$

It is easy to show that $\widetilde{D}$ is an *FD*-graph according to the above definition, properties, and conditions of pushout. Then we use generated morphisms of local pushouts $\mathrm{PUSH}_x\ \ \forall x\in A^N$ to show that both $\widetilde{c}$ and $\widetilde{d}$ are morphisms of *FD*-graphs.

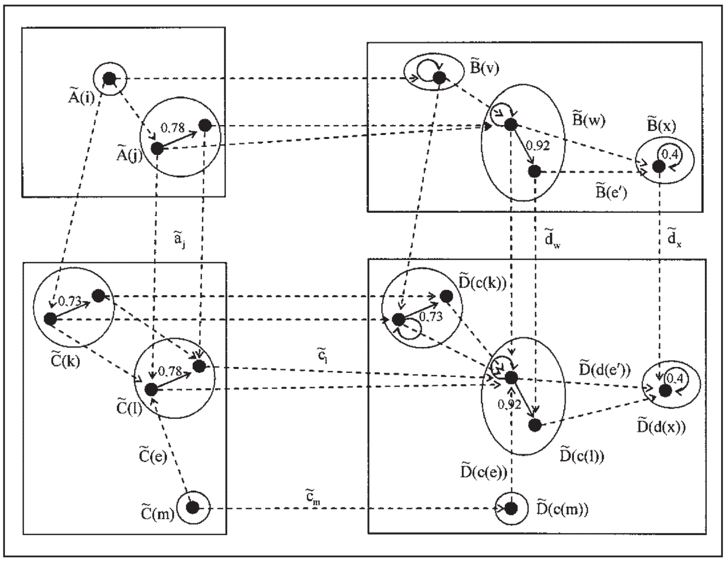

Fig. 8. Pushout structure in the **FDGraph** category.

It is necessary to show pushout properties. The commutativity $\widetilde{c} \circ \widetilde{a} = \widetilde{d} \circ \widetilde{b}$ follows immediately from the construction. For an *FD*-graph $\widetilde{X}$ and morphisms $\widetilde{f}: \widetilde{B} \to \widetilde{X}$, $\widetilde{g}: \widetilde{C} \to \widetilde{X}$ such that $\widetilde{f} \circ \widetilde{b} = \widetilde{g} \circ \widetilde{a}$, the generated morphism $\widetilde{u}$ consists of the generated morphisms for those that underlie pushouts in the **FGraph** category, where they exist. Thus, for all $y \in A^N$, where $c \circ a(y) = x$, $\widetilde{u}_x$ is a morphism generated by PUSH$_x$. For all $z \in C^N - a(A^N)$ such that $c(z) = x\,(v \in B^N - b(A^N))$, where $d(v) = x$), the following holds: $\widetilde{u}_x = \widetilde{g}_z\ (\widetilde{u}_x = \widetilde{f}_v\ )$.

The fact that $\widetilde{u}$ can be determined follows from the universal property of local pushouts and the pushout condition. It is necessary to show that $\widetilde{u}$ is completely defined. This is true in our case since *c* and *d* are jointly surjective. Therefore, there exists a morphism u_x for all $x \in D^N$. Moreover, $\widetilde{u}_j \circ \widetilde{D}(e) = \widetilde{X}(u(e)) \circ \widetilde{u}_i$, where $s(e) = i$ and $t(e) = j$, can be proved based on the universal property of local pushouts and the pushout condition.

Then we should show that $\widetilde{u} \circ \widetilde{c} = \widetilde{g}$. The equality $u \circ c = g$ holds by virtue of the properties of the pushout of *D*. According to the definition of $\widetilde{u}$, we obtain $\widetilde{u}_{c(x)} \circ \widetilde{c}_x = \widetilde{g}_x$ for all $x \in C^N$. The properties of the composition $\widetilde{u} \circ \widetilde{d}$ can be proved in a similar way.

The uniqueness of $\widetilde{u}$, which means that $\widetilde{u} = \widetilde{u}'$ for all $\widetilde{u}': \widetilde{D} \to \widetilde{X}$ such that $\widetilde{u}' \circ \widetilde{c} = \widetilde{g}$ and $\widetilde{u}' \circ \widetilde{d} = \widetilde{f}$, can be obtained immediately from the definition of $\widetilde{u}$. Assume there exists a node $y \in D^N$ such that $\widetilde{u}_y \neq \widetilde{u}'_y$ and there exists $x \in C^N$, where $c(x) = y$. Then $\widetilde{u}_y \circ \widetilde{c}_x = \widetilde{g}_x = \widetilde{u}'_y \circ \widetilde{c}_x$. Therefore, the morphism $\widetilde{u}$ is unique. The proof is completed. ■

Figure 8 shows an example of a pushout on two *n*-injective morphisms of *FD*-graphs in the **FDGraph** category. The pushout conditions are satisfied since $\widetilde{a}_j : \widetilde{A}(j) \to \widetilde{C}(l)$ is bijective. All the other morphisms of fuzzy graphs cannot be injective or surjective. A morphism of fuzzy graphs between $\widetilde{D}(c(k))$ and $\widetilde{D}(c(l))$ is constructed as a generated morphism between both pushout graphs. Since $\widetilde{a}_j$ is bijective, it clear how the morphism of fuzzy graphs $\widetilde{D}(d(e'))$ should be constructed. Such a morphism is defined as $\widetilde{D}(d(e')) = \widetilde{d}_x \circ \widetilde{B}(e') \circ d_w^{-1}$. If, for example, $\widetilde{C}(l)$ contains an additional node, the morphism $\widetilde{d}_w : \widetilde{B}(w) \to \widetilde{D}(c(l))$ is not bijective and cannot be inverted. The morphism $\widetilde{D}(c(e))$ of fuzzy graphs is constructed as $\widetilde{D}(c(e)) = \widetilde{c}_l \circ \widetilde{C}(e) \circ \widetilde{c}_m^{-1}$.

