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NECESSARY AND SUFFICIENT CONDITIONS OF EXISTENCE AND UNIQUENESS OF SOLUTIONS TO INTEGRAL EQUATIONS OF ACTUARIAL MATHEMATICS

B. V. Norkin

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Necessary and sufficient conditions of existence and uniqueness of solutions of integral equations are established for ruin probability as a function of the initial capital of an insurance company. Several sufficient conditions and correctness conditions of the problem of finding ruin probability are also established. A general method of successive approximations for solution of this problem is substantiated.

Keywords: *actuarial mathematics, insurance mathematics, risk process, ruin probability, integral equation, existence and uniqueness of a solution, necessary and sufficient conditions, method of successive approximations, correctness.*

We call a risk process a random process $\xi_{t\geq 0}$ describing the stochastic evolution of the capital of an insurance company and denote by $\xi_0 = u$ the initial capital of the company [1]. We denote by $\Psi(u)$ the ruin probability for the insurance company as a function of $u \geq 0$ and by $\varphi(u) = 1 - \Psi(u) = P\{\xi_t \geq 0 \ \forall t \geq 0\}$ the corresponding probability of its survival. The probability $\varphi(u)$ as a function of the initial capital for these processes satisfies the integral equation (or a system of equations for a risk process in a random environment)

$$\varphi(u) = A\varphi(u) \tag{1}$$

under the boundary condition

$$\varphi(+\infty) = \lim_{u \to +\infty} \varphi(u) = 1, \tag{2}$$

where A is some linear integral operator defined on nondecreasing functions $\varphi(u)$ such that we have $0 \leq \varphi(u) \leq 1$. The computation of the survival probability $\varphi(u)$, i.e., the solution of problem (1), (2) for $\varphi(u)$ is an important scientific and practical problem. For different processes, the operator A assumes the following forms:

- the classical risk process [1, 2]

$$A_1\varphi(u) := 1 - \frac{\lambda\mu}{c} + \frac{\lambda}{c}\int_0^u \varphi(u-z)(1-F(z))dz; \tag{3}$$

- a nonlinear risk process [3, 4]

$$A_2\varphi(u) := \int_0^\infty \int_0^{U(u,t)} \varphi(U(u,t)-z)\ dF(z)\,dK(t),$$

$$\frac{dU}{dt} = c(u,U),\ U(u,0) = u; \tag{4}$$

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- a risk process in the presence of random premiums [5]

$$A_3\varphi(u) := \frac{\lambda_c}{\lambda_c+\lambda_p}\int_0^u \varphi(u-z)dF(z) + \frac{\lambda_p}{\lambda_c+\lambda_p}\int_0^{+\infty}\varphi(u+p)dG(p); \tag{5}$$

- a risk process in the presence of stochastic and determined premiums [5]

$$A_4\varphi(u) := \int_0^\infty \int_0^{U(u,t)} \varphi(U(u,t)-z)\,dF(z)(1-K_p(t))dK_c(t)$$

$$+\int_0^\infty \int_0^{+\infty} \varphi(U(u,t)+z)dG(z)(1-K_c(t))dK_c(t); \tag{6}$$

- a risk process in a Markov environment [6]

$$A_5\varphi_i(u) := \int_0^\infty e^{-t(\lambda_i+\gamma_i)}\left\{\gamma_i \sum_{j=1}^m p_{ij}\,\varphi_j(u+c_i t)\right.$$

$$\left.+\lambda_i \int_0^{u+c_i t} \varphi_i(u+c_i t - z)dF_i(z)\right\}dt, \quad i=1,\ldots,m. \tag{7}$$

Here, the meaning of the parameters of integral operators is as follows (details are given in the corresponding references): $F(\cdot)$ and $F_i(\cdot)$ are right continuous distribution functions of insurance claims, $G(\cdot)$ is the distribution function of random premiums, $K(\cdot)$, $K_c(\cdot)$, and $K_p(\cdot)$ are distribution functions of random time intervals between sequential claims, $\lambda, \lambda_i, \lambda_c$, and λ_p are the intensities of flows of claims and premiums (in the Poisson case), c and c_i are determined intensities of arrival of premiums, $\mu = \int_0^\infty (1-F(z))dz$ is the average value of insurance claims with their distribution function F, γ_i and p_{ij} are the parameters of a Markov random environment in the ith state, and $\varphi_i(u)$ is the survival probability for the company starting from the ith state of the Markov random environment with an initial capital u.

