Cybernetics and Systems Analysis, Vol. 43, No. 1, 2007

METHODS OF SEARCHING FOR GUARANTEEING AND OPTIMISTIC SOLUTIONS TO INTEGER OPTIMIZATION PROBLEMS UNDER UNCERTAINTY

N. V. Semenova

UDC 519.8

The paper studies complex integer optimization problems with inexact coefficients of the linear objective function and convex quadratic constraint functions. Exact and approximate decomposition methods are developed and proved to search for guaranteeing and optimistic solutions to such problems. The methods are based on approximation of initial problems by problems of a simpler structure.

Keywords: *discrete optimization, integer variables, quadratic constraint functions, inexact data, perturbations in initial data.*

INTRODUCTION

Mathematical models of discrete optimization envelop a wide range of applied problems that arise in making optimal design, technological, and economic decisions. Moreover, a series of theoretical problems in mathematics can be formulated as discrete optimization problems. Therefore, there is a pressing need to develop a theory and methods of searching for solutions to discrete optimization problems [1, 2].

Real situations described by discrete optimization models are often of uncertain and random nature. In these cases, the quality of decisions made and their consequences substantially depends on how completely all uncertain factors are taken into account: inexact input information, inadequate mathematical models, round-off errors, computation errors, etc. The modern optimization theory and practice are based on the classical formulation of optimization problems, which assumes that all data are exact. But such a formulation is unsatisfactory for a large number of discrete optimization problems that arise, for example, in economy, biology (genetics, DNA analysis, molecular biology), physics (high-energy physics, X-ray crystallography), statistics (data analysis and reliability), cryptography (constructing error-control codes), mathematics (theory of combinations, graph theory), policy (choice of electoral districts), and social sciences (control of health care, education, and social safety systems). The initial data (the objective function and admissible domain) may vary during optimization. Moreover, Academician V. M. Glushkov believed [3] that main informative essence of optimization for the class of problems under consideration is in their purposeful change.

Continuing the studies reflected in [4–10], we present the results of developing and substantiating methods for exact and approximate solution of problems originating in studying complex integer optimization models with controlled and inexact initial data and based on their approximation by problems of a more simple structure. These methods are decomposition ones, they combine and use the ideas of relaxation [12], linearization [13], and Kelley cutting plane [14] methods.

We will construct and justify exact and approximate decomposition methods of searching for guaranteeing and optimistic solutions to integer optimization problems with convex quadratic constraint functions under data uncertainty. Some classes of uncertainty sets are proposed that describe the initial data of the problems under consideration.

V. M. Glushkov Institute of Cybernetics, National Academy of Sciences of Ukraine, Kyiv, Ukraine. Translated from Kibernetika i Sistemnyi Analiz, No. 1, pp. 103-114, January-February 2007. Original article submitted September 5, 2006.

1. FORMULATION OF THE PROBLEM

An integer optimization problem with convex quadratic constraint functions is given by

$$\max\{\langle c,x\rangle \mid f_i(x)=\langle D_i x,x\rangle+\langle q_i,x\rangle+b_i\le 0,\ i\in N_m=\{1,\dots,m\}\},\tag{1}$$

where the vector of solutions $x\in Z^n$, Z^n is a set of n-dimensional integer vectors from R^n, the parameters $c\in R^n$, $b=(b_1,\dots,b_m)\in R^m$, $q_i\in R^n$, and $D_i\in R^{n\times n}$ are symmetric nonnegative definite matrices for all $i\in N_m$. The statement (1) assumes that data on the vector c and the quadratic constraint functions $f_i(x)$, $i\in N_m$, are known exactly. However, in practice, these parameters are estimated using data that are subject to noise, perturbations, measurement errors, and various forms of uncertainties. Only sets of possible values can be known about them, and any stochastic characteristics are absent. Since integer optimization problems are, as a rule, sensitive to perturbation of parameters, the errors of input data tend to influence the solutions of these problems and often lead to results far from optimal ones.

Thus, let the parameters of model (1) be known not exactly but be defined, for example, by the statistical estimation from available observations. Assume that the a priori information on the vector $c\in R^n$ and the functions $f_i(x)$, $i\in N_m$, consists in representing the sets C and F_i such that $c\in C$ and $f_i\in F_i$, $i\in N_m$. The sets F_i can be specified by the sets S_i to which the data (D_i,q_i,b_i), $i\in N_m$, of problem (1) belong.

An integer optimization problem of searching for guaranteeing solutions with convex quadratic constraint functions under uncertainty can be presented as follows:

$$\max\{\min\{\langle c,x\rangle \mid c\in C\}\mid x\in X_1\},\tag{2}$$

and the problem of searching for optimistic solutions with controlled data as

$$\max\{\max\{\langle c,x\rangle \mid c\in C\}\mid x\in X_2\},\tag{3}$$

where

$$X_1=X_1(S_i,i\in N_m)=\{x\in Z^n \mid \langle D_i x,x\rangle+\langle q_i,x\rangle+b_i\le 0\quad \forall\,(D_i,q_i,b_i)\in S_i,\ i\in N_m\},$$

$$X_2=X_2(S_i,i\in N_m)=\{x\in Z^n \mid \exists (D_i,q_i,b_i)\in S_i:\ \langle D_i x,x\rangle+\langle q_i,x\rangle+b_i\le 0,\ \ i\in N_m\},$$

C is a convex closed set from R^n, and F_i is the set of convex quadratic functions in R^n, $i\in N_m$.