Figure 7 shows the case where a pushout cannot be constructed componentwise. The theorem below states that pushout conditions (i) and (ii) are not only sufficient but also necessary for componentwise construction of a pushout in the **FDGraph** category for *n*-injective morphisms of an *FD*-graph.

THEOREM 2. Let $\tilde{a}:\tilde{A}\to\tilde{C}$ and $\tilde{b}:\tilde{A}\to\tilde{B}$ be *n*-injective morphisms of *FD*-graphs. The pushout of $\tilde{a}$ and $\tilde{b}$ in **FDGraph** can be constructed componentwise if and only if pushout conditions (i) and (ii) are satisfied.

Proof. Theorem 1 states that pushout conditions (i) and (ii) are sufficient to construct the pushout $\tilde{a}$ and $\tilde{b}$ componentwise. It remains to show that the pushout conditions are also necessary for componentwise construction of a pushout. Assume that pushout condition (i) is not satisfied. Then there is a network edge $e\in C^E - a(A^E)$ such that there exists $y\in A^N$, where $a(y)=s(e)$, and $\tilde{b}_y$ is not bijective. If $\tilde{b}_y$ is not surjective, then there exists a node $x\in\tilde{B}(b(y))$ that has no pre-image in $\tilde{A}(y)$. Since $\tilde{d}_{b(y)}$ should be complete, there also exists a node $x'\in\tilde{D}(d(b(y)))=\tilde{D}(c\ \ (s(e)))$. Since $\tilde{D}(c(s(e)))$ is constructed as a graph of pushout of morphisms $\tilde{a}_y$ and $\tilde{b}_y$, the fuzzy graph $\tilde{C}(s(e))$ does not contain a pre-image of x'. According to the componentwise construction, fuzzy graphs such as $\tilde{C}(t(e))$, not having pre-images in $\tilde{A}$, do not change, i.e., $\tilde{D}(c(t(e)))=\tilde{C}(t(e))$. Therefore, it is possible to select a fuzzy graph $\tilde{C}(t(e))=\tilde{D}(c(t(e\ \ )))$ such that a morphism of the fuzzy graph $\tilde{D}(c(e))$: $\tilde{D}(c(s(e)))\to\tilde{D}(c(t(e)))$ does not exist.

If $\approx_{\tilde{b}_y}\not\subset\approx_{\tilde{a}_y}$, then there exists a node $x\in\tilde{B}(b(y))$ that has two pre-images, $z,\ z'\in\tilde{A}(y)$, where $z\neq z'$. Moreover, there exist two nodes, $v,\ v'\in\tilde{C}(s(e))$, such that $\tilde{a}_y(z)=v\neq v'=\tilde{a}_y(z')$. According to the construction of pushout, x should be mapped onto the node $x'\in\tilde{D}(c(s(e)))$. If $\tilde{C}(e\ \ )(v)\neq\tilde{C}(e)(v')$, then it is possible to select a morphism of fuzzy graphs $\tilde{D}(c(e))$:$\tilde{D}(c(s(e)))\to\tilde{D}(c(t(e)))$ such that the node $\tilde{D}(c(s(e)))$ cannot be projected onto a node in $\tilde{D}(c(t(e)))$. ■

Further let us consider a join of *FD*-graphs by arbitrary morphisms of distributed *FD*-graphs. This means that fuzzy graphs inside one distributed graph may be joined. Such a complicated operation is constructed in two steps: first, all local joins are carried out, then *FD*-graphs that precisely correspond to the structure of pushout over *n*-injective morphisms of *FD*-graphs are joined. To perform the construction considering its modularity, we need an additional pushout condition (it is presented below in Definition 14).

Definition 13. The join condition for a network is satisfied for the equivalence relation $\approx$ on an *FD*-graph $\tilde{D}$ if $\forall x\in D^N$, where $x\in[s(e)]$, $\neg\exists e'\in[e]$ for an edge $e\in D^E$, where $s(e')=x$. Let $\tilde{D}_{[s(e)]}$ be a choice of join, and $\tilde{D}'([s(e)])$ is the corresponding graph of the colimit, then $\forall[y]\in\tilde{D}([s(e)])\ \exists y'\in[y]$: $y'\in\tilde{D}(s(e))$.

Definition 14. An additional pushout condition is satisfied for the morphisms $\tilde{a}:\tilde{A}\to\tilde{C}$ and $\tilde{b}:\tilde{A}\to\tilde{B}$ of *FD*-graphs if:

(a) $\tilde{A}$ together with $equiv(\approx_a\cup\approx_b)$ satisfies the join condition for the network;

(b) $\tilde{B}$ together with $equiv(b(\approx_a))$ satisfies the join condition for the network;

(c) $\tilde{C}$ together with $equiv(a(\approx_b))$ satisfies the join condition for the network.