In this article, general necessary and sufficient conditions of existence and uniqueness of a solution of problem (1), (2) are established, the method of successive approximations to the solution of the problem in the general case is substantiated, and the Hadamard conditions of correctness of the problem are established. The existence and uniqueness of the solution of the problem are obvious for the classical risk process since the corresponding integral equation (3) is the Volterra equation. For more intricate risk processes, the problem is not solved since A is a noncompact operator with a noncompact domain and it is not a contracting operator on the entire domain of definition and, hence, well-known results on the existence and uniqueness of solutions of integral equations are inapplicable in the case being considered. This article generalizes the results of [1–6] devoted to concrete models and integral equations of insurance mathematics. In [1–6], concrete sufficient conditions of the existence and uniqueness of the solution of problem (1), (2) are given for each operator and also the results of numerical experiments are presented that demonstrate the efficiency of the method of successive approximations.

We denote by $\mathfrak{F}$ a set of nondecreasing functions $\varphi(u), u \ge 0$, such that we have $0 \le \varphi(u) \le 1$. Let us introduce the following order relation on $\mathfrak{F}$: $\varphi_1 \le \varphi_2$ if we have $\varphi_1(u) \le \varphi_2(u)\ \forall u \ge 0$. We make the following general assumptions on the operator A. As is shown in [2–6], these assumptions are true for each concrete integral operator of insurance mathematics.

Assumption A. The operator A is defined on $\mathfrak{F}$ and

(A1) is linear;

(A2) retains the monotonicity of a function, i.e., if we have $\varphi(u_1) \le \varphi(u_2)\ \forall u_1 \le u_2$, then we obtain $A\varphi(u_1) \le A\varphi(u_2)\ \forall u_1 \le u_2$;

(A3) is not an expanding operator, i.e., we have $\|A\varphi_1 - A\varphi_2\| \le \|\varphi_1 - \varphi_2\|$, where $\|\varphi\| = \sup_{u \ge 0}|\varphi(u)|$;

(A4) is isotonic with respect to the order introduced on $\mathfrak{F}$, i.e., we have $\varphi_1 \le \varphi_2 \Rightarrow A\varphi_1 \le A\varphi_2$;

(A5) is continuous with respect to the pointwise convergence, i.e., we have $\varphi_k(u) \to \varphi(u)\ \ \forall u \ge 0 \Rightarrow\ A\varphi_k(u) \to A\varphi(u)\ \ \forall u \ge 0$.

LEMMA 1. On assumptions (A1)–(A4), the operator A acts from $\mathfrak{F}$ into $\mathfrak{F}$.

Proof. According to condition (A2), the operator A transforms monotone functions into monotone ones. It follows from linearity (A1) that A transforms a zero function into a zero one. This and also the property of isotonicity (A4) imply that it transforms nonnegative functions into nonnegative ones. We denote by $\mathbf{1}(u)$ the function that is identically equal to unity when $u \geq 0$. Since, for any $\varphi \in \mathfrak{F}$, we have $0 \leq \varphi(u) \leq 1$, by virtue of assumption (A4), we obtain $0 \leq A\varphi(u) \leq A\mathbf{1}(u)$, by virtue of assumption (A3), we obtain $A\mathbf{1}(u) \leq ||A\mathbf{1}(u)|| \leq ||\mathbf{1}(u)|| = 1$, and, hence, we have $0 \leq A\varphi(u) \leq 1$. The lemma is proved.