The optimal solution of problem (2) (problem (3), respectively) is the pair $(\underline{c},\bar{x})$ $((\bar{c},\bar{x}))$ whose elements $\underline{c}\in C$, $\bar{x}\in X_1$, $(\bar{c}\in C,\ \bar{x}\in X_2)$ satisfy the condition

$$\langle \underline{c},\bar{x}\rangle=\min\{\langle c,\bar{x}\rangle \mid c\in C\}\ge \min\{\langle c,x\rangle \mid c\in C\},\tag{4}$$

$$(\langle \bar{c},\bar{x}\rangle=\max\{\langle c,\bar{x}\rangle \mid c\in C\}\ge \max\{\langle c,x\rangle \mid c\in C\})\tag{5}$$

for all (c,x) such that $x\in X_1$ $(x\in X_2)$.

If the solution $(\underline{c},\bar{x})$ of problem (2) ($(\bar{c},\bar{x})$ of problem (3), respectively) exists, then it is guaranteeing (optimistic) in the sense that $\langle D_i x,x\rangle+\langle q_i,x\rangle+b_i\le 0\ \forall\,(D_i,q_i,b_i)\in S_i$,

$$(\exists\,(D_i,q_i,b_i)\in S_i:\langle D_i x,x\rangle+\langle q_i,x\rangle+b_i\le 0),\ i\in N_m,$$

and for any $x\in X_1$ $(x\in X_2)$ condition (4) ((5), respectively) is satisfied.

Denote by $X_k^0=X_k^0(S_i,\ i\in N_m)$, $k=1,2$, the sets of optimal (guaranteeing or optimistic) solutions, respectively.

Of interest is to determine uncertainty sets that possess a more simple structure than the sets C and S_i, $i\in N_m$, do and such that do not change the admissible X_k and optimal X_k^0, $k=1,2$, sets of problems (2) and (3), respectively. Here, for example, the following problems arise.

Problem 1. For the given sets C and S_i, $i\in N_m$, find sets $\bar{C}\supset C$, $\bar{S}_i\supset S_i$, $i\in N_m$, that have more simple structures than that of C and S_i, $i\in N_m$, and are such that the equalities $X_k(\bar{S}_i,\ i\in N_m)=X_k(S_i,i\in N_m)$ and $X_k^0(\bar{C},\bar{S}_i,\ i\in N_m)=X_k^0(C,S_i,i\in N_m)$, $k=1,2$, hold.

Problem 2. For the sets C and S_i, $i \in N_m$, determine the sets $\underline{C} \subset C$, $\underline{S}_i \subset S_i$, $i \in N_m$, that have more simple structures than that of C and S_i, $i \in N_m$, and are such that the equalities $X_k(\underline{S}_i, i \in N_m) = X_k(S_i, i \in N_m)$, $X_k^0(\underline{C}, \underline{S}_i, i \in N_m) = X_k^0(C, S_i, i \in N_m)$, $k = 1, 2$ hold.

The solution of the first problem yields boundaries for extending the sets C and S_i, $i \in N_m$, without changing the sets of admissible and optimal solutions of the initial problems (2) and (3), i.e., determines the domains of stability of their solutions when the parameters of these problems are changed. The solution of the second problem allows reducing the amount of calculations by reducing and simplifying the sets C and S_i, $i \in N_m$, which is especially important for organizing an efficient optimization solution of problems (2) and (3). Ramik and Rimanek [15] present some results concerning solution of Problems 1 and 2 with linear constraints of the admissible domain.

The representation of the uncertainty sets C and S_i substantially influences the degree of complexity of the accompanying optimization problems in constructing an optimal or an approximate solution. Note that for some classes of uncertainty sets, integer linear, quadratic, and semidefinite programming problems can easily be reduced to standard optimization problems with exact data.

Let us describe some classes of uncertainty sets for which problems (2) and (3) can be reduced to optimization problems with exact data. Let us consider the case where the set C is given as follows: $C = \{c \mid |c - c_0| \le \varepsilon, \varepsilon > 0\}$, where c_0 and ε are a vector given in space R^n and a given number, respectively; then $\min\{\langle c, x\rangle \mid c \in C\} = -f_C(-x)$, $\max\{\langle c, x\rangle \mid c \in C\} = f_C(x)$, where $f_C(x)$ is the support functional of the convex set C. Since $f_C(x)$ for C is calculated analytically in this case [11], the objective functions of problems (2) and (3) become $\max\{\langle c_0, x\rangle - \varepsilon|x|\}$ and $\max\{\langle c_0, x\rangle + \varepsilon|x|\}$, respectively.

Let us show that for some constraints on the uncertainty sets C and $S_i = \{(D_i, q_i, b_i)\}$, $i \in N_m$, the solutions of problems (2) and (3) can be found by solving once the integer optimization problem (1).