Construction of a pushout over morphisms of *FD*-graphs that may not be *n*-injective stipulates the same construction over *n*-injective morphisms. All the requirements of the morphisms that connect the network are first met. This means that all the fuzzy graphs in $\tilde{A}$ should be joined whose network nodes in *A* are in the same equivalence class generated by $\approx_{ab}$, i.e., the network node indicating this equivalence class is supplemented with respectively joined graphs. Fuzzy graphs are so joined in $\tilde{B}$ and in $\tilde{C}$. Since the same equivalence relation $\approx_{ab}$ is used in $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$, there appears a one-to-one relationship between the corresponding classes and the joined graphs. Therefore, morphisms of fuzzy graphs can be selected as those generated according to the joined graphs. The same ideas are followed in connecting network edges supplemented with morphisms of graphs. The resultant edge class is supplemented with a morphism generated with respect to the source graph join and all the morphisms belonging to edges in this class. By virtue of the additional pushout condition, all the equivalence classes in the source graph join should contain an element in the source graph, which is mapped by a morphism that belongs to the edge in the corresponding class.

In the general case, pushouts exist for each of the two morphisms $\tilde{a}:\tilde{A}\to\tilde{C}$ and $\tilde{b}:\tilde{A}\to\tilde{B}$ of *FD*-graphs in the **FDGraph** category; however, if they do not satisfy the pushout conditions, they cannot be constructed componentwise, i.e., strictly modularwise. We propose here a technique for constructive construction of such a pushout since it is most suitable for the analysis of transformations of distributed systems.

TRANSFORMATIONS OF FUZZY DISTRIBUTED GRAPHS

Let us use a double pushout approach to transformations of fuzzy graphs to analyze transformations of fuzzy distributed graphs. Let us consider transformations of such graphs that do not destroy the structure of distributed graphs and do not cause side effects.

Transformation of fuzzy distributed graphs, where the distributed graph structure varies due to transformations of isolated fuzzy graphs, can be specified by a double pushout over the distributed graphs and morphisms of graphs, i.e., in the **FDGraph** category.

Definition 15. A production over distributed graphs (production of *FD*-graphs) $\tilde{p} = (\tilde{I} \xrightarrow{\tilde{l}} \tilde{L};\ \tilde{I} \xrightarrow{\tilde{r}} \tilde{R})$ consists of *FD*-graphs $\tilde{L}: L \to$ **FGraph**, $\tilde{R}: R \to$ **FGraph**, and $\tilde{I}: I \to$ **FGraph** (called the left-hand side, the right-hand side, and the intermediate *FD*-graph) and morphisms $\tilde{l}$ and $\tilde{r}$ of *FD*-graphs.

If all the $\tilde{p}_x = (\tilde{I}(x) \xrightarrow{\tilde{l}_x} \tilde{L}(l(x));\ \tilde{I}(x) \xrightarrow{\tilde{r}_x} \tilde{R}(r(x)))$ for $x \in I^N$ are left-injective, i.e., all the $\tilde{l}_x$ are injective, then $\tilde{p}$ is also called left-injective. If L, I, and R are the network graphs each consisting of precisely one node, then $\tilde{p}$ is called local. If $L \cong I \cong R$, then $\tilde{p}$ is called synchronized. If $\tilde{p}_x$ is identical (see Definition 3) $\forall x \in I^N$, then $\tilde{p}$ is called a production of network transformation.

A local production describes local action, communication and synchronization can be modeled by synchronized productions. A network of components is controlled by productions of network transformation.

Definition 16. A distributed join condition is satisfied for a production $\tilde{p} = (\tilde{I} \xrightarrow{\tilde{l}} \tilde{L};\ \tilde{I} \xrightarrow{\tilde{r}} \tilde{R})$ of *FD*-graphs and a morphism $\tilde{m}: \tilde{L} \to \tilde{G}$ of *FD*-graphs, which is called the case of matching of *FD*-graph, if the following conditions are satisfied:

(i) the global join condition: $m: L \to G$ satisfies the join condition with respect to $\tilde{p} = (\tilde{I} \xrightarrow{\tilde{l}} \tilde{L};\ \tilde{I} \xrightarrow{\tilde{r}} \tilde{R})$ (the join condition in Definition 4);

(ii) the local join condition: $\forall x \in I^N$: $\tilde{m}_{l(x)}$ satisfies the join conditions with respect to $\tilde{p}_x = (\tilde{I} \xrightarrow{\tilde{l}_x} \tilde{L};\ \tilde{I} \xrightarrow{\tilde{r}_x} \tilde{R})$ and all the $\tilde{p}_x$ are *l*-injective;

(iii) the context condition: let $Context\,(q, n) = H - n(L') + n(l'(I'))$ be a context graph for the production $q = (I' \xrightarrow{l'} L';\ I' \xrightarrow{r'} R')$ and matching $n: L' \to H$, then:

(a) $\forall x, y \in I^N\ \forall e: m(l(x)) \to m(l(y)) \in Context\,(p, m)$:

$$\tilde{G}(e)(Context\,(\tilde{p}_x, \tilde{m}_{l(x)})) \subseteq Context\,(\tilde{p}_y, \tilde{m}_{l(y)}),$$

$$\forall y \in I^N\ \forall e: z \to m(l(y)) \in G^E : \tilde{G}(e)(\tilde{G}(z)) \subseteq Context\,(\tilde{p}_y, \tilde{m}_{l(y)});$$

(b) $\forall x \in I^N\ \forall e: m(l(x)) \to z \in G^E - m(L^E): \tilde{l}_y$ and $\tilde{r}_y$ are bijective;

(iv) the network condition: (a) $\forall x \in L^N - l(I^N)$: $\tilde{m}_x$ is bijective

$$\forall x \in I^N\ \forall e: l(x) \to y \in L^E - l(I^E): \tilde{m}_x \text{ is bijective;}$$

(b) $\forall x \in I^N\ \forall e: r(x) \to y \in R^E - r(I^E)$: $\tilde{m}_{l(y)}$ is bijective.

Assume that $\tilde{m}$ satisfies all the local no-pendant conditions if $\tilde{m}_{l(x)}$ satisfies the no-pendant conditions with respect to $\tilde{p}_x\ \forall x \in \tilde{I}^N$.

Let $\tilde{I}'$ be a subgraph of $\tilde{I}$ such that $I' = \downarrow(p)$ and $\approx_{m'} = \{(x, y) \in I \times I \mid m(l(x)) = m(l(y)) \wedge (x \neq y \Rightarrow x,\ y \notin \downarrow(p))\}$. A morphism $\tilde{m}: \tilde{L} \to \tilde{G}$ of *FD*-graphs is an *FD*-matching for the $\tilde{p}$ if the following distributed-join conditions are satisfied additionally:

(i) the global join condition: $m: L \to G$ satisfies the join condition with respect to $p = (I \xrightarrow{l} L;\ I \xrightarrow{r} R)$ (see Definition 4);

(ii) the disjoint condition: (a) $\forall x \in l(\downarrow(p)): \tilde{m}_x$ is bijective or (b) $\tilde{I}'$, where $equiv(\approx_l)$ satisfies the network join condition, and the graph of the network $\tilde{I}_{[y]}$ should be connected for all $y \in \downarrow(p)$, where $l(y) = x$ and $[y] \in I^N_{/\approx l}$;

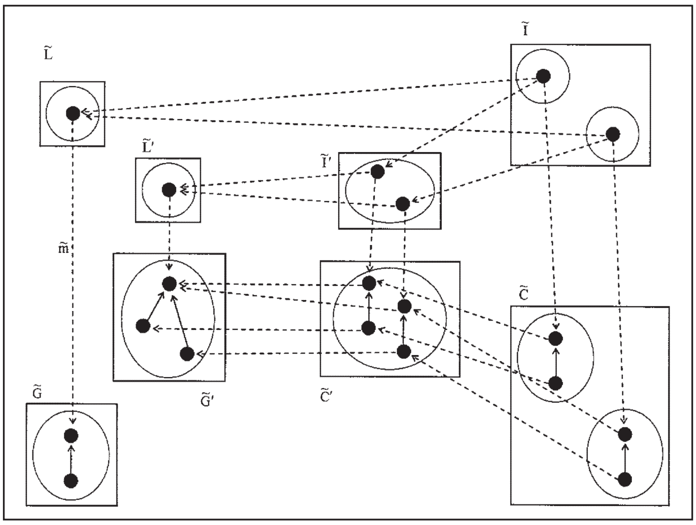

Fig. 9. Morphisms of *FD*-graphs with disjoint impossible.

(iii) the local parallelism condition: $\forall x \in L^N$, where $m(x) = m(y)$ for some $y \in L^N : \tilde{m}_x$ is bijective or (a) $\tilde{I}$, where $equiv(\approx_{m'})$ satisfies the network join condition; (b) $\tilde{L}$, where $equiv(\approx_m)$ satisfies the network join condition; (c) $\tilde{R}$, where $equiv(r(\approx_{m'}))$ satisfies the network join condition;

(iv) the union condition: (a) $\tilde{I}$, where $equiv(\approx_{m'} \cup \approx_r)$ satisfies the network join condition; (b) $\tilde{G}$, where $equiv(m(l(\approx_r)))$ satisfies the network join condition.

The context condition implies the following: (a) an action on the target graph cannot eliminate local elements if some of their copies are not eliminated in coupled source graphs; (b) a local action on the source graph cannot extend it if new elements cannot have images in connected target graphs. Such an action cannot eliminate local elements without eliminating references to the elements in target graphs. Local objects in the source graph cannot be connected if this is not reflected in connected target graphs.

The network condition can be interpreted as follows: (a) network nodes can be eliminated if the fuzzy graph of the node is eliminated as a whole in the same production, i.e., if the current fuzzy graph corresponds to the graph in the production; if a network edge is to be eliminated, its source graph should bijectively correspond to the graph in the production; (b) otherwise, if it is necessary to establish a new connection from the existing source graph, this graph should be a structural equivalent to its correspondent given in the production.

The disjoint condition describes situations of disjoint allowed. Fuzzy graphs to be disjoined are determined completely by a matching or are disjoined in such a way that fuzzy graphs that are copied into the disjoined graphs are connected by interface graphs.