Assumption B. We assume that there are functions $\varphi_* \in \mathfrak{F}$ and $\varphi^* \in \mathfrak{F}$ such that we have

(B1) $\varphi_* \leq \varphi^*$;

(B2) $\lim\limits_{u \to \infty} \varphi_*(u) = \lim\limits_{u \to \infty} \varphi^*(u) = 1$;

(B3) $A\varphi_* \geq \varphi_*$, $A\varphi^* \leq \varphi^*$.

Comment 1. By virtue of Lemma 1, the unit function $\varphi^*(u) \equiv 1$ can be used in the capacity of the function φ^*.

Denote by $\mathfrak{F}^* \subset \mathfrak{F}$ a set of functions $\varphi(u)$ such that we have $\varphi_* \leq \varphi \leq \varphi^*$.

LEMMA 2. Under assumptions (A4) and (B3), the operator A acts from $\mathfrak{F}^*$ into $\mathfrak{F}^*$.

Proof. Let $\varphi \in \mathfrak{F}^*$ and let $\varphi_* \leq \varphi \leq \varphi^*$. Then, by virtue of the property of isotonicity (A4), we have $A\varphi_* \leq A\varphi \leq A\varphi^*$ and, according to assumption (B3), we obtain $\varphi_* \leq A\varphi_* \leq A\varphi \leq A\varphi^* \leq \varphi^*$, which is what had to be proved.

THEOREM 1 (the necessary and sufficient conditions of existence of a solution of problem (1), (2)). In order that there exist a solution of problem (1), (2) with the operator A satisfying assumptions (A), it is necessary and sufficient that condition (B) be true. The sequence of approximations

$$\{\varphi^{k+1}(u) = A\varphi^k(u),\ \varphi^0(u) \equiv \varphi^*(u),\ k = 0, 1 \ldots\}$$

monotonically pointwise converges from above to some solution of problem (1), (2), and the sequence of approximations

$$\{\varphi^{k+1}(u) = A\varphi^k(u), \varphi^0(u) \equiv \varphi_*(u),\ k = 0, 1 \ldots\}$$

monotonically pointwise converges from below to some solution of problem (1), (2).

Proof. The necessity is obvious, and any solution $\varphi(u)$ of problem (1), (2) can be used in the capacity of $\varphi_*(u)$, $\varphi^*(u)$, i.e., we have $\varphi_*(u) \equiv \varphi^*(u) \equiv \varphi(u)$. Let us prove the sufficiency of the conditions of existence of a solution by the construction of a sequence of functions that converges to some solution of the problem. Namely, we will consider the following sequence of approximations:

$$\{\varphi^{k+1}(u) = A\varphi^k(u),\ \varphi^0(u) \equiv \varphi^*(u),\ k = 0, 1 \ldots\}.$$

It follows from assumption (A2) and the monotonicity of $\varphi^*(u)$ that all the functions $\varphi^k(u)$ do not decrease with respect to *u*. The property of isotonicity (A4) of the operator A and the assumption that $A\varphi^*(u) \leq \varphi^*(u)$ imply that the sequence $\{\varphi^k(u),\ k = 0, 1 \ldots\}$ monotonically decreases. Since we have $\varphi^*(u) \geq \varphi_*(u)$, the isotonicity of the operator A and assumption (B3) implies that $\varphi^1 = A\varphi^* \geq A\varphi_* \geq \varphi_*$. Similarly, we obtain by induction that $\varphi^k \geq \varphi_*$. Thus, the sequence of functions $\{\varphi^k(u)\}$ monotonically decreases and is bounded below by the function $\varphi_*(u)$. Therefore, there is a limiting function $\varphi(u) = \lim\limits_{k \to +\infty} \varphi^k(u)$ that, as well as all $\varphi^k(u)$, does not decrease with respect to *u*, $\varphi^*(u) \geq \varphi(u) \geq \varphi_*(u)$, and, hence, according to assumption (B2), we obtain $\lim\limits_{u \to +\infty} \varphi(u) = 1$. Let us pass to the limit for k in the relationship $\varphi^{k+1}(u) = A\varphi^k(u)$. By virtue of the continuity of the operator A with respect to pointwise convergence, the limiting function $\varphi(u)$ satisfies the equation $\varphi = A\varphi$. The monotone convergence (from below) of a sequence of approximations starting from $\varphi^0(u) \equiv \varphi_*(u)$ to some solution of the problem is similarly proved.