Given X, the set W is said to have a maximum element $w^* \in W$ (minimum $w_* \in W$, respectively) if $f(x, w^*) \ge f(x, w)$ $(f(x, w_*) \le f(x, w))$ $\forall x \in X, \forall w \in W$. If w^* (w_*) exists, then it is a solution $w^* \in \arg\max_{w \in W} \min_{x \in X} f(x, w)$ $(w_* \in \arg\min_{w \in W} \max_{x \in X} f(x, w))$.

Let us present examples of sets that have maximum (minimum) elements on the assumption that $S_i = \{D_i\}$, $i \in N_m$.

1. Let for some $i \in N_m$ D_i^0 be known symmetric nonnegative definite matrices, and S_i be closed spheres, with centers in D_i^0, of radius $\rho_i > 0$: $S_i = \{D_i \mid D_i = D_i^{\mathrm{T}}, \ ||D_i - D_i^0||_S \le \rho_i\}$, where $||B||_S = \max\{\langle By, y\rangle \mid ||y|| = 1\}$ is the spectral norm of the matrix B. Then for $0 \le \rho_i \le \lambda_{\min}^i\{D_i^0\}$, where $\lambda_{\min}^i[A]$ is the minimum eigenvalue of the matrix A, S_i is a convex compact set of symmetric nonnegative definite matrices. It is easy to verify that for $D_i^{\max} = D_i^0 + \rho_i E$, $D_i^{\min} = D_i^0 - \rho_i E$ (where E is a unit matrix), the inequalities $f_i(x, D_i^{\max}) \ge f_i(x, D_i)$ and $f_i(x, D_i^{\min}) \le f_i(x, D_i)$ hold for any $x \in Z^n$ and $D_i \in S_i$; therefore, $D_i^{\max}$ and $D_i^{\min}$ are the maximum and minimum elements for any $X_1 \subset Z^n$ $(X_2 \subset Z^n)$, $i \in N_m$.

2. Let S_i, $i \in N_m$, be given by elementwise constraints for the matrices D_i, $S_i = \{D_i \mid \underline{D}_i \le D_i \le \overline{D}_i\}$, where the matrices $\underline{D}_i$ and $\overline{D}_i$ are known and nonnegative definite. If the sets X_k, $k = 1, 2$, contain only nonnegative elements, then $(\overline{D}_i, \overline{q}_i, \overline{b}_i)$ and $(\underline{D}_i, \underline{q}_i, \underline{b}_i)$ are the maximum and minimum elements, respectively, since $f_i(x, \overline{D}_i, \overline{q}_i, \overline{b}_i) \ge f_i(x, D_i, q_i, b_i)$ and $(f_i(x, \underline{D}_i, \underline{q}_i, \underline{b}_i) \le f_i(x, D_i, q_i, b_i))$ $\forall (D_i, q_i, b_i) \in S_i$ for any $x \ge 0$. Note that the sets S_i, $i \in N_m$, thus defined arise, for example, in the confidence-interval estimation of the elements of the unknown covariance matrix D from empirical data.

2. DECOMPOSITION APPROACH TO SEARCHING FOR EXACT AND APPROXIMATE SOLUTIONS AND ITS SUBSTANTIATION

Since each matrix D_i, $i \in N_m$, is nonnegative definite, to solve problem (2) it is possible to apply the decomposition methods proposed in [4]. According to these methods, the problem solution can be reduced to sequential solution of integer optimization problems with linear constraints and linear programming problems.

We develop this approach here to solve problem (3), whose admissible domain X_2 is a union of convex sets and thus can be nonconvex. Following [4], let us consider the MP problem for some $x^j \in Z^n$, $j = 1, 2, \ldots$.

$$\max x_0, \tag{6}$$

under the following conditions:

$$x_0 \le \max_{j \in N_k} \langle c^j, x \rangle, \ c^j \in C, \ k = 1, 2, \dots, \tag{7}$$

$$\min_{(D_i, q_i, b_i) \in S_i} \ \max_{j \in N_l} \ (f_i^j(x^j) + \langle \nabla f_i^j(x^j), x - x^j \rangle) \le 0,$$

$$f_i^j \in F_i, \ i \in I \subset N_m, \ l = 1, 2, \dots, \tag{8}$$

$$x \in Z^n, \tag{9}$$

where $\nabla f_i^j(x^j)$ is the gradient of the function $f_i^j(x)$ at the point x^j.

Let us define the sets

$$Q_i = \{x \in Z^n \mid \exists (D_i, q_i, b_i) \in S_i : \langle D_i x, x \rangle + \langle q_i, x \rangle + b_i \le 0\}, \ i \in N_m.$$

Obviously, $X_2 = \bigcap_{i \in N_m} Q_i$. We assume that the set X_2 is bounded.