The local parallelism condition expresses a situation where local productions can be applied in parallel. The productions applied to a fuzzy graph in parallel should cover it completely or applications of various parallel productions on graphs of the interface should correspond to each other.

The union condition establishes when fuzzy graphs can be united.

Figure 9 describes the situation that allows no disjoint. The fuzzy graph $\tilde{L}$ with one node should be disjoined into two independent fuzzy graphs in $\tilde{I}$ such that the node is copied. Using the matching $\tilde{m}$, which builds in the node into a greater context, this context is copied in the both fuzzy graphs of the *FD*-graph $\tilde{C}$. Following the construction of pushout for $\tilde{l}$ and $\tilde{c}$, we obtain an *FD*-graph $\tilde{G}'$ rather than *G*.

Such a situation is a general problem of disjoining. If a fuzzy graph is not completely disjoined, the remaining context can also be disjoined arbitrarily, which leads to some form of nondeterminacy. Otherwise, the context should be copied into all the parts involved. At the same time, information on this operation is lost if there is no interface graph for the part disjointed, which preserves the information.

$$\begin{tikzcd}
\widetilde{L} \arrow[d, "\widetilde{m}"'] \arrow[dr, phantom, "(1)"] & \widetilde{I} \arrow[l, "\widetilde{l}"'] \arrow[r, "\widetilde{r}"] \arrow[d, "\widetilde{c}"] \arrow[dr, phantom, "(2)"] & \widetilde{R} \arrow[d] \\
\widetilde{G} & \widetilde{C} \arrow[l, "\widetilde{g}"] \arrow[r] & \widetilde{H}
\end{tikzcd}$$

Fig. 10. Diagram of direct transformation of an *FD*-graph.

Definition 17. An *FD*-graph $\widetilde{C}: C \to \mathbf{FGraph}$ is called a graph of *FD*-context of $\widetilde{p}$ and $\widetilde{m}$ if

$$\widetilde{C}(x) = \begin{cases} \text{the greatest subgraph } \widetilde{G}(g(x)) - \cup_{y \in \{z | c(z) = x\}} \widetilde{m}_{l(y)} (\widetilde{L}(l(y)) \\ - \widetilde{l}_y (\widetilde{I}(y))) & \text{if } x \in C^N, \\ \widetilde{g}_{t(x)}^{-1} \circ \widetilde{G}(g(x)) \circ \widetilde{g}_{s(x)} & \text{if } x \in C^E, \end{cases}$$

where $\widetilde{p} = (\widetilde{I} \xrightarrow{\widetilde{l}} \widetilde{L};\ \widetilde{I} \xrightarrow{\widetilde{r}} \widetilde{R})$ are productions of *FD*-graphs, $\widetilde{m}: \widetilde{L} \to \widetilde{G}$ is an *FD*-matching, $C = G - m(L - l(I)) - m(l(\downarrow(p))) + \downarrow(p)$ is the context graph of p and m in the **Graph** category, and the morphisms of the *FD*-graphs $\widetilde{g}: \widetilde{C} \to \widetilde{G}$ and $\widetilde{c}: \widetilde{I} \to \widetilde{C}$ are defined as shown below. Let $c = m \circ l$ and $g = id_{G|C}$, the morphism of *FD*-graphs $\widetilde{c}$ is defined as $\widetilde{c}_y = \widetilde{m}_{l(y)} \circ \widetilde{l}_y\ \forall y \in I^N$, and the morphism of *FD*-graphs $\widetilde{g}_x = id_{\widetilde{G}(x)|\widetilde{C}(x)}\ \ \forall x \in G^N$, otherwise $\widetilde{g}_x = \widetilde{m}_{l(x)} \circ \widetilde{l}_x$.

The complement $(\widetilde{C}(c(y)), \widetilde{g}_{c(y)}, \widetilde{c}_y)$ for $\widetilde{m}_{l(y)}$ and $\widetilde{l}_y$ in the **FGraph** category is constructed for all $y \in I^N$.

LEMMA 12. Let $\widetilde{p} = (\widetilde{I} \xrightarrow{\widetilde{l}} \widetilde{L};\ \widetilde{I} \xrightarrow{\widetilde{r}} \widetilde{R})$ be a left-injective production of an *FD*-graph, $\widetilde{m}: \widetilde{L} \to \widetilde{G}$ be an *FD*-matching, $\widetilde{C}$ be a graph of an *FD*-context, and $\widetilde{g}$ and $\widetilde{c}$ be morphisms of *FD*-graphs, defined according to Definition 17. Then $(\widetilde{G}, \widetilde{m}, \widetilde{g})$ is a pushout of $\widetilde{c}$ and $\widetilde{l}$ in **FDGraph**.