The theorem is proved.

Assumption C. We assume that, for any $u \geq 0$, there is a power $n = n(u)$ that, perhaps, depends on u and is such that the value of the nth power A^n of the operator applied to the unit function $\mathbf{1}(u) \equiv 1$ is less than unity at the point *u*, i.e., we have $A^n\mathbf{1}(u) < 1$.

Example 1. We will consider the operator A_1 from formula (4) in which we have $F(\cdot) < 1$. Then we obtain

$$A_2 \mathbf{1}(u) = \int_0^\infty F(U(u,t))\, dK(t) < 1 \;\; \forall u \geq 0.$$

THEOREM 2 (necessary and sufficient conditions of existence and uniqueness of a solution that is everywhere smaller than unity to problem (1), (2)). In order that there exist a solution that is everywhere smaller than unity to problem (1), (2) with the operator A satisfying assumptions (A), it is necessary and sufficient that conditions (B) and (C) be true.

Proof. Necessity. Let there exist a unique solution to problem (1), (2), and let it be such that we have $\varphi(u) < 1$ for all $u \geq 0$. Taking $\varphi^*(u) \equiv 1$, we obtain $A\varphi^* = A\mathbf{1} \leq \mathbf{1}$ by Lemma 1. We take the solution $\varphi(u)$ of the problem in the capacity of $\varphi_*(u)$. By virtue of equation (1), the inequality $A\varphi(u) \geq \varphi(u)$ is true and, by virtue of condition (2), we have $\lim_{u\to\infty} \varphi(u) = 1$. Thus, condition (B) is satisfied. By Theorem 1, successive approximations

$$\{\varphi^{k+1}(u) = A\varphi^k(u),\; \varphi^0(u) \equiv 1,\; k = 0, 1 \ldots\}$$

pointwise converge to some solution $\widetilde{\varphi}(u)$ to problem (1), (2) and since such a solution is unique, we obtain $\widetilde{\varphi}(u) = \varphi(u) < 1$. Therefore, for any $u \geq 0$, there exists a power $n = n(u)$ such that we have $A^n \mathbf{1}(u) < 1$.

Sufficiency. Let assumptions (A) and (B) be fulfilled. Let us prove that a solution to problem (1), (2) exists, that it is unique, and that it is such that we have $\varphi(u) < 1$ for all $u \geq 0$. By Lemma 1, without loss of generality, we can consider that we have $\varphi^*(u) \equiv 1$. By Theorem 1, the sequence of approximations

$$\{\varphi^{k+1}(u) = A\varphi^k(u),\; \varphi^0(u) \equiv 1,\; k = 0, 1 \ldots\}$$

monotonically pointwise converges to some solution $\varphi(u)$ of problem (1), (2). By virtue of assumption (C), for any $u \geq 0$, we have $\varphi^n(u) = A^n \mathbf{1}(u) < 1$ for some *n*. Then, by virtue of monotone convergence, we have $\varphi(u) \leq \varphi^n(u) < 1$. The existence of a solution $\varphi(u) < 1$ is proved.

Let us prove the uniqueness of such a solution. Let there be two different solutions $\varphi_1(u)$ and $\varphi_2(u)$ to problem (1), (2), $\|\varphi_1 - \varphi_2\| = \sup_{u\geq 0} |\varphi_1(u) - \varphi_2(u)| > 0$.