The quantity $r_i(x) = \min \{\langle D_i x, x \rangle + \langle q_i, x \rangle + b_i \mid (D_i, q_i, b_i) \in S_i\}$, $i \in N_m$, is called the deviation of the point $x \in Z^n$ from the boundary of the set Q_i, and the quantity $r(x) = \max\{r_i(x) \mid i \in N_m\}$ the deviation of the point $x \in Z^n$ from the boundary of the set X_2. Let us define the sets

$$P_i^l = \left\{ x \in R^n \mid \min_{(D_i, q_i, b_i) \in S_i} \ \max_{j \in N_l} \ (f_i^j(x^j) + \langle \nabla f_i^j(x^j), x - x^j \rangle) \le 0, \quad f_i^j \in F_i \right\},$$

$$i \in I \subset N_m, \ l = 1, 2, \ \dots, \tag{10}$$

$$P^l = \bigcap_{i \in I} P_i^l, \tag{11}$$

$$S^k = \left\{ (x_0, x) \in R^{n+1} \mid x_0 \le \max_{j \in N_k} \langle c^j, x \rangle, \ c^j \in C \right\}, \ k = 1, 2, \dots . \tag{12}$$

Thus, we can write the *MP* problem as follows:

$$\max\{x_0 \mid (x_0, x) \in S^k \mid x \in P^l \cap Z^n\}.$$

THEOREM 1. An admissible (optimal) solution (x_0, x) of an *MP* problem is an admissible (optimal) solution of problem (3), where x_0 is the value of the objective function and x is a solution of this problem, if and only if the conditions $r(x) \le 0$ and $x_0 \le \max\{\langle c, x \rangle \mid c \in C\}$ ($x_0 = \max\{\langle c, x \rangle \mid c \in C\}$) are satisfied.

Proof. An admissible solution of the *MP* problem is an admissible solution of problem (3) if and only if $r(x) \le 0$ and $x_0 \le \max\{\langle c, x \rangle \mid c \in C\}$. The necessity of this statement is obvious. The sufficiency follows from the construction of the *MP* problem and the definition of the $r(x)$. Since the *MP* problem is equivalent to problem (3) provided that the conditions of Theorem 1 are satisfied, this statement is proved with respect to the optimal solution.

Developing the decomposition approach to the solution of problem (3) proposed in [4], we reduce problem (3) to a sequential solution of *MP* problems of partially integer optimization with linearized constraints and of linear programming problems $\max\{\langle c, x^j \rangle\} \mid c \in C\}$.

The main idea of the exact and approximate methods proposed here consists in the following. If the solution (x_0, x) of the *MP*-problem is inadmissible in problem (3), i.e., $r(x) > 0$, then it is eliminated from the subsequent consideration by adding a new linear constraint to constraints (8) of the *MP* problem. Thus, constraint (8) now cuts off an inadmissible solution x and a part of the inadmissible domain of problem (3) from all the subsequent considerations. All the added constraints are correct cutting planes, i.e., those that do not cut off any part of the admissible domain of the nonlinear problem (3). If according to Theorem 1 the solution (x_0, x) is optimal for the *MP* problem and $r(x) \le 0$, $x_0 = \max\{\langle c, x \rangle \mid c \in C\}$, then the solution obtained is optimal for problem (3) too.

Algorithm

1. Select $c^1 \in C$, assume $k, l = 1$, $P^1 = \{x \in Z^n\}$, $S^1 = \{x_0, x\} \mid x_0 \le c^1 x, x_0 \in R^1, x \in Z^n\}$.

2. Solve the partially integer *MP* optimization problem. If it is inadmissible, then based on the fact that constraints (8) are a relaxation of the constraints $\exists (D_i, q_i, b_i) \in S_i : \langle D_i x, x\rangle + \langle q_i, x\rangle + b_i \le 0$, $i \in N_m$, of problem (3), conclude that problem (3) is also inadmissible. This case is possible only for $k = 1$. Otherwise obtain the admissible (optimal) solution (x_0^l, x^l) or the information that the objective function is not bounded on the admissible set. Then take sufficiently large values that satisfy the constraints as the coordinates of the vector x^l.

3. Find the deviation $r(x^l)$ from the boundary of the set X_2. If $r(x^l) \le 0$, then according to Theorem 1 (x_0^l, x^l) is the admissible (optimal) solution of problem (3). Go to Step 4 assuming that $(x_0^k, x^k) = (x_0^l, x^l)$. Otherwise find an element $f_i^l \in \bigcap_{i \in I} F_i$ for which $r(x^l) > 0$ and include it in the formation of the set P_i^{l+1}, $i \in I$, according to (10). Replace l with $l + 1$ and go to Step 2.

4. Solve the linear programming problem $\{\langle c, x^k\rangle \mid c \in C\}$. Let c^{k+1} be the optimal solution.

5. If

$$\langle c^{k+1}, x^k\rangle \ge x_0^k, \tag{13}$$

$$(\langle c^{k+1}, x^k\rangle = x_0^k), \tag{14}$$

then stop. According to Theorem 1, the point (x_0^k, x^k) is an admissible (optimal) solution of problem (3). Based on Theorem 1, conclude that (c^{k+1}, x^k) is an approximate (optimal) solution of problem (3) with the value x_0^k of its objective function. Otherwise, form the set S^{k+1} according to (12), replace k with $k+1$, and go to Step 2.

Remark. When using this algorithm to derive an optimal solution of problem (3), one may weaken the optimality requirement for the solution of the *MP* problem. The optimality is only necessary at the final step. Intermediate solutions for constructing new cutting planes should only be the admissible solutions of the *MP* and inadmissible in problem (3), i.e., should not belong to the admissible domain X_2. In this case, deriving the optimal solution of the initial problem will probably need more steps of the algorithm; however, it is possible to apply approximate methods to solve an *MP* subproblem.