Proof. In this case, $C = G - m(L - l(I))$ and $\forall y \in I^N\ \widetilde{C}(x) = \widetilde{G}(x) - \widetilde{m}_{l(y)} (\widetilde{L}(l(y)) - \widetilde{l}_y (\widetilde{I}(y)))$, where $m \circ l(y) = x$. The graph $\widetilde{C}$ is an *FD*-graph since the join condition for all local transformations and the context condition are satisfied. This means that $\widetilde{C}(x)$ is a fuzzy graph for each $x \in C^N$ since the join condition is satisfied for the application of each local production $\widetilde{p}_x$ with respect to the matching $\widetilde{m}_{l(x)}$. For each $e \in C^E$, $\widetilde{C}(e)$ is a complete morphism of a fuzzy graph since the context condition is satisfied for e with respect to $\widetilde{p}_{s(e)}, \widetilde{p}_{t(e)}, \widetilde{m}_{s(e)}$ and $\widetilde{m}_{t(e)}$. Therefore, $\forall x \in \widetilde{C}(s(e))$ there exists $y \in \widetilde{C}(t(e))$, where $\widetilde{G}(e) \circ \widetilde{g}_{s(e)}(x) = y$. It is easy to show that $\widetilde{g}$ are morphisms of an *FD*-graph.

Further, it is necessary to show the properties of the pushout $(\widetilde{G}, \widetilde{m}, \widetilde{g})$. The commutativity $\widetilde{m} \circ \widetilde{l} = \widetilde{g} \circ \widetilde{c}$ follows from Lemma 2. Let us construct the pushout of $\widetilde{l}$ and $\widetilde{c}$ (this is possible since the distributed join conditions contain the pushout conditions, which are a part of network condition (a) and context condition (b)). We get $(\widetilde{X}, \widetilde{f}: \widetilde{L} \to \widetilde{X}, \widetilde{h}: \widetilde{C} \to \widetilde{X})$. Then it is necessary to show that the resultant pushout graph $\widetilde{X}$ is isomorphic to $\widetilde{G}$. According to the pushout properties, there exists a unique morphism $\widetilde{u}: \widetilde{X} \to \widetilde{G}$ of *FD*-graphs, where $\widetilde{u} \circ \widetilde{f} = \widetilde{m}$ and $\widetilde{u} \circ \widetilde{h} = \widetilde{g}$. Vice versa, an appropriate morphism $\widetilde{w}: \widetilde{G} \to \widetilde{X}$ of *FD*-graphs can be defined as shown below. For any node $x \in m(l(I^N))$, $\widetilde{w}_x$ is a generated morphism according to $(\widetilde{G}(x), \widetilde{m}_x, \widetilde{g}_x)$. For all $z \in C^N - c(I^N)$, where $g(z) = x$, $\widetilde{w}_x = \widetilde{h}_z$, and $\forall v \in L^N - l(I^N)$ where $m(v) = x$, the following holds: $\widetilde{w}_x = \widetilde{f}_v \circ m_v^{-1}$. We may use network condition (a) to show that $\widetilde{w}$ is defined correctly.

It is easy to show that $\widetilde{w}\circ\widetilde{g}=\widetilde{h}$ and $\widetilde{w}\circ\widetilde{m}=\widetilde{f}$. Validity of $\widetilde{w}_z\circ\widetilde{g}_z=\widetilde{h}_z\ \forall z\in C^N$ and $\widetilde{w}_v\circ\widetilde{m}_v=\widetilde{f}_v\ \forall v\in L^N$ follows immediately from the pushout properties or definition of $\widetilde{w}$. It is easy to show that $\widetilde{u}\circ\widetilde{w}=id_{\widetilde{G}}$ and $\widetilde{w}\circ\widetilde{u}=id_{\widetilde{X}}$. Therefore, $\widetilde{G}$ and $\widetilde{X}$ are isomorphic. The proof is completed. ■

Definition 18. The complement of the pushout $(\widetilde{C},\widetilde{c}:\widetilde{I}\to\widetilde{C},\ \widetilde{g}:\widetilde{C}\to\widetilde{G})$ for the morphisms $\widetilde{l}:\widetilde{I}\to\widetilde{L}$ and $\widetilde{m}:\widetilde{L}\to\widetilde{G}$ in the **FDGraph** category is semiinitial if:

(i) (C,c,g) is the initial complement of the pushout of l and m in **FGraph**;

(ii) $\forall x\in I^N:(\widetilde{C}(c(x)),\widetilde{c}_x,\widetilde{g}_{c(x)})$ is the initial complement of the pushout over $\widetilde{m}_{l(x)}$ and $\widetilde{l}_x$ in **FGraph**;

(iii) $\forall x\in C^N-c(I^N)$: $\widetilde{g}_x$ is an identical morphism.

THEOREM 3. A complement of the pushout $(\widetilde{C},\widetilde{c}:\widetilde{I}\to\widetilde{C},\ \widetilde{g}:\widetilde{C}\to\widetilde{G})$ for the morphisms $\widetilde{l}:\widetilde{I}\to\widetilde{L}$ and $\widetilde{m}:\widetilde{L}\to\widetilde{G}$ is unique in the class of all semiinitial complements of the pushouts of $\widetilde{l}$ and $\widetilde{m}$.