By the definition of the concept "supremum," there is a sequence $\{u^s\}$ such that we have

$$\lim_{s\to+\infty} |\varphi_1(u^s) - \varphi_2(u^s)| = \|\varphi_1 - \varphi_2\| > 0.$$

It follows from condition (2) that we have $\lim_{u\to\infty} |\varphi_1(u) - \varphi_2(u)| = 0$. This implies that the sequence $\{u^s\}$ is bounded, $u^s \leq u^*$. According to condition (A2), the operator A retains the monotonicity of functions and, obviously, A^n possesses the same property. Therefore, for any $u \leq u^*$ and *n*, we have

$$|\varphi_1(u) - \varphi_2(u)| = |A^n\varphi_1(u) - A^n\varphi_2(u)| \leq A^n |\varphi_1(u) - \varphi_2(u)|$$

$$\leq [A^n \|\varphi_1 - \varphi_2\| \mathbf{1}](u) = \|\varphi_1 - \varphi_2\| A^n \mathbf{1}(u) \leq \|\varphi_1 - \varphi_2\| A^n \mathbf{1}(u^*).$$

By assumption (C), there exists n^* such that we have $A^{n^*} \mathbf{1}(u^*) < 1$. Then, for any $u^s \leq u^*$, we obtain $|\varphi_1(u^s) - \varphi_2(u^s)| \leq \|\varphi_1 - \varphi_2\| A^{n^*} \mathbf{1}(u^*)$. Passing to the limit for $s \to \infty$, we obtain the contradiction $\|\varphi_1 - \varphi_2\| = \lim_{s\to\infty} |\varphi_1(u^s) - \varphi_2(u^s)| < \|\varphi_1 - \varphi_2\|$. The uniqueness of the solution is proved.

The theorem is proved.

COROLLARY 1. Under assumptions (A), (B), and (C), the method of successive approximations $\{\varphi^{k+1}(u) = A\varphi^k(u),\; k = 0, 1 \ldots\}$ converges to the solution of problem (1), (2) for any initial function $\varphi^0(u)$ such that we have $\varphi_*(u) \leq \varphi^0(u) \leq \varphi^*(u)$.

Comment 2. Assumptions (A) and (B) are not sufficient to provide the uniqueness of a solution of problem (1), (2). In fact, let $\varphi^*(u) \equiv 1$, and let $\varphi_*(u)$ be any monotonically nondecreasing function such that we have $\lim_{u\to\infty} \varphi_*(u) = 1$.

Then, for the identity operator A, assumptions (A) and (B) are fulfilled but any monotonically nondecreasing function $\varphi(u)$ such that we have $\varphi_*(u) \le \varphi(u) \le \varphi^*(u)$ is a solution of problem (1), (2) with the identity operator A.

A direct check of condition (C) that guarantees the uniqueness of the solution can be difficult. We will consider one condition that is sufficient for the fulfilment of assumption (C) and regulates the action of the operator A on step functions. Let us define a step function with a parameter ν as follows:

$$\varphi_\nu(u) = \begin{cases} 0, & 0 \le u < \nu, \\ 1, & u \ge \nu. \end{cases}$$

Assumption C1. For any $\nu \ge 0$, there exists a number $\varepsilon > 0$ that is independent of ν and is such that we have $A\varphi_\nu(\nu + \varepsilon) < 1$.

LEMMA 3. Assumptions (A1)–(A4) and (C1) imply assumption (C).