THEOREM 2. The above algorithm converges to an approximate (optimal) solution in a finite number of steps or ends at the first step with the conclusion that problem (3) is inadmissible.

Proof. Since the set X_2 is bounded, problem (3) has a finite optimal solution; thus, beginning with a number k_0, the sequence of points $\{x^k\}$ is contained in a bounded set. Therefore, there exists a converging sequence $\{x^{k_l}\}$. Taking into account the condition $x^i \in Z^n$, $i = 1, 2, \ldots$, we conclude that the point set $\{x^{k_l}\}$ is finite. Therefore, it is possible to select a stationary sequence, i.e., $\exists l_*$ such that $\forall l \ge l_*$ $x^{k_l} = \bar{x}$, and no new constrain is added to (8) in the *MP* problem beginning with an l_*. This is possible only when $\bar{x} \in X_2$. In Step 5 of the algorithms, if either condition (13) or (14) is not satisfied, constraint (7) in the *MP* problem is replaced with a new one. The number of such constraints does not exceed the number of nodes of the polyhedron C since none of the vectors c^{k+1} can be found once again before condition (13) or (14) is satisfied.

The theorem is proved.

Below we present criteria of checking the solution for admissibility in problems (2) and (3) for some uncertainty sets.

3. REPRESENTATION OF UNCERTAINTY SETS

3.1. Discrete and Polyhedral Uncertainty Sets. A discrete set given as

$$S_a = \{(D, q, b) \mid (D, q, b) = (D_j, q_j, b_j), j \in N_k\}$$

is the simplest type of an uncertainty set, which specifies initial data of constraints that describe admissible domains of problems (2) and (3).

Hereafter, D_j, $j \in N_k$, are symmetric nonnegative definite matrices.

The set described by the guaranteeing constraint $\langle Dx, x\rangle + \langle q, x\rangle + b \le 0$ for all $(D, q, b) \in S_a$ is equivalent to an intersection of k sets each being specified by the convex quadratic constraint for each $(D_j, q_j, b_j) \in S_a$, $j \in N_k$,

$$\bigcap_{j \in N_k} \{\langle D_j x, x\rangle + \langle q_j, x\rangle + b_j \le 0\}, \tag{15}$$

which, in turn, is described by the inequality

$$\max\{\langle D_j x, x\rangle + \langle q_j, x\rangle + b_j \mid (D_j, q_j, b_j) \in S_a\} \le 0.$$

The set described by the constraint

$$\exists (D, q, b) \in S_a : \langle Dx, x\rangle + \langle q, x\rangle + b \le 0 \tag{16}$$

in problem (3) is equivalent to the union of sets described by this constraint for each $(D_j, q_j, b_j) \in S_a$, $j \in N_k$,

$$\bigcup_{j \in N_k} \{\langle D_j x, x\rangle + \langle q_j, x\rangle + b_j \le 0\},$$

which is equivalent to

$$\min\{\langle D_j x, x\rangle + \langle q_j, x\rangle + b_j \mid (D_j, q_j, b_j) \in S_a\} \le 0. \tag{17}$$

Discrete uncertainty sets are used when it is necessary to make a decision that is stable (in problem (2)) and optimistic (in problem (3)) with respect to several scenario (each value of (D_j, q_j, b_j) corresponds to a certain scenario).

The convex hull of a discrete uncertainty set is specified by

$$\operatorname{conv} S_a = \{(D, q, b) \mid (D, q, b) = \sum_{j=1}^{k} \lambda_j (D_j, q_j, b_j),\ \lambda_j \ge 0\ \forall j \in N_k,\ \sum_{j=1}^{k} \lambda_j = 1\}.$$

The constraint $\langle Dx, x\rangle + \langle q, x\rangle + b < 0\ \forall (D, q, b) \in \operatorname{conv} S_a$ is equivalent to the following one: $\sum_{j=1}^{k} \lambda_j \langle D_j x, x\rangle + \langle q_j, x\rangle + b_j \le 0\ \forall \lambda_j \ge 0,\ j \in N_k,\ \sum_{j=1}^{k} \lambda_j = 1$, which, in turn, is equivalent to the set of constraints (15). In the case of the uncertainty set $\operatorname{conv} S_a$, constraints (16) are equivalent to (17).

Thus, the following statement is true.

Statement 1. Let $\operatorname{conv} S_i$ be the convex hull of the set S_i, $i \in N_m$, $\bar{X}_1 = \{x \in Z^n \mid \langle D_i x, x\rangle + \langle q_i, x\rangle + b_i \le 0\ \forall (D_i, q_i, b_i) \in \operatorname{conv} S_i, i \in N_m\}$, $\bar{X}_2 = \{x \in Z^n \mid \exists (D_i, q_i, b_i) \in \operatorname{conv} S_i : \langle D_i x, x\rangle + \langle q_i, x\rangle + b_i \le 0,\ i \in N_m\}$. Then $\bar{X}_k = X_k$, $k = 1, 2$.

Based on Statement 1, we assume that the sets S_i, $i \in N_m$, are convex.