Proof. It is easy to show that $(\widetilde{C},\widetilde{c}:\widetilde{I}\to\widetilde{C},\ \widetilde{g}:\widetilde{C}\to\widetilde{G})$ is semiinitial. Assume there exists another semiinitial complement of the pushout $(\widetilde{C}',\widetilde{c}':\widetilde{I}\to\widetilde{C}',\ \widetilde{g}':\widetilde{C}'\to\widetilde{G})$ for the $\widetilde{l}$ and $\widetilde{m}$. There exist morphisms of the graphs $h:C\to C'$ and $h':C'\to C$, where $h\circ h'=id_C$ and $h'\circ h=id_{C'}$ by Definition 18. Therefore, the graph C is isomorphic to the graph C'. There exist morphisms of the fuzzy graphs $\widetilde{h}_x:\widetilde{C}(x)\to\widetilde{C}'(h(x))$ and $\widetilde{h}'_{h(x)}:\widetilde{C}'(h(x))\to\widetilde{C}(x)\ \forall x\in c(I^N)$. Due to property (ii) in Definition 18, $\widetilde{h}_x\circ\widetilde{h}'_{h(x)}=id_{\widetilde{C}(x)}$ and $\widetilde{h}'_{h(x)}\circ\widetilde{h}_x=id_{\widetilde{C}'(h(x))}$ for all the pairs $\widetilde{h}_x$ and $\widetilde{h}'_{h(x)}$. Property (iii) in the same Definition and the properties of initial complements of pushouts are used to show that $\widetilde{h}$ and $\widetilde{h}'$ are morphisms of *FD*-graphs. There hold $\widetilde{h}\circ\widetilde{h}'=id_{\widetilde{C}}$ and $\widetilde{h}'\circ\widetilde{h}=id_{\widetilde{C}'}$. Therefore, $\widetilde{C}$ and $\widetilde{C}'$ are isomorphic. ■

Definition 19. Direct transformation of an *FD*-graph $\widetilde{G}\Rightarrow_{di}\widetilde{H}$ by means of the production of *FD*-graphs $\widetilde{p}=(\widetilde{I}\xrightarrow{\widetilde{l}}\widetilde{L};\ \widetilde{I}\xrightarrow{\widetilde{r}}\widetilde{R})$ and the *FD*-matching $\widetilde{m}:\widetilde{L}\to\widetilde{G}$ (or $\widetilde{G}\overset{p,m}{\Rightarrow}_{di}\widetilde{H}$) from the *FD*-graph $\widetilde{G}$ into the *FD*-graph $\widetilde{H}$ is specified by pushout diagrams (1) and (2) in the **FDGraph** category as shown in Fig. 10.

A sequence of transformations $\widetilde{G}\overset{P\ *}{\Rightarrow}_{di}\widetilde{H}$ of *FD*-graphs is a sequence of $n\geq 0$ transformations of *FD*-graphs $\widetilde{G}=\widetilde{G}_0\Rightarrow_{di}\widetilde{G}_1\Rightarrow_{di}\dots\ \Rightarrow_{di}\widetilde{G}_n=\widetilde{H}$ in terms of productions of *FD*-graphs from the set *P*; $\widetilde{H}$ is also called *FD*-deducible from $\widetilde{G}$ with respect to *P*.

THEOREM 4. Let $\widetilde{p}=(\widetilde{I}\xrightarrow{\widetilde{l}}\widetilde{L};\ \widetilde{I}\xrightarrow{\widetilde{r}}\widetilde{R})$ be productions of *FD*-graphs and $\widetilde{m}:\widetilde{L}\to\widetilde{G}$ be an *FD*-matching. The transformation $\widetilde{G}\overset{\widetilde{p},\widetilde{m}}{\Rightarrow}_{di}\widetilde{H}$ of *FD*-graphs exists and is unique.

Proof. According to Lemma 11, the morphism $\widetilde{l}$ of *FD*-graphs can uniquely be decomposed into the epimorphism $\widetilde{l}_e:\widetilde{I}\to\widetilde{I}_{\equiv}$ and the monomorphism $\widetilde{l}_i:\widetilde{I}_{\equiv}\to\widetilde{L}$. Since all the $\widetilde{l}_x$ are injective for $x\in I^N$, all the $\widetilde{l}_{ex}$ are bijective. By Lemma 12, there exists a complement $(\widetilde{C}',\widetilde{g}_i:\widetilde{C}'\to\widetilde{G},\ \widetilde{c}':\widetilde{I}_{\equiv}\to\widetilde{C}')$ for the morphisms $\widetilde{m}$ and $\widetilde{l}_i$ of *FD*-graphs. Since $\widetilde{m}$ satisfies the distributed join condition with respect to $\widetilde{p}$, it can be shown easily that $\widetilde{c}'$ also satisfies it with respect to $\widetilde{p}'=(\widetilde{I}\xrightarrow{\widetilde{l}_e}\widetilde{I}_{\equiv};\ \widetilde{I}\xrightarrow{\widetilde{r}}\widetilde{R})$. Therefore, according to Theorem 3, there exists a semiinitial complement. Thus, the pushout $(\widetilde{G},\widetilde{m},\widetilde{g}=\widetilde{g}_i\circ\widetilde{g}_{e})$ for $\widetilde{l}$ and $\widetilde{c}$ exists and is defined uniquely.

There exists a pushout of $\widetilde{c}$ and $\widetilde{r}$, and pushout condition (i) is a part of the context condition, pushout condition (ii) corresponds to the network condition, and the additional pushout condition, to the local-parallelism condition and the join condition. Since pushouts in arbitrary categories are unique up to isomorphism, it is arguable that $\widetilde{G}\overset{p,m}{\Rightarrow}_{di}\widetilde{H}$ is unique. The proof is completed. ■

We assume that distributed actions are performed simultaneously. Such parallel distributed actions can be expressed by a parallel production on *FD*-graphs.