Proof. We denote $\delta_\nu = 1 - A\varphi_\nu(\nu + \varepsilon) > 0$. For any $\nu \ge 0$, the function $A\varphi_\nu(u)$ is a monotonically nondecreasing function and we have $A\varphi_\nu(u) \le 1$. Therefore, for any $u \ge 0$, the inequality $A\varphi_\nu(u) \le (1 - \delta_\nu) + \delta_\nu \varphi_{\nu+\varepsilon}(u)$ is true. This implies that

$$A\mathbf{1}(u) \le (1 - \delta_0) + \delta_0 \varphi_\varepsilon(u),$$
$$A^2\mathbf{1}(u) \le A[(1 - \delta_0) + \delta_0 \varphi_\varepsilon(u)] = (1 - \delta_0)A\mathbf{1}(u) + \delta_0 A\varphi_\varepsilon(u)$$
$$\le (1 - \delta_0) + \delta_0((1 - \delta_\varepsilon) + \delta_\varepsilon \varphi_{2\varepsilon}(u)) \le 1 - \delta_0\delta_\varepsilon + \delta_0\delta_\varepsilon\varphi_{2\varepsilon}(u),$$
$$A^k\mathbf{1}(u) \le 1 - \delta_0\delta_\varepsilon \ldots \delta_{(k-1)\varepsilon} + \delta_0\delta_\varepsilon \ldots \delta_{(k-1)\varepsilon}\varphi_{k\varepsilon}(u).$$

Thus, we obtain $A^k\mathbf{1}(u) \le 1 - \delta_0\delta_\varepsilon \ldots \delta_{(k-1)\varepsilon} < 1$ for all $u \in [0, k\varepsilon)$. This implies that, for any $u \ge 0$, there exists k such that we have $A^k\mathbf{1}(u) < 1$, i.e., assumption (C) is fulfilled. The lemma is proved.

Example 2. We will show that assumption (C1) is fulfilled for the operator A_3 in which $F(0) < 1$. In fact, we have

$$A_3\varphi_\nu(u) = \frac{\lambda_c}{\lambda_c + \lambda_p}\int_0^u \varphi_\nu(u - z)dF(z) + \frac{\lambda_p}{\lambda_c + \lambda_p}\int_0^{+\infty} \varphi_\nu(u + p)dG(p)$$

$$= \frac{\lambda_c}{\lambda_c + \lambda_p}(F(\max\{0, u - \nu\}) - F(0)) + \frac{\lambda_p}{\lambda_c + \lambda_p}(1 - G(\max\{0, \nu - u\})).$$

We choose any $\varepsilon > 0$ such that we have $F(\varepsilon) < 1$. Then we obtain

$$A_3\varphi_\nu(\nu + \varepsilon) \le \frac{\lambda_c}{\lambda_c + \lambda_p}F(\varepsilon) + \frac{\lambda_p}{\lambda_c + \lambda_p}(1 - G(0)) < 1.$$

Assumption D. Let, $\forall \varepsilon > 0, \exists q^*(\varepsilon),\ 0 \le q^*(\varepsilon) < 1$: $\forall \varphi_1, \varphi_2 \in \mathfrak{F}^*$, with a distance $||A\varphi_1 - A\varphi_2|| \ge \varepsilon$, we have

$$||A\varphi_1 - A\varphi_2|| \le q^*(\varepsilon)||\varphi_1 - \varphi_2||.$$

THEOREM 3 (on the rate of convergence of the method of successive approximations). Under assumptions (A), (B), (C), and (D), the method of successive approximations that starts from $\varphi^0(u) \in \mathfrak{F}^*$ uniformly converges to any ε-vicinity of the solution of problem (1), (2) in a finite number of iterations with the rate of the geometrical progression with the ratio equal to $q^*(\varepsilon)$. If some approximation turns out to be in the ε-vicinity of the solution, all the subsequent approximations also belong to it.