The uncertainty sets S_a and $\operatorname{conv} S_a$ can be extended to the following polyhedral uncertainty set:

$$S_b = \{(D, q, b) \mid (D, q, b) = \sum_{j=1}^{k} \lambda_j (D_j, q_j, b_j),\ A\lambda = d, \lambda \ge 0\},$$

where $\{\lambda \in R^k \mid A\lambda = d,\ \lambda \ge 0|\} \ne \varnothing$.

Let us introduce a vector $g = (g_1, \ldots, g_k) \in R^k$, $g_j = \langle D_j x, x\rangle + \langle q_j, x\rangle + b_j$, $j \in N_k$.

Statement 2. A vector $x \in Z^n$ satisfies the constraint $\langle Dx, x\rangle + \langle q, x\rangle + b \le 0$ for all $(D, q, b) \in S_b$ if and only if there exists a vector $u \in R^k$ such that $A^T u \ge g$, $\langle d, u\rangle \le 0$ hold.

Proof. Fix an *x*. Then the constraint $\langle Dx, x\rangle + \langle q, x\rangle + b \le 0$ for all $(D, q, b) \in S_b$ is equivalent to the system of linear inequalities

$$\langle g, \lambda\rangle \le 0\ \forall \lambda \ge 0 \text{ such that } A\lambda = d. \tag{18}$$

According to the theory of duality of linear programming, relations (18) are equivalent to the following ones: $\exists u \in R^k$ such that $A^T u \ge g$, $\langle d, u\rangle \le 0$.

Statement 3. A vector $x \in Z^n$ satisfies the constraint $\exists (D, q, b) \in S_b : \langle Dx, x\rangle + \langle q, x\rangle + b \le 0$ if and only if

$$\langle d, u\rangle \le 0 \ \forall u \ge 0 : A^T u \le g. \tag{19}$$

Proof. The constraint $\exists (D, q, b) \in S_b : \langle Dx, x\rangle + \langle q, x\rangle + b \le 0$ for the given vector $x \in Z^n$ is equivalent to the following one: $\exists \lambda \in R^k$, $\lambda \ge 0$, such that $A\lambda = d$, $\langle g, \lambda\rangle \le 0$. According to the theory of duality of linear programming, these relations are equivalent to (19).

3.2. Uncertainty Sets that Contain Perturbations Bounded in Norm. Further, let us describe two closed uncertainty sets that contain perturbations bounded in norm and are bounded versions of generalized ellipsoidal sets [15]. In the first uncertainty set S_c, all the elements (D, q, b) are given by

$$S_c = \{(D, q, b) \mid (D, q, b) = (D_0, q_0, b_0) + \sum_{j=1}^{k} u_j (D_j, q_j, b_j),\ u \ge 0,\ ||u||_p \le 1\},$$

where $||u||_p$ is an L_p-norm for some real $p \ge 1$, D_j, $j \in \{0\} \cup N_k$ are symmetric nonnegative definite matrices in $R^{n\times n}$.

Statement 4. A vector $x \in Z^n$ satisfies the constraint $\exists (D, q, b) \in S_c : \langle Dx, x\rangle + \langle q, x\rangle + b \le 0$ if and only if there exists a vector $h \in R^k$, $h \ge 0$, $h \ge -g$, that satisfies the inequality

$$\langle D_0 x, x\rangle + \langle q_0, x\rangle + b_0 - ||h||_q \le 0, \tag{20}$$

where $1/p + 1/q = 1$.

Proof. The constraint $\exists (D, q, b) \in S_c : \langle Dx, x\rangle + \langle q, x\rangle + b \le 0$ is equivalent to

$$\langle D_0 x, x\rangle + \langle q_0, x\rangle + b_0 + \min_{\{u:\, u \ge 0,\, ||u||_p \le 1\}} \left\{ \sum_{j=1}^{k} u_j \left(\langle D_j x, x\rangle + \langle q_j, x\rangle + b_j \right) \right\} \le 0 \tag{21}$$

or to the inequality

$$\langle D_0 x, x\rangle + \langle q_0, x\rangle + b_0 - \max_{\{u:\, u \ge 0,\, ||u||_p \le 1\}} \left\{ \sum_{j=1}^{k} u_j \left(-\langle D_j x, x\rangle - \langle q_j, x\rangle - b_j \right) \right\} \le 0.$$

Denote $y_j = -g_j = -\langle D_j x, x\rangle - \langle q_j, x\rangle - b_j$ and $z_j = (y_j)^+ = \max\{y_j, 0\}$, $i \in N_k$. Assume that $p > 1$. Then if $z \ne 0$, the optimal solution u^* of the problem $\max\{\langle u, y\rangle \mid u \ge 0, ||u||_p \le 1\}$ is $u_j^* = (z_j)^{1/p-1} / ||z||_q^{1/p-1}$ otherwise $u_j^* = 0$, $j \in N_k$. Therefore, (21) is equivalent to

$$\langle D_0 x, x\rangle + \langle q_0, x\rangle + b_0 - ||z||_q \le 0. \tag{22}$$

For $p = 1$, $\max\limits_{1 \le j \le k} \{z_j\}$ is the optimal solution of the problem $\max\{\langle u, y\rangle \mid u \ge 0, ||u||_p \le 1\}$. Thus, (21) is equivalent to (22). Moreover, since $z \ge 0$, $||h||_q \ge ||z||_q$ for all $h_j \ge z_j$ (i.e., $h_j \ge 0$ and $h_j \ge y_j$), $j \in N_k$, and (22) holds if and only if there exists a vector $h \in R^k$, $h \ge 0$, $h \ge y$, such that it satisfies inequality (20).