Definition 20. A production $\widetilde{p}_1+\widetilde{p}_2=(\widetilde{I}_1+\widetilde{I}_2\xrightarrow{\widetilde{l}_1+\widetilde{l}_2}\widetilde{L}_1+\widetilde{L}_2;\ \widetilde{I}_1+\widetilde{I}_2\xrightarrow{\widetilde{r}_1+\widetilde{r}_2}\widetilde{R}_1+\widetilde{R}_2)$ of *FD*-graphs is called a parallel production of *FD*-graphs that consists of $\widetilde{p}_1$ and $\widetilde{p}_2$, where $\widetilde{p}_1=(\widetilde{I}_1\xrightarrow{\widetilde{l}_1}\widetilde{L}_1;\ \widetilde{I}_1\xrightarrow{\widetilde{r}_1}\widetilde{R}_1)$, $\widetilde{p}_2=(\widetilde{I}_2\xrightarrow{\widetilde{l}_2}\widetilde{L}_2;\ \widetilde{I}_2\xrightarrow{\widetilde{r}_2}\widetilde{R}_2)$ are productions of *FD*-graphs, $\widetilde{L}_1+\widetilde{L}_2$, $\widetilde{I}_1+\widetilde{I}_2$, $\widetilde{R}_1+\widetilde{R}_2$ are coproducts of the graphs, and $\widetilde{l}_1+\widetilde{l}_2$ and $\widetilde{r}_1+\widetilde{r}_2$ are generated morphisms.

Construction of a parallel production is associative since it is based on deriving coproducts. Denote a parallel production $((\ldots(\widetilde{p}_1 + \widetilde{p}_2) + \ldots) + \widetilde{p}_n)$ by $\Sigma_1^n \widetilde{p}_i$, which simplifies the notation but does not imply that coproducts are unique.

Construction of parallel productions can repeat, which results in productions $\widetilde{p}_1 + \widetilde{p}_2 + \ldots + \widetilde{p}_n$, where $n \geq 1$, that are also called parallel productions, which can be written as

$$\Sigma_1^n \widetilde{p}_i = \left(\Sigma_1^n \widetilde{I}_i \xrightarrow{\Sigma \widetilde{l}_i} \Sigma_1^n \widetilde{L}_i;\ \Sigma_1^n \widetilde{I}_i \xrightarrow{\Sigma \widetilde{r}_i} \Sigma_1^n \widetilde{R}_i \right).$$

Definition 21. A grammar of fuzzy distributed graphs $FDG = (\widetilde{S}, P)$ is a set consisting of the source *FD*-graph $\widetilde{S}$ and the set P of productions of the *FD*-graphs.

Let P^+ be the least extension of the set *P*, including all parallel productions $\widetilde{p}_1 + \widetilde{p}_2$ of *FD*-graphs for $\widetilde{p}_1, \widetilde{p}_2 \in P^+$. The operational semantics OS(FDG) of the grammar FDG is represented by the class of all possible transformations that begin with $\widetilde{S}$ and use the set P^+ of productions of *FD*-graphs, i.e., $\mathrm{OS}(FDG) = \{\widetilde{S} \overset{P^+}{\Rightarrow}{}^{*}_{di} \widetilde{G}\}$.

As an operational semantics, we take all possible transformations of *FD*-graphs for which distributed productions can be applied in parallel.

CONCLUSIONS

A transformation approach to fuzzy distributed graphs provides a powerful and flexible resource for modeling dynamic distributed systems, which is based on applying transformations to network structures of fuzzy components. Their interaction and synchronization can be modeled by combining a fuzzy graph and transformations on common parts of such a graph.

The categorical approach is used in the paper to formalize transformations of *FD*-graphs that are structured transformations of fuzzy graphs. We formally defined the necessary and sufficient conditions for the transformations of *FD*-graphs that do not violate the integrity of their structure and a constructive (componentwise) construction of pushout in the **FDGraph** category is possible. Such a structure of pushout is used to analyze a direct transformation of *FD*-graphs that specifies the result of application of productions (including parallel ones) to fuzzy distributed graphs. We have introduced distributed graph grammars, which generalize fuzzy graph grammars. They make it possible to describe admissible transformations of *FD*-graphs.

Changing the form of network during a transformation step allows modeling dynamic network structures of software systems. We have shown that nodes and edges of fuzzy graphs can not only be united but also be disjoined into isolated nodes and edges during a step of transformation of *FD*-graphs. We have formalized the distributed operations "disjoin" and "join," as well as parallel transformations of *FD*-graphs, by corresponding morphisms of networks of components. A parallel transformation of a fuzzy graph is considered as a special case of a transformation of an *FD*-graph for which the graphs subject to transformations are undistributed, i.e., unstructured.

The results presented in the paper can be applied, in particulars, to the design and analysis of models of fuzzy information technologies based on MDA (Model Driven Architecture) [15], which can function within a high-performance parallel medium of the SKIT cluster system created at the V. M. Glushkov Institute of Cybernetics, National Academy of Sciences of Ukraine.

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