Proof. Under assumptions (A), (B), and (C), by Theorem 2, there exists a unique solution $\varphi(u)$ to the problem. Let us consider a sequence of approximations $\{\varphi^{k+1}(u) = A\varphi^k(u),\ k = 0, 1 \ldots\}$. We fix an arbitrary $\varepsilon > 0$. If $||\varphi^n - \varphi|| \le \varepsilon$ for some n, then we obtain

$$||\varphi^{n+1} - \varphi|| = ||A\varphi^n - A\varphi|| = ||A(\varphi^n - \varphi)|| \le ||\varphi^n - \varphi|| \le \varepsilon$$

and, hence, for all $k \ge n$, we have $||\varphi^k - \varphi|| \le \varepsilon$. Thus, if we have $||\varphi^0 - \varphi|| \le \varepsilon$, then we obtain $||\varphi^k - \varphi|| \le \varepsilon$ for all

$k \geq 0$. Let us consider the case when $||\varphi^0 - \varphi|| \geq \varepsilon$ and, hence, we have $||\varphi^k - \varphi|| \geq \varepsilon$ when $0 \leq k \leq n$ for some $n \geq 1$. When $0 \leq k \leq n$, by virtue of assumption (D), we have

$$\varepsilon \leq ||\varphi^k - \varphi|| \leq (q^*(\varepsilon))^k ||\varphi^0 - \varphi||.$$

The inequality $q^*(\varepsilon) < 1$ implies the existence of $n = n(\varepsilon)$ such that we have $||\varphi^k - \varphi|| \geq \varepsilon$ and $||\varphi^k - \varphi|| \leq (q^*(\varepsilon))^k ||\varphi^0 - \varphi||$ when $0 \leq k \leq n(\varepsilon)$ and $||\varphi^k - \varphi|| \leq \varepsilon$ when $k > n(\varepsilon)$.

The theorem is proved.

The most intricate and informal element of investigating a particular problem of the form (1), (2) is the determination of the lower bound $\varphi_*(u)$ in condition (B3). For example, for the operator A_1, such a bound is the Cramer–Lundberg bound $\varphi_*(u) = 1 - e^{-Ru}$, where R is the positive root of the equation

$$\frac{\alpha}{c}\int_0^{+\infty} e^{Rz}[1 - F(z)]dz = 1.$$

Such a root exists on the condition that $\lambda\mu / c < 1$. Examples of exponential lower bounds $\varphi_*(u)$ for other operators are given in [2–6].

The integral operators in (3)–(7) depend on parameters such as distribution functions of claims, premiums, time intervals between sequential claims or intensities of flows of claims, etc. For an operator A that depends on an abstract parameter α, we denote such a dependence by the index α, A_α. For the operator A_α, the solution φ_α of the corresponding problem (1), (2) also depends on α. An important distinctive feature of these operators is that they monotonically depend on parameters and that the solutions of corresponding problems (1), (2) monotonically depend on parameters. For example, it would be appear natural that the smaller insurance claims, the more the survival probability.

Assumption E. Let a partial order be defined on the set of parameter values, and let the inequality $A_\alpha\varphi \geq A_\beta\varphi$ be true for all $\varphi \in \mathfrak{F}$ when $\alpha \leq \beta$.

The following lemma provides a tool for the choice of lower bounds φ_* under assumption (B).

LEMMA 4 (on the choice of lower bounds φ_*). Let the operator $A = A_\alpha$ satisfy assumptions (A). Let a lower bound φ_β exist for the some $\beta \geq \alpha$ and the corresponding operator A_β, and let this bound be such that we have $A_\beta\,\varphi_\beta \geq \varphi_\beta$ and $\lim_{u\to\infty}\varphi_\beta(u) = 1$. Then, under additional assumption (E), the function φ_β can be taken as the lower bound of φ_* from condition (B3) for the operator $A = A_\alpha$ and there exists a solution $\varphi_\alpha \geq \varphi_\beta$ of problem (1), (2) with the operator $A = A_\alpha$.