Statement 5. A vector $x \in Z^n$ satisfies the constraint $\langle Dx, x\rangle + \langle q, x\rangle + b \le 0$ for all $(D, q, b) \in S_c$ if and only if there exists a vector $h \in R^k$, $h \ge 0$, $h \ge g$, such that it satisfies the inequality

$$\langle D_0 x, x\rangle + \langle q_0, x\rangle + b_0 + ||h||_q \le 0, \tag{23}$$

where $1/p + 1/q = 1$.

To prove this statement, note that the constraint $\langle Dx, x\rangle + \langle q, x\rangle + b \le 0 \ \ \forall (D, q, b) \in S_c$ is equivalent to the constraint

$$\langle D_0 x, x\rangle + \langle q_0, x\rangle + b_0 + \max_{\{u:\, u \ge 0,\, ||u||_p \le 1\}} \left\{ \sum_{j=1}^{k} u_j \left(\langle D_j x, x\rangle + \langle q_j, x\rangle + b_j \right) \right\} \le 0. \tag{24}$$

Arguing as in the proof of Statement 4, we prove that constraint (24) holds if and only if there exists a vector $h \in R^k$, $h \ge 0$, $h_j \ge g_j$, $j \in N_k$, such that it satisfies inequalities (23).

In the uncertainty set S_c, perturbations in the quadratic D and affine (q, b) terms are specified by the same parameter u. However, these perturbations are independent in many applications, i.e., can be specified by different parameters. Let us consider the following uncertainty set:

$$S_d = \left\{ \begin{array}{l} (D, q, b) | D = D_0 + \sum_{j=1}^{k} u_j D_j, \ ||u||_p \le 1, \\ (q, b) = (q_0, b_0) + \sum_{j=1}^{k} w_j (q_j, b_j), \ ||w||_r \le 1, \end{array} \right\},$$

where p and r are real numbers no less than 1.

Remark. Since u has no constraint on the sign, and the matrices D_j, $j \in \{0\} \cup N_k$ are nonnegative definite, we conclude that the worst case of the perturbation is $u^* \ge 0$ and the best one is $u^* \le 0$.

Statement 6. A vector $x \in Z^n$ satisfies the constraint $\exists (D, q, b) \in S_d : \langle Dx, x\rangle + \langle q, x\rangle + b \le 0$ if and only if there exist vectors $h \in R^k$ and $v \in R^k$ such that

$$\langle D_0 x, x\rangle + \langle q_0, x\rangle + b_0 - ||h||_q - ||v||_S \le 0, \tag{25}$$

where $h_j = -\langle D_j x, x\rangle$, $v_j = -\langle q_j, x\rangle - b_j$, $j \in N_k$, $1/p + 1/q = 1$, $1/r + 1/s = 1$.

Proof. The constraint $\exists (D, q, b) \in S_d : \langle Dx, x\rangle + \langle q, x\rangle + b \le 0$ is equivalent to the following one:

$$\langle D_0 x, x\rangle + \langle q_0, x\rangle + b_0 + \min_{\{u:\, u \le 0,\, \|u\|_p \le 1\}} \left\{ \sum_{j=1}^{k} u_j \langle D_j x, x\rangle \right\} + \min_{\{w:\, \|w\|_r \le 1\}} \left\{ \sum_{j=1}^{k} w_j (\langle q_j, x\rangle + b_j) \right\} \le 0,$$

which is equivalent to

$$\langle D_0 x, x\rangle + \langle q_0, x\rangle + b_0 - \max_{\{u:\, u \le 0,\, \|u\|_p \le 1\}} \left\{ \sum_{j=1}^{k} u_j (-\langle D_j x, x\rangle) \right\} - \max_{\{w:\, \|w\|_r \le 1\}} \left\{ \sum_{j=1}^{k} w_j (-\langle q_j, x\rangle - b_j) \right\} \le 0. \tag{26}$$

With the Cauchy–Bunyakovskii inequality, we get the equalities

$$\max_{\{u:\, u \le 0,\, \|u\|_p \le 1\}} \left\{ \sum_{j=1}^{k} u_j (-\langle D_j x, x\rangle) \right\} = ||h||_p, \qquad \max_{\{w:\, \|w\|_r \le 1\}} \left\{ \sum_{j=1}^{k} w_j (-\langle q_j, x\rangle - b_j) \right\} = ||v||_S .$$

Thus, (26) is equivalent to (25). Since the matrices D_j are nonnegative definite $\forall j \in N_k$, it is easy to verify that the equality $h_j = -\langle D_j x, x\rangle$ can be weakened up to $h_j \le -\langle D_j x, x\rangle$, $j \in N_k$, without influence on the constraint (25).