Proof. By virtue of assumption (E), we have $A_\alpha\varphi \geq A_\beta\varphi$ for all $\varphi \in \mathfrak{F}$ and, in particular, for $\varphi = \varphi_\beta$; hence, we obtain $A_\alpha\,\varphi_\beta \geq A_\beta\,\varphi_\beta \geq \varphi_\beta$. Thus, the function φ_β is the lower bound of φ_* from condition (B3) for the operator A_α. We take $\varphi^*(u) \equiv 1$ in the capacity of the upper bound φ^* from condition (B3) for the operator A_α. By Theorem 1, successive approximations $\{\varphi^{k+1}(u) = A_\alpha\,\varphi^k(u),\ \varphi^0(u) \equiv 1,\ k = 0, 1\ldots\}$ pointwise converge to some solution φ_α of problem (1), (2) with the operator $A = A_\alpha$ and, at the same time, the inequality $\varphi_\alpha \geq \varphi_\beta$ is true.

COROLLARY 2 (on the monotone dependence of the solution of problem (1), (2) on parameters). Let $\alpha \leq \beta$, and let assumptions (A), (B), (C), and (E) be fulfilled for the operators A_α and A_β. Then, for the corresponding solutions φ_α and φ_β of problem (1), (2), we have $\varphi_\alpha \geq \varphi_\beta$.

In conclusion, we formulate a statement on the correctness (a continuous dependence of solutions on parameters) of problem (1), (2).

THEOREM 4. Let, in the parameter space of problem (1), (2), some topology be defined, and let operators A_{α_k} strongly converge to some operator A_β continuous with respect to pointwise convergence,

$$\lim_{\alpha_k\to\beta}||A_{\alpha_k} - A_\beta|| = \lim_{\alpha_k\to\beta}\sup_{\varphi\in\mathfrak{F}}||A_{\alpha_k}\varphi - A_\beta\,\varphi|| = 0.$$

Let, for all operators A_{α_k} and A_β, assumptions (A), (C), and (B) be fulfilled with a common monotone lower bound φ_* such that we have $0 \leq \varphi_* \leq 1$, $\lim_{u\to\infty}\varphi_*(u) = 1$, $A_{\alpha_k}\varphi_* \geq \varphi_*$, and $A_\beta\varphi_* \geq \varphi_*$. Then, for the corresponding solutions φ_{α_k} and φ_β of problem (1), (2), we have $\lim_{\alpha_k\to\beta}\varphi_{\alpha_k}(u) = \varphi_\beta(u)\ \forall u \geq 0$.

Proof. By Theorem 2, solutions $\varphi_{\alpha_k} \geq \varphi_*$ of problem (1), (2) exist. By the Helly theorem, there is a subsequence α'_k such that the sequence $\varphi_{\alpha'_k}$ pointwise converges to some monotone function $\widetilde{\varphi} \geq \varphi_*$ on the set of continuity points $\widetilde{\varphi}$. Since the discontinuing set for the monotonous function $\widetilde{\varphi}$ is no more than countable, a subsequence α'_k can be selected so that $\varphi_{\alpha'_k}$ is everywhere convergent to $\widetilde{\varphi}$. We write $A_{\alpha'_k} \varphi_{\alpha'_k} = A_\beta \varphi_{\alpha'_k} + (A_{\alpha'_k} \varphi_{\alpha'_k} - A_\beta \varphi_{\alpha'_k})$. Passing to the limit as $\alpha'_k \to \beta$ and taking into account that $\lim_{\alpha'_k \to \beta} (A_{\alpha'_k} \varphi_{\alpha'_k}(u) - A_\beta \varphi_{\alpha'_k}(u)) = 0 \ \forall u \geq 0$, we obtain

$$\widetilde{\varphi} = \lim_{\alpha'_k \to \beta} A_{\alpha'_k} \varphi_{\alpha'_k} = \lim_{\alpha'_k \to \beta} A_\beta \varphi_{\alpha'_k} = A_\beta \lim_{\alpha'_k \to \beta} \varphi_{\alpha'_k} = A_\beta \widetilde{\varphi},$$

i.e., $\widetilde{\varphi}$ is the solution of problem (1), (2) with the operator $A = A_\beta$. But since this solution is unique, we have $\widetilde{\varphi} = \varphi_\beta$ and the entire sequence φ_{α_k} pointwise converges to φ_β.

The theorem is proved.

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