Statement 7. A vector $x \in Z^n$ satisfies the constraint $\langle Dx, x\rangle + \langle q, x\rangle + b \le 0 \ \ \forall (D, q, b) \in S_d$ if and only if there exist vectors $h \in R^k$ and $v \in R^k$ such that the following inequalities hold:

$$\langle D_0 x, x\rangle + \langle q_0, x\rangle + b_0 + ||h||_q + ||v||_S \le 0, \tag{27}$$

where $v_j = \langle q_j, x\rangle + b_j$, $h_j = \langle D_j x, x\rangle$, $j \in N_k$, $1/p + 1/q = 1$, $1/r + 1/s = 1$.

Proof. The constraint $\langle Dx, x\rangle + \langle q, x\rangle + b \le 0 \ \ \forall (D, q, b) \in S_d$ is equivalent to

$$\langle D_0 x, x\rangle + \langle q_0, x\rangle + b_0 + \max_{\{u:\, u \ge 0,\, \|u\|_p \le 1\}} \left\{ \sum_{j=1}^{k} u_j \langle D_j x, x\rangle \right\} + \max_{\{w:\, \|w\|_r \le 1\}} \left\{ \sum_{j=1}^{k} w_j (\langle q_j, x\rangle + b_j) \right\} \le 0. \tag{28}$$

With the Cauchy–Bunyakovskii inequality, we arrive at the equalities

$$\max_{\{u:\, u \ge 0,\, \|u\|_p \le 1\}} \left\{ \sum_{j=1}^{k} u_j \langle D_j x, x\rangle \right\} = ||h||_q, \qquad \max_{\{w:\, \|w\|_r \le 1\}} \left\{ \sum_{j=1}^{k} w_j (\langle q_j, x\rangle \ b_j) \right\} = ||v||_S .$$

Therefore, (28) is equivalent to (27), and the equality $h_j = \langle D_j x, x \rangle$ can be weakened up to $h_j \ge \langle D_j x, x \rangle$, $j \in N_k$, without influence on the constraint (27).

Some uncertainty sets that describe the initial data for problems (2) with continuous variables and their applications are presented in [16].

CONCLUSIONS

We analyzed complex integer optimization problems with inexact coefficients of the linear objective function and quadratic constraint functions. We considered some uncertainty sets that describe the initial data of problems for which such problems can be formulated as problems with exact data. We constructed and justified exact and approximate decomposition methods to find guaranteeing and optimistic solutions to these problems under data uncertainty, based on their constructive approximations with problems of simpler structures. Further study will be aimed at developing the methods for solving complex integer optimization problems with controlled initial data, which will lead to formulations of problems on preconvex sets. Admissibility and optimality criteria for solutions will be obtained, their stability will be analyzed, and algorithms for data analysis will be developed.

REFERENCES

1. I. V. Sergienko, Mathematical Models and Methods of Solving Discrete Optimization Problems [in Russian], Naukova Dumka, Kyiv (1988).
2. I. V. Sergienko and V. P. Shilo, Discrete Optimization: Problems, Solution Methods, Analysis [in Russian], Naukova Dumka, Kyiv (2003).
3. V. M. Glushkov, "Systemwise optimization," Cybernetics, **16**, No. 5, 731–732 (1980).
4. N. V. Semenova, "Solving integer optimization problems on convex sets under uncertainty," Teor. Optym. Rishen', No. 4, 107–112 (2005).
5. N. V. Semenova, "Solution of a generalized integer-valued programming problem," Cybernetics, **20**, No. 5, 641–651 (1984).
6. I. V. Sergienko, V. O. Roshchin, and N. V. Semenova, "Solving inexact integer programming problems," Dop. AN URSR, Ser. A, No. 61–64 (1988).
7. V. A. Roshchin, N. V. Semenova, and I. V. Sergienko, "Solution and investigation of one class of inexact integer programming problems," Cybernetics, **25**, No. 2, 185–192 (1989).
8. V. A. Roshchin, N. V. Semenova, and I. V. Sergienko, "Decomposition approach to solving some integer programming problems with inexact data," Zh. Vych. Mat. Mat. Fiz., **29**, No. 5, 786–791 (1990).
9. I. V. Sergienko and N. V. Semenova, "Integer programming problems with inexact data: exact and approximate solutions," Cybern. Syst. Analysis, **31**, No. 6, 842–851 (1995).
10. I. V. Sergienko, V. A. Roshchin, and N. V. Semenova, "Some integer programming problems with ambiguous data and their solution," Probl. Upravl. Inform., No. 6, 116–123 (1998).
11. R. Rockafellar, Convex Analysis, Princeton Univ. Press (1970).
12. L. S. Lasdon, Optimization Theory for Large Systems, Macmillan, New York (1970).
13. B. N. Pshenichnyi, Linearization Method [in Russian], Nauka, Moscow (1983).
14. J. Kelley, "The cutting plane method for solving convex program," SIAM J., **8,** No. 4, 703–712 (1960).
15. J. Ramik and J. Rimanek, "Linear constraints with inexact data," Izv. AN SSSR, Tech. Kibern., No. 2, 41–48 (1987).
16. D. Goldfarb and G. Iyengar, "Robust convex quadratically constrained programs," Math. Program., No. 3, 135–141 (2003).